I Z A Institute of Labor Economics

Initiated by Deutsche Post Foundation

DISCUSSION PAPER SERIES

IZA DP No. 16809

A General Measure of Bargaining Power for Non-cooperative Games

Joseph-Simon Görlach
Nicolas Motz

FEBRUARY 2024

I Z A Institute
of Labor Economics

Initiated by Deutsche Post Foundation

DISCUSSION PAPER SERIES

IZA DP No. 16809

A General Measure of Bargaining Power for Non-cooperative Games

Joseph-Simon Görlach
Bocconi University, BIDSA, CEPR, IGIER and IZA;

Nicolas Motz
Universidad Complutense de Madrid

FEBRUARY 2024

ISSN: 2365-9793

IZA – Institute of Labor Economics

Schaumburg-Lippe-Straße 5–9
53113 Bonn, Germany

Phone: +49-228-3894-0
Email: publications@iza.org

www.iza.org

ABSTRACT

A General Measure of Bargaining Power for Non-cooperative Games*

Despite recent advances, no general methods for computing bargaining power in non-cooperative games exist. We propose a number of axioms such a measure should satisfy and show that they characterise a unique function. The principle underlying this measure is that the influence of a player can be assessed according to how much changes in this player's preferences affect outcomes. Considering specific classes of games, our approach nests existing measures of power. We present applications to cartel formation, the non-cooperative model of the household, and legislative bargaining.

JEL Classification: C72, C78, D01

Keywords: bargaining power, non-cooperative games

Corresponding author:
Joseph-Simon Görlach
Bocconi University
Via Sarfatti 25
20136 Milan
Italy
E-mail: josephsimon.goerlach@unibocconi.it

* We are grateful for many comments and suggestions that have greatly improved the paper. In particular, we would like to thank Joaqun Artes, Antonio Cabrales, Luis Corchón, Philipp Denter, Martin Dumav, Boris Ginzburg, Angel Hernando-Veciana, Matias Iaryczower, Nenad Kos, Antoine Loeper, Nicola Maaser, Massimo Morelli, Salvatore Nunnari, Antonio Romero-Medina, and Roberto Serrano for detailed feedback and discussions.

1 Introduction

Bargaining power and its sources have long interested economists and social scientists more generally. Examples include bargaining between buyers and sellers (Dunlop & Higgins 1942, Taylor 1995, Loertscher & Marx 2022), cartel members (Napel & Welter 2021), employers and labour unions (Hamermesh 1973, Svejnar 1986, Manning 1987), husband and wife (Basu 2006, Browning et al. 2013, Anderberg et al. 2016), the members of a political alliance (Diermeier et al. 2003, Francois et al. 2015), or legislators (Snyder et al. 2005, Kalandrakis 2006, Napel & Widgrén 2006, Ali et al. 2019, Nunnari 2021). In cooperative game theory, a vast literature deriving power indices exists with the Shapley-Shubik index (Shapley & Shubik 1954) or the Penrose-Banzhaf index (Penrose 1946, Banzhaf 1965, Dubey & Shapley 1979) being the most famous examples. Cooperative game theory, however, does not model the process through which players interact with one another and is thus not able to answer questions such as how the bargaining power of a player depends on their ability to make a counter offer, delay agreement, or veto certain outcomes. Understanding the role of such features is important for the purpose of institutional design, for instance. In non-cooperative game theory, on the other hand, the structure of the interaction between players forms an explicit part of a game, but in this context much less effort has been invested in developing measures of power. A common approach is to assume transferable utility (henceforth TU) and self-interested players, in which case bargaining power can be measured by the expected share of the surplus that each participant receives. But if utility is non-transferable or at least one player feels some degree of altruism, the utility a player achieves in equilibrium need not be informative about this player's bargaining power. To see this, consider the following example: Three countries form a military alliance and need to decide how to respond to foreign aggression. Country A is hawkish, Country B is dovish, and Country C prefers a measured response. If the agreed policy coincides with that favoured by Country C, it is not clear whether this outcome is due to the dominance of Country C or represents a compromise between countries A and B. How can we put a number on the bargaining power of each country?

In this paper, we provide a measure of bargaining power that can be applied to any non-cooperative game that features a conflict of interest among players, including games of incomplete information, but also to mechanisms or even social choice functions. As the example in the previous paragraph shows, the outcome of the game alone may not fully reveal each players' bargaining power. The fundamental idea underlying our approach is that we can instead calculate a player's power based on the effect of hypothetical changes in this player's preferences, holding all other aspects of the game fixed. In the case of the military alliance, for example, we can consider what would happen to the agreement among countries if country C was dovish or hawkish instead of moderate. If a shift in the position of country C leaves the agreement almost unchanged, it appears that Country C has little influence. If, in contrast, the outcome of the game always coincides with the one preferred by Country C, this player can be considered decisive.

Despite the simplicity of this idea, any number of measures of bargaining power can be constructed on its basis. To guide our choice between these measures, we specify five Axioms that such a function should satisfy. These axioms reflect the basic principle outlined above: The Axiom of Null players, for instance, states that a player should be assigned a bargaining power of zero if changes in their utility function never have any effect on the outcome of a game. The Axiom of Local Dictators, on the other hand, posits that some player n should be assigned a power of one if, starting from the vector of players' actual utility function, any shift in the utility function of player n produces the same outcome as if all other players' preferences were aligned with those of player n. A third axiom, Proportionality, requires the measure of bargaining power to be based on a comparison of cause and effect. Simply put, if in one game a small shift in a player's utility function has a comparable effect on the outcome of the game as a larger shift in another game, then the bargaining power of the player should be proportionally higher in the first game.

The Axiom of Invariance to Irrelevant Extensions addresses the question of which hypothetical changes in preferences should form the basis of the calculation of bargaining power. Since considering shifts of a player's preferences to *any* possible alternative preference ordering is often not practicable, delimiting the range of relevant preferences requires a judgement call on the side of the researcher analysing a specific application. Nevertheless, expanding the set of hypothetical preference orderings is in general always possible, and doing so does not affect the equilibrium of the game given players' *actual* preferences. The axiom therefore requires bargaining power to be unaffected by such extensions of the set of preferences taken into consideration.

To specify the final axiom, we introduce the concept of a compound game, which is a lottery that determines the game through which players interact. The Axiom of Compound games essentially states that the bargaining power assigned to a player in a compound game should be a weighted average of the bargaining power in each constituent game.

Our main result establishes that these five axioms characterize a unique function. This function is calculated based on a limited number of equilibria and has a clear interpretation. Specifically, the measure calculates how much the outcome of the game is affected if the utility function of a player is replaced with that of another player, with the actual utility function of the player serving as a metric that is used to quantify the size of the impact. The effect of the shift in the player's preferences is then expressed relative to the one that would occur if the player was a local dictator. The final measure is the average of this quantity across shifts to the utility function of each other player. The bargaining power of a player calculated in this way thus answers the question of how much a player is able to influence the outcome of the game compared to a local dictator.

We establish conditions under which, for games of transferable utility, our measure is equal to the expected share of the total surplus a player receives in equilibrium and thus equivalent to the conventional approach to calculating bargaining power in this setting. Whereas the two approaches often coincide, they can also produce notably different results as illustrated by the following example: Suppose there are two players who need to divide a cake among themselves and each player's utility is given by the share of the cake they receive. With probability .9 the whole cake is given to player 1 and the game ends. With the remaining probability, player 2 is given the opportunity to propose a split. If player 1 accepts such an offer, the split proposed by player 2 is implemented. If player 1 rejects, both players receive nothing. In the unique subgame perfect equilibrium of this game, player 2 proposes to keep the whole cake and player 1 accepts. The share of the cake (and of the available surplus) that player 1 receives in expectation is therefore equal to .9. However, the preferences of player 1 do not matter for the outcome. For example, the outcome of the game would not change even if player 1 preferred to give all of the cake to player 2. Given that our measure is based on the degree to which changes in a player's preferences lead to changes in the outcome, it assigns player 1 a bargaining power of zero rather than 0.9. While arguments in favour of either approach to measuring bargaining power exist, the key advantage of our method is that it is not limited to TU settings and can be applied to any game of bargaining.

The bargaining power our measure assigns to a player is conditional on players' preferences, which is in line with the well-known fact that aspects of preferences, such as impatience or risk aversion, can matter for a player's ability to achieve favourable outcomes. It can also be of interest to abstract from preferences and evaluate power as determined by the rules of the game only, for example when designing institutions before players' preferences are known. Such an ex ante measure of power can be constructed based on our ex post measure by specifying a distribution that players' preferences are drawn from and then calculating expected ex post power under said distribution. When applied to weighted voting games, we show that under suitable choices of the distribution of players' preferences the ex ante version of our measure reproduces the Shapley-Shubik index and the Penrose-Banzhaf index.

We provide three additional applications of our theory, the first of which is cartel formation. If firms are unable to make transfers between cartel members due to the risk of being caught out, firms may negotiate over individual production quantities. Knowing the influence that each firm had on the agreement can provide a basis for apportioning compensation in case of conviction, for instance. We show that under mild assumptions our measure of bargaining power takes a particularly simple form in this setting and becomes equal to a firm's profit in equilibrium divided by the profit this firm would achieve if it was a monopolist. With asymmetric costs or demand elasticities the latter number may differ widely between cartel members. Even a firm with a small market share may thus turn out to wield considerable influence.

The second application we consider is intra-household decision-making. The literature of the economics of the household has an intrinsic interest in the distribution of power between husband and wife and its underlying determinants. While the collective model of the household features explicit bargaining weights, in non-cooperative models power is an implicit product of the entire environment. Our measure can be used to quantify bargaining power in this setting and reveal the driving factors through comparative statics. We illustrate this in the context of a model analysed by Bertrand et al. (2020) and show that even a small gender wage gap can lead to wide differences in the bargaining power of husband and wife.

As a third example, we examine bargaining power in a legislative context. The decision-making power conferred onto the members of an institution is a crucial aspect of the adopted rules of procedure. We calculate the power of the players of two classic models of legislative bargaining, which can be seen as variations of a common benchmark model. This example illustrates how applying our measure to slightly modified extensive forms can reveal which aspects of the rules of the game give a player more or less influence.

The remainder of this paper is organised as follows: In Section 2, we place our study in the context of the literature. Section 3 derives our measure of bargaining power and explores its properties. Some extensions of the basic theory are introduced in Section 4. Section 5 presents applications, while Section 6 concludes.

2 Related Literature

Our main contribution to the literature is to provide a method for calculating the bargaining power of a player that can be applied to any non-cooperative model of bargaining. In cooperative game theory, a vast literature exists that develops power indices for so-called simple games with a particular interest in voting games (see, for example, Penrose 1946, Shapley & Shubik 1954, Banzhaf 1965, Deegan & Packel 1978, Johnston 1978, Holler 1982, Owen & Shapley 1989). Since a non-cooperative game can generally not be expressed as an in some sense equivalent cooperative game,[1] there is no general way to apply power indices intended for cooperative games to non-cooperative games. In non-cooperative game theory, in contrast, the only approach to measuring power that is widely applied is to assume transferable utility and selfish players, in which case power can be measured by the share of the total surplus a player receives (Taylor 1995, Haller & Holden 1997, Kambe 1999, Fréchette et al. 2005, Snyder et al. 2005, Kalandrakis 2006, Ali et al. 2019). Yet, transferable utility is a strong assumption since it requires that players have access to a common currency with constant marginal utility (Myerson 1991, p. 384). When utility is non-transferable, it is in some cases possible to express the equilibrium of the bargaining game as a weighted mean of each player's most preferred outcome, either in terms of physical outcomes or in terms of utilities. In games with more than two players such weights are often not unique, however, as in the example of the military alliance we provide in the introduction. Larsen & Zhang (2021) follow this approach to derive a measure of bargaining power for two-player games. Their measure is outcome-based in the sense that it assigns a player a high bargaining power if their utility is close to their best-possible outcome. The same is not necessarily true for our measure, as illustrated by the example in the introduction where player 1 is given a high share of the surplus regardless of their choices and thus assigned a bargaining power of zero.

[1] Papers that connect cooperative and non-cooperative game theory typically seek to provide a non-cooperative justification for a cooperative solution concept by finding a specific non-cooperative game that generates the same distribution of payoffs as the cooperative solution. See, for example, Hart & Mas-Colell (1996), Krishna & Serrano (1996), Serrano & Vohra (1997) and Laruelle & Valenciano (2008).

Steunenberg et al. (1999) develop a power measure for games where players' utilities are a function of the distance between the outcome and their ideal point. They assume a distribution that players' preferences and the status quo are drawn from and that the power of a player is inversely proportional to the average distance between their ideal point and the outcome across all possible draws. This procedure cannot calculate power conditional on a specific constellation of preferences.

Napel & Widgrén (2004) introduce the idea of measuring power based on shifts in players' preferences. They propose a measure for games with a one-dimensional outcome space and suggests different ways in which their approach can potentially be generalised. While our measure can be applied to a wider set of games, another key difference between our approach and theirs is that Napel & Widgrén focus on marginal shifts in preferences, while we shift player's preferences to match those of other players. A drawback of marginal shifts is that they may not reveal the full extent of a player's influence. To see this, consider the following example: Two players need to agree on a point on the real line. Each players' utility is equal to minus the distance between the chosen point and their ideal point. The ideal point of player 1 is equal to 1, that of player 2 equal to 2, and there is a status quo given by 2.5. The game simply consists in player 1 making a take-it-or-leave-it offer to player 2. Player 2 only accepts if the offer is weakly above 1.5 and player 1 thus offers 1.5. A marginal shift in the ideal point of player 1 leaves the outcome unchanged and the measure of Napel & Widgrén thus assigns player 1 a bargaining power of zero. However, player 1 clearly has an influence on the outcome of the game. Our measure assigns both players a bargaining power of .5.

Lojkine (2022) proposes to calculate the set of all outcomes that can arise in the equilibrium of a game under any possible preference ordering of a player. The power of a player is then given by the measure of this set. While this idea can in principle be applied to any game, doing so in practice requires non-trivial choices. Calculating the set of equilibria under all possible preference orderings is typically not feasible and restrictions therefore need to be imposed. The power of a player also depends on the chosen measure over outcomes, with no obvious options if the outcome space is unbounded. Lojkine (2022) only discusses outcome sets that are either finite or closed intervals.

We thus go beyond the existing literature by providing a new measure of bargaining power, which is the first measure that can be applied to any non-cooperative game of bargaining. Furthermore, we provide the first axiomatization of a measure of bargaining power in the field of non-cooperative game theory.

3 A Measure of Bargaining Power

In this section we present our approach to measuring bargaining power. We start by formally defining the setting in which we develop our theory.

3.1 Theoretical Framework

Let Γ be a bargaining game. Play of Γ leads to a physical outcome o, such as a distribution of resources, a contract, or a law. The set of all possible outcomes is given by O and contains at least two elements, that is, $|O| \geq 2$. $\mathcal{N}$ denotes the set of players with $N = |\mathcal{N}|$ and $2 \leq N < \infty$. The preferences of player n over the set O are represented by a utility function u_n. We assume that u_n attains a maximum on O, that is, there exists an outcome $\bar{o} \in O$ such that $u_n(\bar{o}) \geq u_n(o)$ for any $o \in O$. In addition, $u_n(o) > -\infty$ for any $o \in O$. Denote by $\mathcal{U}$ the set of all such utility functions that are considered relevant to the issue being modelled. This definition is intentionally vague and the role of the set $\mathcal{U}$ will become clear below.

Due to possible moves of nature or mixed strategies, an equilibrium of Γ generates a probability distribution over the set of outcomes O. We assume there exists a function μ^* that maps vectors of utility functions $\mathbf{u} \in \mathcal{U}^N$ into probability measures over the set of outcomes O. This assumption is satisfied if the equilibrium of Γ is always unique, possibly subject to some method of equilibrium selection. We provide an extension to games with multiple equilibria in Section 4.2.[2]

The indirect utility function of player n is defined as the expected utility of the player under the equilibrium distribution $\mu^*(\mathbf{u})$ over outcomes, that is,

$$v_n(u_n, \mathbf{u}) = \int_O u_n(o)\ d\mu^*(\mathbf{u})\ .$$

Note that the utility function of player n appears twice in the definition of the indirect utility function: once explicitly and once as part of the vector $\mathbf{u}$. Importantly, we do not require these utility functions to coincide. The indirect utility function can thus be used to evaluate "hypothetical" outcomes that would occur if the utility function of player n contained in $\mathbf{u}$ was different from the first argument u_n. To avoid confusion, we henceforth follow the convention that (vectors of) utility functions such as u_n or $\mathbf{u}$ refer to the utility functions contained in the definition of the game Γ. We call these the "endowed" utility functions and

[2]We abstract from issues such as equilibrium existence or measurability, which may require additional restrictions on utility functions in practice.

denote the set of endowed utility functions by $\mathcal{U}_E \subseteq \mathcal{U}$. In contrast, symbols such as u' or $\mathbf{u}'$ denote arbitrary (vectors of) utility functions drawn from the set $\mathcal{U}$. Since we never consider indirect utilities where the first argument is different from player n's endowed utility function, we simplify notation by suppressing dependence on the first argument and simply write $v_n(\mathbf{u})$.

We refer to the indirect utilities that arise if all players were to share the same preferences as agreement payoffs. To define these formally, let $\mathbf{1}_{u'}$ be an N-vector such that each element is equal to the same utility function $u' \in \mathcal{U}$.

Definition 1 (Agreement Payoffs)**.** *An agreement payoff of player n is an indirect utility of the form $v_n(\mathbf{1}_{u'})$ for some $u' \in \mathcal{U}$.*

Under the assumptions placed on u_n we have $u_n(\bar{o}) \geq v_n(\mathbf{1}_{u'}) > -\infty$ for any $u' \in \mathcal{U}$. In many games, the agreement payoff $v_n(\mathbf{1}_{u_n})$ under agreement on player n's endowed utility function represents the best feasible payoff from player n's perspective. In a public goods game, for example, agreement on player n's utility function would imply an equilibrium where all players apart from player n contribute.

All games we consider satisfy the following assumption:

Assumption 1 (Conflict of Interest)**.** *For any player n there exists a player m such that $v_n(\mathbf{1}_{u_n}) > v_n(\mathbf{1}_{u_m})$.*

Assumption 1 states that every player strictly prefers agreement on their endowed utility function over agreement on the endowed utility function of at least one other player. This assumption requires not only that there are two players with distinct preferences, but also that players collectively have at least some influence on the outcome. Assumption 1 thus rules out any "game" where the outcome is independent of any players' choices. On the other hand, a game where all players have the same most-preferred alternative can satisfy Assumption 1 as long as players do not have the ability to implement the mutually preferred outcome with certainty and some players disagree in their ranking of other outcomes. Assumption 1 could thus be summarised as requiring that there is a conflict of interest between players regarding the outcomes that are actually achievable. Since bargaining is a way to resolve a conflict of interest, Assumption 1 represents an essential feature of a bargaining game.

We furthermore assume that if the endowed utility functions of two players are not identical, then neither are the corresponding agreement payoffs.

Assumption 2 (Regularity)**.** *Suppose $u_n \neq u_m$ for $n, m \in \mathcal{N}$. Then $v_n(\mathbf{1}_{u_n}) \neq v_n(\mathbf{1}_{u_m})$.*

The measure of bargaining power that we derive based on a list of axioms below can be applied to games that violate Assumption 2 and in this sense the assumption is not essential. However, the axioms produce a unique function only when restricting the set of games under consideration to those satisfying both Assumption 1 and Assumption 2. Denote the set of indirect utility functions of player n generated by all such games by $\mathcal{V}_n$.

Given a suitable method of equilibrium selection, the above framework covers a very broad range of games. For example, representing a game of incomplete information in the above form is possible by making the state of the world a part of the outcome of the game.

TU-games can be defined in our context as follows:

Definition 2 (TU-Games)**.** *A game Γ satisfies transferable utility if $O = \{o \in [0,1]^N | \sum_{n=1}^{N} o_n \leq 1\}$ and each player's utility function is given by $u_n(o) = o_n$.*

The outcome of a TU-game is a vector that assigns each player a share of the available surplus and each player's utility is equal to the share they receive. Any such game satisfies Assumption 1 and Assumption 2 as long as for any two players n and m it holds that the share of the surplus that player n receives if all players agree that player n should receive the entire surplus is infinitesimally larger than the share that player n receives if all players agree that player m should receive the entire surplus.

We further illustrate the concepts by means of the following example:

Example 1. *Consider a game with outcome space $O = [0,1]$ and three players. The utility function of player $n \in \{1,2,3\}$ is given by $u_n(o) = 1-(o-i_n)^2$, where i_n is the ideal point of player n. Let $i_1 = 0$, $i_2 = 0.6$, and $i_3 = 1$. The set of endowed utility functions is thus given by*

$$\mathcal{U}_E = \left\{u(o) = 1-(o-i)^2 | i \in \{0, .6, 1\}\right\} .$$

The set of all relevant utility functions is

$$\mathcal{U} = \left\{u(o) = 1-(o-i)^2 | i \in [0,1]\right\} .$$

It is then possible to write the indirect utilities as functions of ideal points: $v_n(i_1, i_2, i_3)$.

The game starts with a move of nature that determines which, if any, of the players can subsequently freely choose the outcome of the game. Player n is chosen with probability λ_n. With probability λ_4, however, nature determines the outcome to be equal to 0.

Since each player implements their own ideal point if given the opportunity, the expected outcome of the games described in Example 1 is equal to $.6 \cdot \lambda_2 + \lambda_3$. Clearly, the influence of player 2 over the outcome of the game is increasing in λ_2. Given various values of this parameter, Panel (a) of Figure 1 shows how the indirect utility of player 2 depends on this player's ideal point. If λ_2 equals zero, hypothetical changes in i_2 would not affect the outcome of the game and the utility of player 2 under their endowed ideal point remains constant. In general, the more influence player 2 has, the more their indirect utility responds to changes in their ideal point.

Panel (b) of Figure 1 plots agreements payoffs of player 2 as a function of the ideal outcome that players agree on. Since all players implement the same outcome in this scenario, only changes in the parameter λ_4 are of consequence. In general, player 2 prefers agreement on their own ideal point over agreement on any other ideal point. However, the best feasible outcome from player 2's perspective given the rules of the game only coincides with their ideal point if $\lambda_4 = 0$. The example indicates that the difference between some player n's best-possible outcome $u_n(\bar{o})$ and the best-feasible outcome $v_n(\mathbf{1}_{u_n})$ is a measure of the degree of control that players collectively have over the outcome of the game.

Any of the games given in Example 1 satisfy Conflict of Interest and Regularity if and only if $\lambda_4 < 1$: as long as at least one player has some control over the outcome, players strictly prefer agreement on their own ideal point over agreement on any other ideal point. While the choice of the set $\mathcal{U}$ of relevant utility functions seems natural, it would be equally possible to include additional functions with different curvatures or various local maxima, for example.

3.2 Axioms

Our aim is to derive a function $\rho_n : \mathcal{V}_n \to \mathbb{R}$ that uses the information contained in the indirect utility function of a player to assign this player a number that can be interpreted as their bargaining power.[3] Below we introduce axioms that this

[3]Note that it would in principle be possible to let the bargaining power of a player depend on all players' indirect utility functions rather than just their own. Doing so would have the potential advantage that the sum of bargaining powers can be normalized to equal one, for

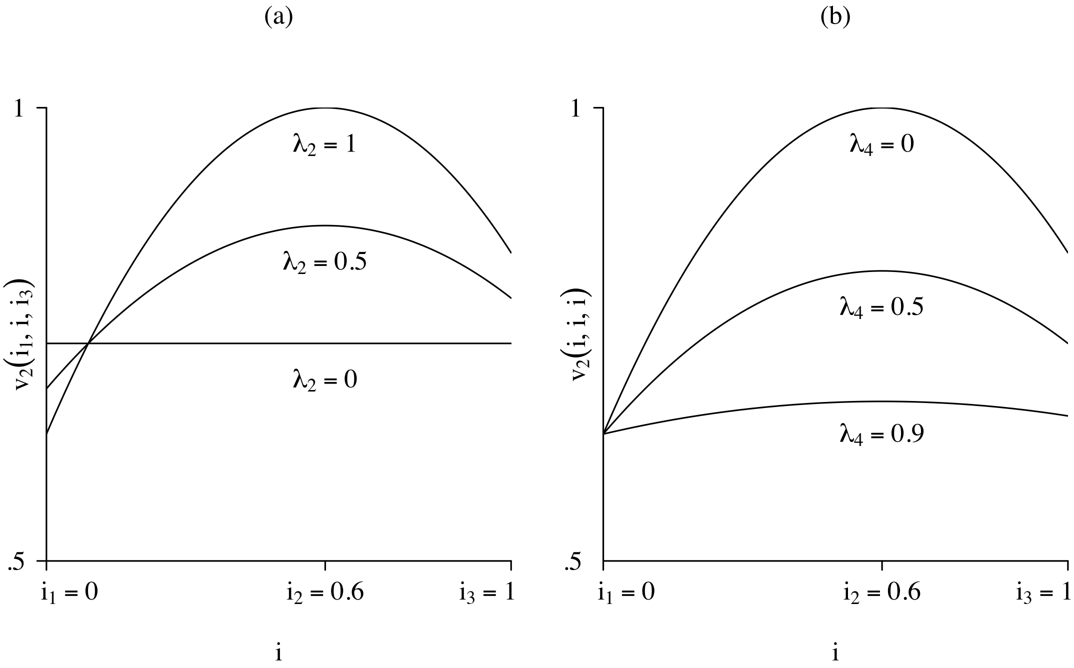

Figure 1: An Illustration of the Indirect Utility Function of Player 2 in Example 1

Notes: Panel (a) plots the indirect utility of player 2 as a function of this player's (counterfactual) ideal point for different values of λ_2, assuming $\lambda_1 = \lambda_3 = (1 - \lambda_2)/2$ and $\lambda_4 = 0$. Panel (b) plots agreement payoffs of player 2 as a function of the ideal point that players agree on for different values of λ_4. (Under agreement, the relative values of λ_1, λ_2, and λ_3 do not affect the outcome.)

function should satisfy, which require the following definitions. Throughout, we refer to Example 1 for illustrative purposes.

First, a player n is a local dictator if—given the endowed utility functions of the remaining players—the outcome of the game is always equal to the outcome that would arise if all other players shared the preferences of player n, no matter what the utility function of player n actually is. Let $\mathbf{u}'_{u'_n \leftarrow u''}$ represent the vector of utility functions created by taking some vector $\mathbf{u}'$ and replacing the utility function of player n with some function $u'' \in \mathcal{U}$.

Definition 3 (Local Dictator)**.** *Player n in some game Γ is said to be a local dictator if $\mu^*(\mathbf{u}_{u_n \leftarrow u'}) = \mu^*(\mathbf{1}_{u'})$ for any $u' \in \mathcal{U}$.*

We refer to a player satisfying Definition 3 as a local dictator rather than simply as a dictator since the property holds only for a specific vector of other players'

example. However, as we argue in Section 3.4, such a normalisation would not be compatible with our axioms in any case. Without a clear reason to include other players' payoffs, we instead opt for a simpler measure.

preferences rather than for any such vector. In Example 1, a player n satisfies the definition of a local dictator if and only if $\lambda_n = 1 - \lambda_4$. The definition therefore does not imply that a local dictator has the ability to implement their most preferred outcome with certainty. Instead, the defining property of a local dictator is that their influence over the outcome is equal to the collective influence of all players.

A null player, on the other hand, is a player who never has any impact on the outcome of the game.

Definition 4 (Null Player)**.** *Player n in some game Γ is said to be a null player if $\mu^*(\mathbf{u}') = \mu^*(\mathbf{u}'_{u'_n \leftarrow u''})$ for any $\mathbf{u}' \in \mathcal{U}^N$ and $u'' \in \mathcal{U}$.*

Assumption 1 rules out that a player could simultaneously be a local dictator and a null player.[4] In Example 1, player n is a null player if and only if $\lambda_n = 0$, which implies that player n is both null and a local dictator if and only if $\lambda_4 = 1$. As explained above, the latter case would, however, violate the Assumption of Conflict of Interest.

Finally, a compound game is a game that starts with a random draw that determines which of a number of other games is played. Importantly, all players are aware of which game is selected and—given that equilibrium is assumed to be unique—the behaviour of players is thus identical to the case where each game is played in isolation. The constituent games of a compound game need to be compatible in the sense that they share the same sets of outcomes and players.

Definition 5 (Compound Game)**.** *Γ is said to be a compound game if*

i. there exists a finite set of games $\mathbf{\Gamma} = \{\Gamma_1, \Gamma_2, ..., \Gamma_G\}$ possessing equal sets of outcomes O and players $\mathcal{N}$, and

ii. the game Γ begins with a commonly-observed move of nature that selects one game from $\mathbf{\Gamma}$ to be played subsequently, and each game $\Gamma_g \in \mathbf{\Gamma}$ is chosen with probability λ_g.

We write

$$\Gamma = \sum_{g=1}^{G} \lambda_g \Gamma_g \ .$$

[4] A player can be both a local dictator and a null player only if $\mu^*(\mathbf{1}_{u'}) = \mu^*(\mathbf{1}_{u''})$ for any $u', u'' \in \mathcal{U}$. To see this, suppose there exist $u', u'' \in \mathcal{U}$ such that $\mu^*(\mathbf{1}_{u'}) \neq \mu^*(\mathbf{1}_{u''})$. Then n being a local dictator implies $\mu^*(\mathbf{u}_{u_n \leftarrow u'}) = \mu^*(\mathbf{1}_{u'}) \neq \mu^*(\mathbf{1}_{u''}) = \mu^*(\mathbf{u}_{u_n \leftarrow u''})$. It follows that n is not null, which would require $\mu^*(\mathbf{u}_{u_n \leftarrow u'}) = \mu^*(\mathbf{u}_{u_n \leftarrow u''})$. Assumption 1 is thus sufficient to ensure that a player cannot be a local dictator and a null player at once since it implies that not all agreement outcomes are equal.

Any of the games in Example 1 such that $\lambda_n < 1$ holds for any $n \in \{1, 2, 3, 4\}$ can be seen as a compound game.

We now state and discuss the axioms that we impose on the measure of bargaining power ρ_n. Our goal is to measure bargaining power based on shifts in a player's preferences. The set $\mathcal{U}$ of relevant utility functions determines which possible shifts in preferences can be taken into consideration. For the practical purpose of calculating bargaining power, including all utility functions corresponding to any possible preference ordering over O in the set $\mathcal{U}$ is often not feasible. The specification of the set $\mathcal{U}$ is then an essentially arbitrary choice of the researcher. Given this element of arbitrariness, we impose that adding utility functions to the set $\mathcal{U}$ should not affect bargaining power. An additional reason for this restriction is that extending the set $\mathcal{U}$ by including additional utility functions has no effect on the actual equilibrium of the game as long as the set of players remains unchanged.

Axiom A1 (Invariance under Irrelevant Extensions)**.** *Given some game Γ, construct a second game Γ_+ by adding elements to the set of relevant utility functions $\mathcal{U}$ while keeping all other aspects of the game fixed. Denote the indirect utilities of some player n corresponding to the two games by v_n and $v_{+,n}$, respectively. Then $\rho_n(v_n) = \rho_n(v_{+,n})$.*

In general, adding utility functions to the set $\mathcal{U}$ means that the indirect utilities of the extended game contain information about how the outcome of the game behaves under additional combinations of players' preferences. An implication of Axiom A1 is that this additional information is not taken into account. A measure satisfying this axiom is thus a "local" measure of bargaining power, specific to players' endowed utility functions. For example, a player's bargaining power in a war of attrition would depend on their and their competitor's valuation of the prize, rather than being a "global" measure of power across all possible combinations of valuations. In Section 4.1, we consider a method that uses our local measure of power to calculate power across a range of possible preferences.

Axiom A2 (Null Players)**.** *If player n is a null player in a game Γ with their associated indirect utility function given by v_n, then $\rho_n(v_n) = 0$.*

Axiom A3 (Local Dictators)**.** *If player n is a local dictator in a game Γ with their associated indirect utility function given by v_n, then $\rho_n(v_n) = 1$.*

Axioms A2 and A3 impose that a local dictator is assigned a higher bargaining power than a null player and further normalise the power of such players to one and zero, respectively.

Axiom A4 (Compound Games)**.** *Let* $\Gamma = \sum_{g=1}^{G} \lambda_g \Gamma_g$ *and denote by* v_n*,* $v_{1,n}$*, ...,*$v_{G,n}$ *the corresponding indirect utility functions of some player* n*. If all constituent games* Γ_1 *to* Γ_G *share the same agreement payoffs, then for any player* n

$$\rho_n(v_n) = \sum_{g=1}^{G} \lambda_g \rho_n(v_{g,n}) \ .$$

The Axiom of Compound Games states that the bargaining power of a player in a compound game Γ should be equal to a weighted average of the bargaining power of this player in each of the constituent games of Γ. This property is desirable since equilibrium uniqueness and the assumption that players are aware of which game is selected ensure that behaviour in each constituent game is the same as if this game were played on its own. The outcome of the game as a whole is thus a weighted average of the outcomes in each constituent game, as are the indirect utility functions. Furthermore, the assumption of equal agreement payoffs included in the axiom implies that players collectively have the same degree of control over the outcome of each game. The meaning of being a local dictator is thus the same across games. In Example 1, consider the case that $\lambda_4 = 0$, which implies that the players have full control over the outcome of the game. Then the probability that player n is able to choose the outcome, λ_n, is an obvious measure of this player's bargaining power. The example indicates that it is natural to think of the bargaining power of a player in a compound game as their expected power across constituent games as required by the axiom.

Axiom A5 (Proportionality)**.** *Let* v_n *be an indirect utility corresponding to a game where player* n *is a null player. Denote by* v'_n *and* v''_n *indirect utilities corresponding to two alternative games, where* v'_n *and* v''_n *are identical to* v_n *except that* $v'_n(\mathbf{u}_{u_n \leftarrow u'}) = v_n(\mathbf{u}_{u_n \leftarrow u'}) - c$ *and* $v''_n(\mathbf{u}_{u_n \leftarrow u''}) = v_n(\mathbf{u}_{u_n \leftarrow u''}) - c$ *for some* $c \neq 0$ *and* $u', u'' \in \mathcal{U}_E \setminus u_n$*. Then*

$$\frac{\rho_n(v'_n)}{\rho_n(v''_n)} = \frac{v''_n(\mathbf{1}_{u_n}) - v''_n(\mathbf{1}_{u''})}{v'_n(\mathbf{1}_{u_n}) - v'_n(\mathbf{1}_{u'})} \ .$$

The central idea underlying our approach to measuring bargaining power is that changes in the preferences of a player reveal information about this player's power through the effect that such a change has on the outcome of the game. Axiom A5 formalizes the intuition that if in one game a small shift in a player's utility function has a comparable effect on the outcome of the game as a larger shift in another game, then the bargaining power of the player should be proportionally higher in the first game. The starting point of Axiom A5 is a game where player n is a null player and replacing the endowed utility function of this player with any other utility function accordingly has no effect on the outcome. In each of the games corresponding to the indirect utilities v'_n and v''_n, on the other hand, exactly one such shift has an impact on the outcome: replacing u_n with an endowed utility function u' in the game leading to indirect utility v'_n and replacing u_n with the endowed utility function u'' in the case of the indirect utility v''_n. Furthermore, measured in utils of player n, the size of the impact is the same in both games. However, the size of the underlying shift in the preferences of player n may differ across games. Specifically, u' may represent a bigger change in the preferences of player n relative to u_n than u'' does, or vice versa. This raises the question of how to quantify the size of such a shift. Note that for a player with a *given* degree of power, a larger change in preferences produces a stronger impact on the outcome. To fix the power of a player, we can consider the scenario where the player is a local dictator and calculate how much a change in preferences would affect the outcome of the game in this case. If player n was a local dictator, replacing their endowed utility function with some utility function u' would shift the outcome from $\mu^*(\mathbf{1}_{u_n})$ to $\mu^*(\mathbf{1}_{u'})$. Expressed in utils of player n, the size of the shift in preferences can thus be measured as $v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})$. Accordingly, the ratio

$$\frac{v''_n(\mathbf{1}_{u_n}) - v''_n(\mathbf{1}_{u''})}{v'_n(\mathbf{1}_{u_n}) - v'_n(\mathbf{1}_{u'})}$$

compares the size of the preference shifts from u_n to u' and from u_n to u'' and Axiom A5 imposes that the bargaining power of player n is proportionally higher in the game where the relevant shift in preferences is smaller.

3.3 The Main Result

For the purpose of stating the main result, denote by $\mathcal{U}_{\neq n}$ the set of endowed utility functions such that agreement on any of these functions generates a different level of utility for player n than agreement on their own endowed utility function would, that is,

$$\mathcal{U}_{\neq n} = \{u' \in \mathcal{U}_E \mid v_n(\mathbf{1}_{u'}) \neq v_n(\mathbf{1}_{u_n})\} .$$

In the context of games satisfying Assumption 2 it holds that $\mathcal{U}_{\neq n} = \mathcal{U}_E \setminus u_n$.

We can now state our main result:

Theorem 1. *A function $\rho_n : \mathcal{V}_n \to \mathbb{R}$ satisfies axioms A1, A2, A3, A4, and A5* *if and only if*

$$\rho_n(v_n) = \frac{1}{|\mathcal{U}_{\neq n}|} \sum_{u' \in \mathcal{U}_{\neq n}} \frac{v_n(\mathbf{u}) - v_n(\mathbf{u}_{u_n \leftarrow u'})}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})} \ . \tag{1}$$

Proof. See Appendix A. □

The measure of bargaining power introduced by Theorem 1 has a straightforward interpretation. Each of the terms of the sum calculates the effect that a change in the preferences of player n has on the outcome, with the endowed utility function of player n serving as a metric. The effect is then expressed as a share of the one that would occur if player n was a local dictator, which is given by $v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})$. Final bargaining power is calculated as a simple average of these individual terms across the relevant range of preferences, where the latter consists of the utility functions of other players that differ from that of player n in the sense that they generate different agreement payoffs. The question answered by the function ρ_n is therefore simply how much influence player n has on the outcome of the game relative to that of a local dictator.

As pointed out above, it holds for games satisfying Assumption 2 that $\mathcal{U}_{\neq n} = \mathcal{U}_E \setminus u_n$. In such cases the sum in Equation (1) could accordingly be expressed equivalently over elements of the latter set. However, summing over elements of the set $\mathcal{U}_{\neq n}$ ensures that the value of ρ_n is well-defined also in the context of games violating Assumption 2.

In Appendix A we present the proof of Theorem 1 as a series of lemmas that clearly show the additional restrictions that each axiom imposes on the shape of the function ρ_n. First, the Axiom of Invariance under Irrelevant Extensions implies that ρ_n can only depend on indirect utilities $v_n(\mathbf{u}')$ that are functions of vectors of endowed utility functions, that is, $\mathbf{u}' \in \mathcal{U}_E^N$. This is the case since, according to the axiom, adding or eliminating utility functions from the set $\mathcal{U}$ should not affect bargaining power as long as all other aspects of the game remain unchanged. All utility functions that are not endowed utility functions can therefore be deleted without affecting the value of ρ_n. Eliminating endowed utility functions, on the other hand, is not possible since doing so would imply a change in the set of players. Since the set of endowed utility functions is finite, ρ_n thus depends on a finite number of indirect utilities.

Next, the Axiom of Compound Games has the consequence that ρ_n must be an affine function on a class of games sharing the same outcome sets, sets of players, and agreement payoffs. To see this, note that the indirect utilities of a player in a compound game $\Gamma = \sum_{g=1}^{G} \lambda_g \Gamma_g$ are a weighted average of the indirect utilities of each constituent game: $v_n = \sum_{g=1}^{G} \lambda_g v_{g,n}$. If all constituent games share the same agreement payoffs, the Axiom of Compound Games requires

$$\rho_n \left(\sum_{g=1}^{G} \lambda_g v_{g,n} \right) = \sum_{g=1}^{G} \lambda_g \rho_n(v_{g,n}) \ .$$

Given that ρ_n is a functions of a finite number of utilities, which are real numbers, affinity implies the functional form

$$\rho_n(v_n) = \beta + \sum_{\mathbf{u}' \in \mathcal{U}_E^N} \alpha(\mathbf{u}') \ v_n(\mathbf{u}') \ ,$$

where β and each $\alpha(\mathbf{u}')$ are real numbers. The value of these coefficients must be constant across games with equal agreement payoffs, but may differ between such classes of games. In other words, the coefficients may be functions of agreement payoffs.

The Axiom of Null Players imposes $\rho_n(v_n) = 0$ if player n is a null player. The definition of a null player implies that any indirect utilities $v_n(\mathbf{u}')$ and $v_n(\mathbf{u}'_{u'_n \leftarrow u''})$, which differ only in the included utility function of player n, take the same value. However, the definition does not pin down the level of these payoffs. For ρ_n to take the value zero in any game in which player n is a null player, it is thus necessary that ρ_n can be expressed as a function of differences of indirect utilities $v_n(\mathbf{u}') - v_n(\mathbf{u}'_{u'_n \leftarrow u''})$. Since any such difference is equal to zero when n is null, the constant β must also be equal to zero.

Note that $\mathbf{u}_{u_n \leftarrow u'}$ is a vector of utility functions that differs from the vector of endowed utility functions only in the utility function of player n. The definition of a local dictator restricts any utility $v_n(\mathbf{u}_{u_n \leftarrow u'})$ to be equal to $v_n(\mathbf{1}_{u'})$. n being a local dictator does not, however, restrict the values of other indirect utilities where the utility functions of players other than n differ from their endowed utility functions. To ensure that $\rho_n(v_n) = 1$ if n is a local dictator as required by the Axiom of Local Dictators, ρ_n thus cannot depend on indirect utilities other than those of the form $v_n(\mathbf{u}_{u_n \leftarrow u'})$.[5]

The above arguments establish that ρ_n takes the shape

$$\rho_n(v_n) = \sum_{(u', u'') \in \mathcal{U}_E^2} \alpha(u', u'') \ [v_n(\mathbf{u}_{u_n \leftarrow u'}) - v_n(\mathbf{u}_{u_n \leftarrow u''})] \ . \tag{2}$$

[5] As was pointed out above, the coefficients used to calculate ρ_n may depend on the values of agreement payoffs. The final expression for ρ_n given in Theorem 1 therefore contains agreement payoffs in additions to indirect utilities of the form $v_n(\mathbf{u}_{u_n \leftarrow u'})$.

It is then possible to factor out an arbitrary non-zero number C in the form

$$\rho_n(v_n) = \ C \sum_{(u',u'')\in\,\mathcal{U}_E^2} \frac{\alpha(u',u'')}{C}\ [v_n(\mathbf{u}_{u_n\leftarrow u'}) - v_n(\mathbf{u}_{u_n\leftarrow u''})]\ .$$

Since the values of coefficients are at this point undetermined, we can redefine their values to include the division by C. In order to satisfy the Axiom of Local Dictators, the constant C multiplying the sum must be equal to one divided by the value that the remaining part of the expression takes in case player n is a local dictator, that is,

$$\rho_n(v_n) = \frac{\sum_{(u',u'')\in\,\mathcal{U}_E^2} \alpha(u',u'')\ [v_n(\mathbf{u}_{u_n\leftarrow u'}) - v_n(\mathbf{u}_{u_n\leftarrow u''})]}{\sum_{(u',u'')\in\,\mathcal{U}_E^2} \alpha(u',u'')\ [v_n(\mathbf{1}_{u'}) - v_n(\mathbf{1}_{u''})]}\ .$$

If the set of endowed utility functions contains only two elements—either because there are only two players or because various players share the same utility function—the preceding expression simplifies to the form given by Theorem 1. The role of the Axiom of Proportionality is thus to pin down the values of the α-coefficients in the case of more than two endowed utility functions. The remainder of the proof relies on the functional form for ρ_n given by Equation (2). Denote by v_n' and v_n'' indirect utilities as defined in the statement of the Axiom of Proportionality. By the construction of the indirect utility v_n', any of the utility differences in Equation (2) involving the payoff $v_n'(\mathbf{u}_{u_n\leftarrow u'})$ are equal to c or $-c$ while any other utility differences are equal to zero. It follows that

$$\rho_n(v_n') = c \sum_{u'''\in\,\mathcal{U}_E\backslash u'} [\alpha(u''',u') - \alpha(u',u''')]\ .$$

Denoting the sum in the preceding expression as $\tilde{\alpha}(u')$, the Axiom of Proportionality therefore implies

$$\frac{\rho_n(v_n')}{\rho_n(v_n'')} = \frac{\tilde{\alpha}(u')}{\tilde{\alpha}(u'')} = \frac{v_n''(\mathbf{1}_{u_n}) - v_n''(\mathbf{1}_{u''})}{v_n'(\mathbf{1}_{u_n}) - v_n'(\mathbf{1}_{u'})}\ .$$

The fact that such an equality must hold for any pair of utility functions $u', u'' \in \mathcal{U}_E \backslash u_n$ is sufficient to determine the value of each coefficient $\tilde{\alpha}$ up to multiplication by a common constant δ. More specifically, it must hold that

$$\tilde{\alpha}(u') = \delta/[v_n'(\mathbf{1}_{u_n}) - v_n'(\mathbf{1}_{u'})] \tag{3}$$

for any $u' \in \mathcal{U}_E \setminus u_n$. Note that Equation (2) can be rearranged as follows:

$$\begin{aligned}\rho_n(v_n) &= -\sum_{u'\in\,\mathcal{U}_E}\left[\sum_{u'''\in\,\mathcal{U}_E\setminus u'} \alpha(u''',u') - \alpha(u',u''')\right]\; v_n(\mathbf{u}_{u_n\leftarrow u'}) \\ &= -\sum_{u'\in\,\mathcal{U}_E} \tilde{\alpha}(u')\; v_n(\mathbf{u}_{u_n\leftarrow u'})\;.\end{aligned}$$

After using Equation (3) to substitute for every $\tilde{\alpha}(u')$ such that $u' \in \mathcal{U}_E \setminus u_n$, there then remain two unknowns: the constant δ and the coefficient $\tilde{\alpha}(u_n)$. The Axiom of Null Players and the Axiom of Local Dictators provide two equations that can be solved for these unknowns, yielding

$$\rho_n(v_n) = \frac{1}{|\mathcal{U}_E \setminus u_n|} \sum_{u'\in\,\mathcal{U}_E\setminus u_n} \frac{v_n(\mathbf{u}) - v_n(\mathbf{u}_{u_n\leftarrow u'})}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})}\;.$$

Recalling that under Assumption 2 it holds that $\mathcal{U}_E \setminus u_n = \mathcal{U}_{\neq n}$ completes the proof.

3.4 Additional Properties

In this section, we discuss properties of the function ρ_n introduced by Theorem 1 that are not directly stated in the axioms. For example, the Axiom of Compound Games implies that ρ_n is a continuous function when restricted to a class of games that share equal agreement payoffs. In fact, ρ_n turns out to be a continuous function in general, which follows since Assumption 1 guarantees that the denominator in Equation (1) is not equal to zero for any $v_n \in \mathcal{V}_n$. This is an attractive property since it implies that players are assigned a similar bargaining power in games that generate similar indirect utility functions. Furthermore, the function ρ_n is invariant under affine transformations of players' utility functions, which is reassuring since such transformations do not affect behaviour.

It is also instructive to compare the properties of our measure of bargaining power to those of the Shapley value. The Shapley value is a solution concept for cooperative games and thus assigns each player a payoff, while our measure is intended for non-cooperative games. Nevertheless, both are functions that take a description of a game and assign a real number to each player and two of the four axioms that define the Shapley value are in fact related to axioms imposed by us. In particular, both approaches rely on an Axiom of Null Players and the definition of a null player is similar in both contexts. In addition, our Axiom of Compound

Games is a weaker version of the Axiom of Linearity imposed on the Shapley value. As a consequence, ρ_n is not a linear function and only affine on subsets of games sharing the same agreement payoffs. Shapley's Axiom of Anonymity is not required for our result, even though the function ρ_n is also invariant to the re-labelling of players. On the contrary, the axioms of Invariance under Irrelevant Extensions, Local Dictators, and Proportionality are unique to our setting. The clearest point of departure, however, is that the Axiom of Efficiency requires the payoffs assigned to players by the Shapley value to add up to one. Such a normalisation is not compatible with our axioms, in particular those that require the measure to be local in the sense of depending on players' endowed utility functions and the corresponding equilibrium of the game. The reason is that locally all players may be indistinguishable from null players in the sense that no individual player could change the outcome even if they tried. All players are then assigned a bargaining power of zero. A situation of this type can arise, for example, in an equilibrium of a voting game where no player's ballot can swing the outcome. An advantage of not normalizing the sum of power coefficients is that this sum reveals information about the nature of the game, namely the degree to which players mutually block each other from affecting the outcome.

A final characteristic that we want to highlight in this section is the relationship between our measure and the share of the surplus that a player receives in a TU-game, which is commonly used to assess a player's bargaining power in that setting. As the following result demonstrates, the two approaches coincide under certain conditions.

Proposition 1. *In a TU-game, $\rho_n(v_n) = v_n(\mathbf{u})$ if the outcomes $\mu^*(\mathbf{u}_{u_n \leftarrow u'})$, $\mu^*(\mathbf{1}_{u_n})$, and $\mu^*(\mathbf{1}_{u'})$ are Pareto efficient for any $u' \in \mathcal{U}_{\neq n}$.*

Proof. See Appendix B. □

The proof of Proposition 1 proceeds by using the definition of a TU-game and the assumption of Pareto efficient outcomes to determine the values of the indirect utilities entering ρ_n. First, Pareto efficiency implies that one player receives all resources if all players agree that this would be the ideal outcome. Accordingly, $v_n(\mathbf{1}_{u_n}) = 1$ and $v_n(\mathbf{1}_{u'}) = 0$ for any $u' \neq u_n$. In addition, under the vector of utility functions $\mathbf{u}_{u_n \leftarrow u'}$ all players prefer to redistribute resources from player n to some other player, and Pareto efficiency therefore implies $v_n(\mathbf{u}_{u_n \leftarrow u'}) = 0$. Substituting accordingly in Equation (1) yields the desired result. The intuition behind this result is that efficiency of the agreement payoffs implies that the players collectively have full control over the allocation of the surplus and so would a local dictator.

A player's bargaining power thus depends on what share of the total surplus they have under their individual control. Efficiency of the outcomes $\mu^*(\mathbf{u}_{u_n \leftarrow u'})$ further implies that any part of the surplus that player n receives in equilibrium is actually due to their influence, rather than simply assigned to them due to some feature of the rules of the game (recall the example in the introduction). The player's surplus share then fully reflects their bargaining power. Inefficiency of any of the outcomes listed in Proposition 1 implies that $\rho_n(v_n) = v_n(\mathbf{u})$ does not hold in general, even though the equality can arise coincidentally.

3.5 An Example

Before presenting more substantive applications of our measure of bargaining power in Section 5, we illustrate its use through a simpler example.

Example 2. *Consider a game in which three players need to agree on a point in the set* $[0, 1]$ *and bargaining takes place with an infinite time horizon. In the first period, player 1 makes an offer to player 2. If player 2 accepts, this offer is implemented and the game ends. If player 2 rejects the initial offer, player 2 plays Rubinstein bargaining with player 3 from the second period onwards. More specifically, players 2 and 3 alternate in making offers until an offer is accepted by the other player, with player 2 making the first offer. The utility of player* $n \in \{1, 2, 3\}$ *depends on the accepted offer* o *and the period of agreement* T*, and is given by*

$$u(o, T) = \delta^T (1 - |o - i_n|) \ ,$$

where δ *is a common discount factor and* i_n *is the ideal point of player* n*, with* $i_1 = 0$*,* $i_2 = .5$ *and* $i_3 = 1$*.*

The unique subgame perfect equilibrium of the game in Example 2 in the limit as δ approaches 1 can be characterized as follows: In the subgame starting in period 2, player 2 offers the outcome $(i_2 + i_3)/2$ and player 3 accepts (Rubinstein 1982). In period 1, player 2 is willing to accept any offer they like at least as much as the outcome in period 2. Player 1 offers their preferred outcome among those that player 2 accepts and the game ends in period 1. Under the ideal points given above, the outcome of the game is equal to .25. This game puts player 3 in a peculiar situation: if player 3 could commit to accepting any offer from player 2 in period 2, the outcome of the game would be equal to 0.5.[6] Player 3's attempt to

[6] Player 2 would then propose the outcome 0.5 in period 2. If player 2 is perfectly patient, player 1 has no other option than offering the same outcome in period 1.

achieve a favourable result in the bargaining with player 2 thus backfires, leading to an outcome that is overall worse for player 3.

Note that the game described in the preceding paragraph is similar in terms of the constellation of ideal points to the example of the military alliance in the introduction. As in that example, the outcome of the game by itself allows only limited insights into the distribution of bargaining power. In the current example one can conclude that player 1 must have some influence, but the relative importance of players 2 and 3 remains unclear. After going through the calculations required to apply Equation (1),[7] the results are $\rho_1 = .5$, $\rho_2 = .5$, and $\rho_3 = -.25$. The most eye-catching aspect of these results is likely the negative bargaining power of player 3, which reflects that player 3's actions are to their own detriment as pointed out above. Players 1 and 2, on the other hand, are found to be equally powerful: player 1 has the ability to make the first offer, but this does not give them the upper hand over player 2 since the latter is patient and has the option to negotiate with player 3 instead.

4 Extensions

4.1 Ex Ante Power and Relation to Voting Power Indices

Our measure of bargaining power calculates power based on the endowed utility functions and power may depend on preferences. In some sense this is natural: for example, it is generally held that more impatient negotiators are at a disadvantage. In some cases, and in particular for the purpose of institutional design, it can nevertheless be of interest what degree of influence the rules of the game assign to each player independently of preferences. Napel & Widgrén (2004) distinguish in this context between an ex ante and an ex post perspective, that is, assessments of power before or after players' preferences have been revealed. Following their approach, we can use our ex post measure to calculate power from an ex ante perspective. Doing so requires specifying a distribution F that players' preferences are drawn from and ex ante power is simply equal to expected ex post power under F. Depending on the chosen distribution, it may be possible to calculate this

[7]Take the calculation of ρ_1 as an example. In this setting, replacing one player's utility function with that of another player simply requires shifting the former player's ideal point to match that of the latter. When players' ideal points coincide, the outcome of the game is equal to the common ideal point with certainty. When the ideal point of player 1 is set equal to that of player 2, the outcome is equal to 0.5, while if the ideal point of player 1 is shifted to equal that of player 3, the outcome is .75. We thus have $v_1(\mathbf{u}) = .75$, $v_1(\mathbf{u}_{u_1 \leftarrow u_2}) = .5$, $v_1(\mathbf{u}_{u_1 \leftarrow u_3}) = .25$, $v_1(\mathbf{1}_{u_1}) = 1$, $v_1(\mathbf{1}_{u_2}) = .5$, and $v_1(\mathbf{1}_{u_3}) = 0$, which is all the information need to calculate ρ_1.

expectation exactly, such as when F has finite support. Otherwise, expected power can be calculated numerically by drawing preferences, calculating ex post power, and repeating this process until the mean across draws converges. Denote by $\bar{\rho}_n(F)$ the ex ante bargaining power of player n under the distribution F calculated based on the ex post measure ρ_n.

In practice, care needs to be taken with respect to preference profiles that violate Assumption 1, since the value of ρ_n is not defined in such cases. One option is to specify F such that such cases do not occur. Alternatively, it may be possible to resolve the problem by assigning a default value when ρ_n is not defined. For example, if players' utility functions are identical, it may be reasonable to assign each player a power of zero or of $1/N$. In other games, such as the example that follows, a natural extension of ρ_n exists.

We now use the ex post and ex ante measures ρ_n and $\bar{\rho}_n$ to investigate the relationship between our theory and the literature on voting power indices, which calculate the power of players in weighted voting games. In such games, a committee decides whether to accept or reject a proposal. The outcome space is equal to $\{0, 1\}$, where 1 corresponds to acceptance of the proposal, while 0 indicates rejection. It is typically assumed that players have strict preferences over the two outcomes and it is then without loss of generality to let all players' utility functions be given either by u^0 or by u^1, where $u^i(o) = 1$ if $o = i$ and $u^i(o) = 0$ otherwise. Beyond the set of players, a weighted voting game is characterised by a voting rule, which consists of a quota $q > 0$ and a vector of weights $w \in \mathbb{R}^N_+$, one for each member of the committee. Players simply vote in favour of or against the proposal and the proposal is accepted if and only if the sum of all players' weights who vote in favour is at least equal to q. All players voting in favour is sufficient for acceptance, that is, $\sum_{n=1}^N w_n \geq q$. Assume players vote sincerely. Denote by $S \subseteq \mathcal{N}$ the set of players who prefer acceptance under the endowed utility functions $\mathbf{u}$. In the language of cooperative game theory, the players in S form a coalition and the value V of the game indicates whether a coalition wins: $V(S) = 1$ if $\sum_{n \in S} w_n \geq q$ and $V(S) = 0$ otherwise.

Under any given constellation of preferences $\mathbf{u}$ and the corresponding profile of votes, player n is said to be pivotal if them changing their vote would change the outcome of the game. Since such a player satisfies the definition of a local dictator, the measure ρ_n assigns them a power of 1. If a player is not pivotal, their preferences do not matter for the outcome and $\rho_n = 0$. Note, however, that agreement among the players implies that Assumption 1 is violated and the value of ρ_n is not defined. It seems natural to introduce the convention that in such unanimous games (that is, $S = \emptyset$ or $S = \mathcal{N}$), $\rho_n = 1$ if player n is pivotal and $\rho_n = 0$ otherwise. Under this convention, we have the following result.

Proposition 2. *Let v_n^S denote the indirect utility of player n corresponding to a weighted voting game where the set of players S prefers acceptance. Assume $\rho_n(v_n^{S=\emptyset}) = 1$ if $w_n \geq q$ and $\rho_n(v_n^{S=\emptyset}) = 0$ otherwise. Also assume $\rho_n(v_n^{S=\mathcal{N}}) = 1$ if $\sum_{m \in S \setminus n} w_m < q$ and $\rho_n(v_n^{S=\mathcal{N}}) = 0$ otherwise. Then there exist distributions F_{PB} and F_{SS} such that $\bar{\rho}_n(F_{PB})$ is equal to the Penrose-Banzhaf index and $\bar{\rho}_n(F_{SS})$ is equal to the Shapley-Shubik index.*

Proof. See Appendix B. □

Under suitable choices of the distribution of preferences F, $\bar{\rho}_n(F)$ is thus equal to the Shapley-Shubik index or the Penrose-Banzhaf index. These indices are based on cooperative game theory, and showing that they are equivalent to $\bar{\rho}_n(F)$ is possible since a weighted voting game is a rare case of a game that can naturally be expressed in a cooperative or a non-cooperative form. In general, however, voting power indices cannot be applied to non-cooperative games, for which our measure is intended.

4.2 Games with Multiple Equilibria

Above we considered games with a unique equilibrium under any of the possible constellations of players' utility functions, or at least games where equilibrium uniqueness applies under some suitable refinement. Instead of assuming that we can assign a probability of one to a particular equilibrium, we can choose a more general approach and specify a probability distribution over possible equilibria. If the set of equilibria is finite, for example, it is possible to assume that every equilibrium is equally likely. Since an equilibrium in our context is essentially a distribution over outcomes, let $\Sigma(\mathbf{u}')$ denote the set of probability measures that correspond to the equilibria that exist under some vector of utility functions $\mathbf{u}'$. Assuming that we can specify a probability measure $\sigma_{\mathbf{u}'}$ on each $\Sigma(\mathbf{u}')$, we can define the indirect utility of player n as

$$v_n(\mathbf{u}') = \int_{\Sigma(\mathbf{u}')} \int_O u_n(o) \; d\mu^* \; d\sigma_{\mathbf{u}'} \; .$$

The measure of bargaining power of Theorem 1 can then be computed based on this indirect utility function without any further adjustments. What is more, the definitions and axioms presented above can be adapted to this more general setting with only minor changes and the proof of Theorem 1 applies verbatim. For example, the definition of a Null Player in a game with multiple equilibria would require that changes in this player's utility function have no effect on the set of equilibria.

5 Applications

5.1 Cartel Formation

The formation of a cartel arguably constitutes a setting of non-transferable utility since monetary transfers could be used as evidence of collusion in court. Since the production levels that maximise joint profits may imply wide disparities between the profits of individual cartel members, quantities may be subject to negotiation. Suppose, for example, that N firms produce a homogeneous good, where each firm has a constant marginal cost c_n that differs between firms. In this case the sum of profits would be maximised if only the firm with the lowest cost produces, but in the absence of a means to redistribute these profits the remaining firms clearly have no incentive to agree to such terms. If the firms are later found by the authorities to have engaged in collusive behaviour, the relative influence of each firm in bringing about the agreement could be used for the purpose of apportioning compensation.[8] In order to determine this relative influence, it may not be sufficient to know the market share or cost structure of each firm, for instance because a relatively small or inefficient firm could be pulling above its weight due to political clout or connections to organised crime. Non-cooperative cartel formation is a subject of ongoing research (Abe 2021, Korsten & Samuel 2023) and providing a fully-specified model is beyond the scope of this paper. Yet, our measure of bargaining power takes a particularly simple form in this setting under weak assumptions about the underlying process. These assumptions are i) that a firm's profit is fully reflective of its payoff in the game, which is reasonable if other forms of compensation are not possible, and ii) that if a firm's utility function is replaced with that of another firm, it ceases production, implying a profit of zero. Under these conditions our measure of bargaining power becomes equal to a firm's equilibrium profit divided by this firm's individual monopoly profit. Simply

[8] Napel & Welter (2021, 2022) respectively propose using the Shapley-Shubik index and the Shapley value to assign relative responsibility for damages to the members of a cartel. The drawback of these approaches is that one has to assume that a cartel among any subgroup of firms is associated with a unique vector of production quantities, precluding bargaining among cartel members.

relying on market shares or shares of total profits may thus not accurately reflect a firm's role in the formation of the cartel. The reason is that total production or total profits do not provide a relevant benchmark at the individual level. The highest-possible profit an inefficient firm could hope for may be substantially lower than that of a competitor with lower costs. To illustrate, consider a case with three firms and unit costs that are given by $c_1 = 0.1$, $c_2 = 0.2$, and $c_3 = 0.3$ and an inverse market demand equal to $P = 1 - Q$ where Q is total production. Then individual monopoly profits are given by 0.20, 0.16 and 0.12 in ascending order of costs. The best possible payoff thus differs substantially across firms and dividing individual by total equilibrium profits would overstate the bargaining power of efficient firms and understate that of inefficient firms.

5.2 Household Bargaining

The literature on intra-household decision making has an inherent interest in the determinants of the balance of power between spouses. One approach, namely the collective model of the household (Chiappori 1988, 1992), assumes efficient outcomes while the distribution of resources is determined by explicit parameters for male and female bargaining power. The main competitor is the non-cooperative model of the household (Lundberg & Pollak 1994, Konrad & Lommerud 1995, Browning et al. 2010, Lechene & Preston 2011), which instead assumes that husband and wife play a Nash equilibrium. In this case, bargaining power is an implicit product of the decision-making environment. We use an application of this framework presented in Bertrand et al. (2020) to demonstrate how our approach can be used to evaluate the bargaining power of household members. We focus on the second period of the model, after a man and a woman have decided to form a household. At this point of the game, husband and wife simultaneously decide how to allocate one unit of time between remunerated work and the production of a public good within the household. For simplicity, we assume that there are no spillovers from private consumption. The utility of household member $g \in \{m, f\}$ is then given by

$$u_g(t_g, t_{-g}) = (1 - t_g)w_g + \beta \log(t_m + t_f) \ ,$$

where $t_g \in [0, 1]$ is the share of time spent on producing the public good, w_g is the gender-specific wage, and β determines the weight of public good consumption relative to private consumption. We follow Bertrand et al. (2020) and assume a gender wage gap, $w_f < w_m$, and $\beta < w_m$. Under these assumptions the man works full-time while the woman stays home if $w_f < \beta$ and works part-time otherwise.

In order to calculate players' bargaining powers, we also need to determine the equilibrium if the husband maximises the utility of the wife and vice versa. Without spillovers from earnings, maximising the utility of the partner implies dedicating all available time to producing public goods. If the wife shares the utility function of the husband, the latter always works full time. In the reverse situation, the wife also stays home if her wage is sufficiently low and works part-time or full-time for higher wages. Given that the husband's behaviour differs across these two scenarios for all parameter constellations under consideration, the agreement payoffs of both players are not equal and the game satisfies the Assumption of Conflict of Interest. For β sufficiently large, on the other hand, both partners would always prefer to stay home and there is no disagreement.

Figure 2 plots the bargaining powers of husband and wife as a function of the female wage w_f for the cases $\beta = 0.2$ and $\beta = 0.6$, assuming $w_m = 1$. For $w_f < \beta$, the wife devotes all her time to the production of public goods, which is also the behaviour that maximises the utility of the husband. Accordingly, the husband is assigned a bargaining power of one and the wife a bargaining power of zero. Once her wage becomes sufficiently high, the wife finds it attractive to work part time. Doing so increases her utility and lowers that of her husband, leading to a more equal distribution of power. However, the power of the wife is substantially lower than that of the husband even if her wage is almost equal to his. The reason is that even a slightly lower opportunity cost of domestic labour on part of the wife allows the husband to free-ride on her effort. For $w_f = w_m$, the equilibrium remains unique under agreement on one player's utility function. However, multiple equilibria exist under the endowed utility functions and bargaining power depends on the probability assigned to each equilibrium (see Section 4.2). The figure assigns probability one to the equilibrium where the husband works full-time, which may be due to a social convention. Assigning the same probability to all equilibria, in contrast, would lead to equal bargaining power and a discontinuity at $w_f = w_m$.

As Figure 2 shows, a higher value of the public good β polarises the distribution of bargaining power, since the wife reduces her labour supply while the husband continues to free-ride. A possible interpretation is that modern appliances that generate a more quickly declining marginal productivity of housework lead to greater equality within the household.

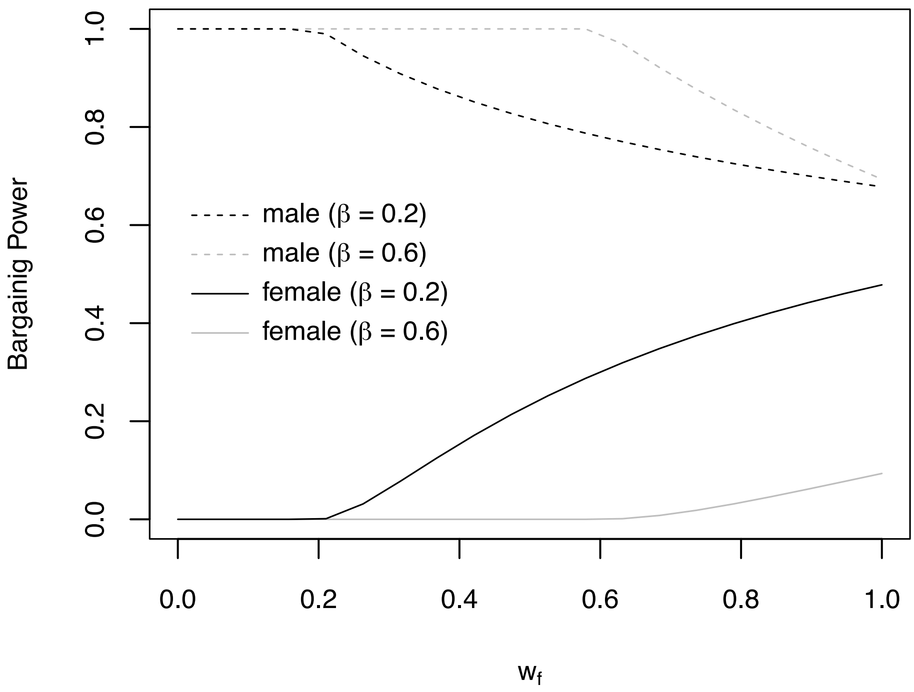

Figure 2: Bargaining Power of Husband and Wife

Notes: The figure plots the bargaining power of husband (dashed lines) and wife (solid lines) against the wife's wage w_f, assuming the husband's wage w_m is equal to 1. Black lines correspond to a value of β of 0.2, while grey lines correspond to $\beta = 0.6$.

5.3 Legislative Bargaining

In this section, we apply our theory to two classic models of legislative bargaining: the agenda setter model of Romer & Rosenthal (1978) and the gatekeeper model of Denzau & Mackay (1983). We choose these examples since they feature non-transferable utility and the outcome of the game is therefore not fully informative about bargaining power. In both models, a committee brings a bill to the floor of a legislative body, which then deliberates and eventually votes on the proposal. A bill is a point x in the interval $[0, 1]$ and if accepted, the bill replaces the status quo $q \in [0, 1]$. The utility of each player from the final outcome o is given by $-|o - i_n|$, where i_n is the ideal outcome of player n. In the agenda setter model, the committee puts forward a bill under a closed rule, that is, the bill cannot be amended and the legislature simply votes subject to simple majority whether to accept the proposal. In the gatekeeper model, an open rule is in place, meaning that any legislator can propose amendments. The finally accepted proposal is then always equal to the ideal point of the median legislator since such a proposal defeats any other. However, the committee has the ability to refuse to put forward a bill, keeping the status quo in place. We introduce a third model as a benchmark, where the committee has to present a proposal under an open rule, which renders the committee powerless. Compared to this benchmark model, the former two models each differ in one aspect of the rules of procedure: the agenda setter model replaces the open rule with a closed rule, while the gatekeeper model gives the committee the ability to withhold the bill. The committee is represented by a single player and we assume here that the committee is not itself a member of the legislature. Any influence of the committee accordingly derives from its choice regarding the initial proposal. The set of players thus consists of one committee and $N-1$ legislators.

We follow Section 4.1 and calculate power in an ex ante sense. To do so, we assume that ideal points and the status quo are drawn uniformly at random from an evenly spaced grid between 0 and 1 with 100 elements. For each draw, we calculate the bargaining power of the committee and of the legislators. We then repeat this procedure until the average across draws converges.[9] The results are presented in Table 1. Note that the legislators are ex ante symmetric and thus have the same bargaining power.

In the benchmark model, the committee is a null player and accordingly assigned a bargaining power of zero. In contrast, the committee has a positive influence in both the agenda-setter and the gatekeeper model. Not surprisingly, the closed rule of the agenda-setter model increases the power of the committee relative to that of a legislator more than the mere ability to withhold legislation in the gatekeeper model. The number of legislators decreases the influence of each individual legislator, since each legislator becomes less likely to occupy the median position, but has a minor effect on the power of the committee. Since only the position of the median legislator is of relevance for the decision of the committee, one may ask why the influence of the latter depends on the number of legislators at all. The reason is that an increase in the number of legislators makes the median legislator more moderate in expectation.[10] A more moderate median legislator,

[9]We could also freely draw from the interval $[0, 1]$. However, in this case large values of ρ_n can occur due to numeric issues when a denominator in Equation 1 becomes very small. Such outliers would slow convergence of the average. Cases where the value of ρ_n is not defined due to a failure of the Assumption of Conflict of Interest occur if and only if the ideal points of all players coincide. Given the low probability of such draws, there is no need to specify a corresponding default value for ρ_n.

[10]To be precise, while the expected position of the median legislator is always equal to one-half, the expected distance of the median legislator from one-half is decreasing in the number of legislators.

	Benchmark Model		Agenda-Setter Model		Gatekeeper Model	
	Com.	*Leg.*	*Com.*	*Leg.*	*Com.*	*Leg.*
$N = 4$	0	.47	.64	.11	.24	.24
$N = 6$	0	.28	.62	.07	.23	.15
$N = 10$	0	.15	.61	.04	.22	.08

Table 1: Bargaining Power in Three Models of Legislative Bargaining

Notes: N is the number of players: one committee and $N-1$ legislators. Columns titled *Com.* show the bargaining power of the proposing committee, whereas columns titled *Leg.* contain the bargaining power of each legislator.

in turn, is located closer to the status quo on average. Since the legislature never accepts a proposal that is further away from the median legislator than the status quo, the ability of the committee to affect the outcome is therefore decreasing in the number of legislators.[11]

6 Conclusion

Bargaining power is a key element of economic, political and social relations. Many central questions in these fields are analysed through the lenses of non-cooperative games, for which measures of bargaining power, however, have been proposed only for specific settings. This paper introduces a novel method for measuring power in any non-cooperative game of bargaining. The power of a player is calculated as the extent to which shifts in this player's preferences change the outcome of the game relative to the change that would occur if the player in question was a dictator. We show that no other function satisfies a number of axioms. For the special case of TU-games, we compare our measure to the more conventional approach of interpreting the expected surplus share of a player as their bargaining power. The two approaches coincide when the equilibria of the game are Pareto efficient, but generally yield different results when they are not. Intuitively, inefficiencies imply that players collectively do not have full control over the distribution of the surplus and our measure calculates bargaining power relative to the share of the surplus that players can freely allocate. The measure can also be averaged over a possible

[11]This observation is true both when considering the raw numbers in Table 1 and the ratio between the bargaining power of the committee and the sum of bargaining powers of the legislators.

distribution that players' preferences are drawn from, which makes it possible to evaluate bargaining power in an ex ante sense, before players' preferences are known. We show that in the context of a weighted voting game, this ex ante measure reproduces the Shapley-Shubik or the Penrose-Bhanzaf power index for suitable choices of the distribution of preferences.

Given that non-cooperative games are explicit about the process of bargaining, our measure is particularly valuable when assessing features of this process and their role in determining the influence of a player. Such insights are crucial, for example, when designing institutions that aim to achieve a specific distribution of power among agents. We illustrate the usefulness of our approach through applications to cartel formation, the non-cooperative model of the household and legislative bargaining.

References

Abe, T. (2021), 'Cartel formation in cournot competition with asymmetric costs: A partition function approach', *Games* **12**(1).

Ali, N., Bernheim, D. & Fan, X. (2019), 'Predictability and power in legislative bargaining', *The Review of Economic Studies* **86**(2), 500–525.

Anderberg, D., Rainer, H., Wadsworth, J. & Wilson, T. (2016), 'Unemployment and Domestic Violence: Theory and Evidence', *Economic Journal* **126**(597), 1947–1979.

Banzhaf, J. (1965), 'Weighted voting doesn't work: A mathematical analysis', *Rutgers Law Review* **19**(2), 317–343.

Basu, K. (2006), 'Gender and say: A model of household behaviour with endogenously determined balance of power', *Economic Journal* **116**(511), 558–580.

Bertrand, M., Cortes, P., Olivetti, C. & Pan, J. (2020), 'Social Norms, Labour Market Opportunities, and the Marriage Gap Between Skilled and Unskilled Women', *The Review of Economic Studies* **88**(4), 1936–1978.

Browning, M., Chiappori, P.-A. & Lechene, V. (2010), 'Distributional effects in household models: Separate spheres and income pooling', *The Economic Journal* **120**(545), 786–799.

Browning, M., Chiappori, P.-A. & Lewbel, A. (2013), 'Estimating consumption economies of scale, adult equivalence scales, and household bargaining power', *The Review of Economic Studies* **80**(4), 1267–1303.

Chiappori, P.-A. (1988), 'Rational household labor supply', *Econometrica* **56**(1), 63–90.

Chiappori, P.-A. (1992), 'Collective labor supply and welfare', *Journal of Political Economy* **100**(3), 437–467.

Deegan, J. & Packel, E. (1978), 'A new index of power for simple n-person games', *International Journal of Game Theory* **7**, 113–123.

Denzau, A. T. & Mackay, R. J. (1983), 'Gatekeeping and monopoly power of committee: An analysis of sincere and sophisticated behavior', *American Journal of Political Science* **27**(4), 740–761.

Diermeier, D., Eraslan, H. & Merlo, A. (2003), 'A structural model of government formation', *Econometrica* **71**(1), 27–70.

Dubey, P. & Shapley, L. (1979), 'Mathematical properties of the Banzhaf power index', *Mathematics of Operations Research* **4**(2), 99–131.

Dunlop, J. & Higgins, B. (1942), '"Bargaining power" and market structures', *Journal of Political Economy* **50**(1), 1–26.

Francois, P., Rainer, I. & Trebbi, F. (2015), 'How is power shared in Africa?', *Econometrica* **83**(2), 465–503.

Fréchette, G., Kagel, J. & Morelli, M. (2005), 'Nominal bargaining power, selection protocol, and discounting in legislative bargaining', *Journal of Public Economics* **89**(8), 1497–1517.

Haller, H. & Holden, S. (1997), 'Ratification requirement and bargaining power', *International Economic Review* **38**(4), 825–851.

Hamermesh, D. (1973), 'Who 'wins' in wage bargaining?', *Industrial and Labor Relations Review* **26**(4), 1146–1149.

Hart, S. & Mas-Colell, A. (1996), 'Bargaining and value', *Econometrica* **64**(2), 357–380.

Holler, M. (1982), 'Forming coalitions and measuring voting power', *Political Studies* **30**(2), 262–271.

Johnston, R. (1978), 'On the measurement of power: Some reactions to Laver', *Environment and Planning A: Economy and Space* **10**(8), 907–914.

Kalandrakis, T. (2006), 'Proposal rights and political power', *American Journal of Political Science* **50**(2), 441–448.

Kambe, S. (1999), 'Bargaining with imperfect commitment', *Games and Economic Behavior* **28**(2), 217–237.

Konrad, K. & Lommerud, K. E. (1995), 'Family policy with non-cooperative families', *The Scandinavian Journal of Economics* **97**(4), 581–601.

Korsten, R. & Samuel, A. (2023), 'Cartel formation and detection: The role of information costs and disclosure', *European Journal of Law and Economics* **56**(1), 117–153.

Krishna, V. & Serrano, R. (1996), 'Multilateral bargaining', *The Review of Economic Studies* **63**(1), 61–80.

Larsen, B. & Zhang, A. (2021), Quantifying bargaining power under incomplete information: A supply-side analysis of the used-car industry. Mimeo.

Laruelle, A. & Valenciano, F. (2008), 'Noncooperative foundations of bargaining power in committees and the Shapley-Shubik index', *Games and Economic Behavior* **63**(1), 341–353.

Lechene, V. & Preston, I. (2011), 'Noncooperative household demand', *Journal of Economic Theory* **146**(2), 504–527.

Loertscher, S. & Marx, L. (2022), 'Incomplete information bargaining with applications to mergers, investment, and vertical integration', *American Economic Review* **112**(2), 616–649.

Lojkine, U. (2022), A general power index. Mimeo.

Lundberg, S. & Pollak, R. (1994), 'Noncooperative bargaining models of marriage', *The American Economic Review* **84**(2), 132–137.

Manning, A. (1987), 'An integration of trade union models in a sequential bargaining framework', *The Economic Journal* **97**(385), 121–139.

Myerson, R. (1991), *Game theory: Analysis of conflict*, Harvard University Press, Cambridge, MA.

Napel, S. & Welter, D. (2021), 'Simple voting games and cartel damage proportioning', *Games* **12**(4).

Napel, S. & Welter, D. (2022), 'Relative responsibility for cartel damages', *Journal of Institutional and Theoretical Economics* **178**(3), 231–257.

Napel, S. & Widgrén, M. (2004), 'Power measurement as sensitivity analysis: A unified approach', *Journal of Theoretical Politics* **16**(4), 517–538.

Napel, S. & Widgrén, M. (2006), 'The inter-institutional distribution of power in EU Codecision', *Social Choice and Welfare* **27**(1), 129–154.

Nunnari, S. (2021), 'Dynamic legislative bargaining with veto power: Theory and experiments', *Games and Economic Behavior* **126**, 186–230.

Owen, G. & Shapley, L. (1989), 'Optimal location of candidates in ideological space', *International Journal of Game Theory* **18**(3), 339–356.

Penrose, L. (1946), 'The elementary statistics of majority voting', *Journal of the Royal Statistical Society* **109**(1), 53–57.

Romer, T. & Rosenthal, H. (1978), 'Political resource allocation, controlled agendas, and the status quo', *Public Choice* **33**(4), 27–43.

Rubinstein, A. (1982), 'Perfect equilibrium in a bargaining model', *Econometrica* **50**(1), 97–109.

Serrano, R. & Vohra, R. (1997), 'Non-cooperative implementation of the core', *Social Choice and Welfare* **14**(4), 513–525.

Shapley, L. & Shubik, M. (1954), 'A method for evaluating the distribution of power in a committee system', *The American Political Science Review* **48**(3), 787–792.

Snyder, J., Ting, M. & Ansolabehere, S. (2005), 'Legislative bargaining under weighted voting', *The American Economic Review* **95**(4), 981–1004.

Steunenberg, B., Schmidtchen, D. & Koboldt, C. (1999), 'Strategic power in the European Union: Evaluating the distribution of power in policy games', *Journal of Theoretical Politics* **11**(3), 339–366.

Svejnar, J. (1986), 'Bargaining power, fear of disagreement, and wage settlements: Theory and evidence from U.S. industry', *Econometrica* **54**(5), 1055–1078.

Taylor, C. (1995), 'The long side of the market and the short end of the stick: Bargaining power and price formation in buyers', sellers', and balanced markets', *The Quarterly Journal of Economics* **110**(3), 837–855.

Appendix

A Proof of Theorem 1

This appendix contains the proof of Theorem 1. The proof is presented in the context of a fixed class of indirect utility functions $\bar{\mathcal{V}}_n \subset \mathcal{V}_n$ that correspond to games sharing a common outcome space, set of players, and agreement payoffs.

We start by presenting three lemmas that successively introduce sharper restrictions on the function $\rho_n(v_n)$.

Lemma 1. *A function $\rho_n : \bar{\mathcal{V}}_n \to \mathbb{R}$ satisfies axioms A1 and A4 if and only if*

$$\rho_n(v_n) = \beta + \sum_{\mathbf{u}' \in \mathcal{U}_E^N} \alpha(\mathbf{u}')\ v_n(\mathbf{u}') \ ,$$

where β and all $\alpha(\mathbf{u}')$ are real numbers.

Proof. Start by considering the implications of Axiom A1. Note that for any game Γ_+, eliminating all but the endowed utility functions from the corresponding set of relevant utility functions $\mathcal{U}_+$ creates a second game Γ such that this pair of games satisfies the conditions set out in Axiom A1. The indirect utility functions corresponding to the two games, $v_{+,n}$ and v_n, differ only in their domains: $\mathcal{U}_+^N$ and $\mathcal{U}^N = \mathcal{U}_E^N$, respectively. Accordingly, it holds that $v_{+,n}\big|_{\mathcal{U}_E^N} = v_n$ and it then follows from Axiom A1 that $\rho_n(v_{+,n}) = \rho_n(v_n) = \rho_n(v_{+,n}\big|_{\mathcal{U}_E^N})$. Conversely, assume $\rho_n(v_n) = \rho_n(v_n\big|_{\mathcal{U}_E^N})$ holds for any $v_n \in \bar{\mathcal{V}}_n$. For any $v_{+,n}$ created by adding elements to the set of relevant utility functions $\mathcal{U}$ to create the set $\mathcal{U}_+$ it holds that $v_n\big|_{\mathcal{U}_E^N} = v_{+,n}\big|_{\mathcal{U}_{+,E}^N}$, since both games share the same endowed utility functions. It then follows that $\rho_n(v_n) = \rho_n(v_n\big|_{\mathcal{U}_E^N}) = \rho_n(v_{+,n}\big|_{\mathcal{U}_{+,E}^N}) = \rho_n(v_{+,n})$. The function ρ_n therefore satisfies Axiom A1 if and only if $\rho_n(v_n) = \rho_n(v_n\big|_{\mathcal{U}_E^N})$ holds for any $v_n \in \bar{\mathcal{V}}_n$. Since the set $\mathcal{U}_E^N$ is finite, ρ_n is therefore a function of a finite vector of real numbers.

Let $\Gamma = \sum_{g=1}^{G} \lambda_g \Gamma_g$ with corresponding indirect utility functions v_n, $v_{1,n}$, ..., $v_{G,n} \in \bar{\mathcal{V}}_n$. Given that the class $\bar{\mathcal{V}}_n$ was defined to contain indirect utilities sharing the same agreement payoffs, Axiom A4 requires

$$\begin{aligned}\sum_{g=1}^{G} \lambda_g\ \rho_n(v_{g,n}) &= \rho_n(v_n)\\ &= \rho_n\left(\sum_{g=1}^{G} \lambda_g v_{g,n}\right) \ ,\end{aligned}$$

where the second equality follows since the indirect utilities of a compound game are a convex combination of the indirect utilities of the constituent games. ρ_n is therefore an affine function on $\bar{\mathcal{V}}_n$. Given that it was established above that ρ_n is a function of a finite vector of real numbers, affinity of ρ_n is satisfied if and only if

$$\rho_n(v_n) = \beta + \sum_{\mathbf{u}' \in \mathcal{U}_E^N} \alpha(\mathbf{u}')\ v_n(\mathbf{u}')\ ,$$

where β and each $\alpha(\mathbf{u}')$ are real numbers. $\square$

Lemma 2. *A function $\rho_n : \bar{\mathcal{V}}_n \to \mathbb{R}$ satisfies axioms A1, A2, and A4 if and only if*

$$\rho_n(v_n) = \sum_{\substack{(\mathbf{u}',u'') \\ \in\, \mathcal{U}_E^{N+1}}} \alpha(\mathbf{u}', u'')\ [v_n(\mathbf{u}') - v_n(\mathbf{u}'_{u'_n \leftarrow u''})]\ ,$$

where all $\alpha(\mathbf{u}', u'')$ are real numbers.

Proof. Given the functional form of ρ_n established in Lemma 1, it needs to be shown what additional restrictions Axiom A2 imposes. It will be shown that it must be possible to formulate ρ_n as a function of differences in payoffs of the form $v_n(\mathbf{u}') - v_n(\mathbf{u}'_{u'_n \leftarrow u''})$. To see this, suppose that after rearranging the terms of ρ_n to form pairs of utilities of the preceding kind, there remains one payoff $v_n(\tilde{\mathbf{u}})$ for some $\tilde{\mathbf{u}} \in \mathcal{U}_E^N$ with non-zero coefficient $\alpha(\tilde{\mathbf{u}})$ for which no pair can be formed. Let v_n correspond to a game where player n is a null player and thus $\rho_n(v_n) = 0$. Since all differences in payoffs of the form $v_n(\mathbf{u}') - v_n(\mathbf{u}'_{u'_n \leftarrow u''})$ are equal to zero if player n is null, we have

$$\rho_n(v_n) = \beta + \alpha(\tilde{\mathbf{u}})\ v_n(\tilde{\mathbf{u}}) = 0\ . \tag{4}$$

If there exist multiple games in $\bar{\mathcal{V}}_n$ such that n is null and the payoff $v_n(\tilde{\mathbf{u}})$ differs across some of these games, then the preceding equality cannot hold for all such games and Axiom A2 would be violated. Suppose therefore that n being null implies a fixed value of $v_n(\tilde{\mathbf{u}})$ across all elements of $\bar{\mathcal{V}}_n$. It will be shown that this assumption can only be satisfied if the utility functions of all players other then n contained in $\tilde{\mathbf{u}}$ are equal. To the contrary, suppose that there exist players m and k such that $\tilde{u}_m \neq \tilde{u}_k$. Then we can construct two games, Γ_m and Γ_k, for which it holds, respectively, that $\mu^*_m(\mathbf{u}') = \mu^*_m(\mathbf{1}_{u'_m})$ and $\mu^*_k(\mathbf{u}') = \mu^*_k(\mathbf{1}_{u'_k})$ for any $\mathbf{u}' \in \mathcal{U}^N$. It follows that player n is null in both games. Furthermore, since it must hold either that $\tilde{u}_n \neq \tilde{u}_m$ or that $\tilde{u}_n \neq \tilde{u}_k$, Assumption 2 implies that the payoff of player n under the vector $\tilde{\mathbf{u}}$ differs across the two games, which is the desired contradiction. n being null can therefore only imply a fixed value of

$v_n(\tilde{\mathbf{u}})$ if the utility functions of all players other than n contained in $\tilde{\mathbf{u}}$ are equal to some $u' \in \mathcal{U}_E$. In this case it follows from the definition of a null player that $v_n(\tilde{\mathbf{u}}) = v_n(\tilde{\mathbf{u}}_{\tilde{u}_n \leftarrow u'}) = v_n(\mathbf{1}_{u'})$. Equation (4) then implies $\beta = -\alpha(\tilde{\mathbf{u}})\ v_n(\mathbf{1}_{u'})$, which contradicts that it is impossible to pair the payoff $v_m(\tilde{\mathbf{u}})$ with another of the form $v_n(\tilde{\mathbf{u}}_{\tilde{u}_n \leftarrow u})$. Given that all such pairs are zero if player n is null, it further follows that ρ_n cannot contain any additional constant. $\square$

Lemma 3. *A function $\rho_n : \bar{\mathcal{V}}_n \to \mathbb{R}$ satisfies axioms A1, A2, A3, and A4 if and only if*

$$\rho_n(v_n) = \frac{\sum_{(u',u'') \in \mathcal{U}_E^2} \ \alpha(u',u'')\ [v_n(\mathbf{u}_{u_n \leftarrow u'}) - v_n(\mathbf{u}_{u_n \leftarrow u''})]}{\sum_{(u',u'') \in \mathcal{U}_E^2} \ \alpha(u',u'')\ [v_n(\mathbf{1}_{u'}) - v_n(\mathbf{1}_{u''})]} \ ,$$

where all $\alpha(u',u'')$ are real numbers such that the denominator in the preceding expression is not equal to zero.

Proof. As a first step, it will be shown that an additional restriction implied by Axiom A3 is that ρ_n can only depend on indirect utilities of the form $v_n(\mathbf{u}_{u_n \leftarrow u'})$ for some $u' \in \mathcal{U}_E$, that is, indirect utilities under vectors of utility functions that differ from the vector of endowed utility functions only in the utility function of player n. Given the functional form established by Lemma 2, suppose that ρ_n depends on a pair of indirect utilities $v_n(\mathbf{u}') - v_n(\mathbf{u}'_{u'_n \leftarrow u''})$ with a non-zero coefficient, where the utility function of some player other than n included in the vector $\mathbf{u}'$ differs from their endowed utility function. At least one of these payoffs is not an agreement payoff and, without loss of generality, let this be the payoff $v_n(\mathbf{u}')$. Suppose v_n corresponds to a game where player n is a local dictator and $\rho_n(v_n) = 1$. Since n being a local dictator does not restrict the payoff $v_n(\mathbf{u}')$, we can construct a second indirect utility v'_n where n continues to be a local dictator by changing this payoff while holding v_n otherwise constant. Given the already established functional form of ρ_n, the perturbation in $v_n(\mathbf{u}')$ increases or decreases the value of $\rho_n(v'_n)$ relative to $\rho_n(v_n)$, violating Axiom A3.

We have thus established that

$$\rho_n(v_n) = \sum_{(u',u'') \in \mathcal{U}_E^2} \alpha(u',u'')\ [v_n(\mathbf{u}_{u_n \leftarrow u'}) - v_n(\mathbf{u}_{u_n \leftarrow u''})] \ . \tag{5}$$

Let $C \neq 0$ be some real number. We can rewrite

$$\rho_n(v_n) = \ C \sum_{(u',u'') \in \mathcal{U}_E^2} \frac{\alpha(u',u'')}{C}\ [v_n(\mathbf{u}_{u_n \leftarrow u'}) - v_n(\mathbf{u}_{u_n \leftarrow u''})] \ .$$

Since the exact values of the coefficients are as of yet undetermined, we can redefine them to include the division by C and simply write

$$\rho_n(v_n) = C \sum_{(u',u'')\in\,\mathcal{U}_E^2} \alpha(u',u'')\,\left[v_n(\mathbf{u}_{u_n\leftarrow u'}) - v_n(\mathbf{u}_{u_n\leftarrow u''})\right]\,. \tag{6}$$

Under Axiom A3, n being a local dictator implies

$$C \sum_{(u',u'')\in\,\mathcal{U}_E^2} \alpha(u',u'')\,\left[v_n(\mathbf{1}_{u'}) - v_n(\mathbf{1}_{u''})\right] \;=\; 1\,.$$

Solving for C and substituting back into Equation (6) yields the desired result. Any such function satisfies Axiom A3 as long as the coefficients are chosen such that the value of C is not equal to zero. □

It needs to be shown that the function given in the statement of Theorem 1 is the only function among those given by Lemma 3 that satisfies Axiom A5.

If $|\mathcal{U}_E| = 2$, Lemma 3 pins down a unique function given by

$$\rho_n(v_n) = \frac{v_n(\mathbf{u}) - v_n(\mathbf{u}_{u_n\leftarrow u_{-n}})}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u_{-n}})}\,,$$

where u_{-n} is the unique element of the set $\mathcal{U}_E \setminus u_n$. It remains to consider the case $|\mathcal{U}_E| > 2$.

For what follows, it is convenient to revert back to the functional form established by Equation (5) and write

$$\rho_n(v_n) = \sum_{(u',u'')\in\,\mathcal{U}_E^2} \alpha(u',u'')\,\left[v_n(\mathbf{u}_{u_n\leftarrow u'}) - v_n(\mathbf{u}_{u_n\leftarrow u''})\right]\,. \tag{7}$$

Let the indirect utilities v_n, v'_n and v''_n correspond to the definitions given in the statement of Axiom A5. The only payoffs that differ between these functions are those corresponding to the vectors of utility function $\mathbf{u}_{u_n\leftarrow u'}$ and $\mathbf{u}_{u_n\leftarrow u''}$. These payoffs are not agreement payoffs since the vector $\mathbf{u}$ contains more than two distinct utility functions by the assumption that $|\mathcal{U}_E| > 2$. The indirect utility functions v_n, v'_n and v''_n therefore belong to the same class $\bar{\mathcal{V}}_n$ and the coefficients used to calculate the corresponding values of ρ_n are identical.

By construction, it holds that $v'_n(\mathbf{1}_{u_n\leftarrow u^-}) - v'_n(\mathbf{1}_{u_n\leftarrow u^=}) = 0$ for any $u^-, u^= \in \mathcal{U}_E \setminus u'$ while $v'_n(\mathbf{1}_{u_n\leftarrow u'''}) - v'_n(\mathbf{1}_{u_n\leftarrow u'}) = c$ and $v'_n(\mathbf{1}_{u_n\leftarrow u'}) - v'_n(\mathbf{1}_{u_n\leftarrow u'''}) = -c$ for any $u''' \in \mathcal{U}_E \setminus u'$. Based on Equation (7), it follows that

$$\rho_n(v_n') = c \sum_{u''' \in \mathcal{U}_E \setminus u'} [\alpha(u''', u') - \alpha(u', u''')] \ .$$

Repeating an analogous derivation for the indirect utility v_n'', we have

$$\frac{\rho_n(v_n')}{\rho_n(v_n'')} = \frac{\tilde{\alpha}(u')}{\tilde{\alpha}(u'')} \ ,$$

where

$$\tilde{\alpha}(u') := \sum_{u''' \in \mathcal{U}_E \setminus u'} [\alpha(u''', u') - \alpha(u', u''')]$$

for any $u' \in \mathcal{U}_E$. Axiom A5 thus implies

$$\frac{\tilde{\alpha}(u')}{\tilde{\alpha}(u'')} = \frac{v_n''(\mathbf{1}_{u_n}) - v_n''(\mathbf{1}_{u''})}{v_n'(\mathbf{1}_{u_n}) - v_n'(\mathbf{1}_{u'})} \ ,$$

or, equivalently,

$$\tilde{\alpha}(u') = \frac{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u''})}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})} \tilde{\alpha}(u'') \ ,$$

since all involved games share the same agreement payoffs. Given that u' and u'' are arbitrary elements of the set $\mathcal{U}_E \setminus u_n$, the preceding equality must hold for any such pair, implying

$$\tilde{\alpha}(u') = \frac{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u''})}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})} \tilde{\alpha}(u'') = \frac{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'''})}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})} \tilde{\alpha}(u''')$$

for any $u''' \in \mathcal{U}_E \setminus \{u_n, u', u''\}$. It follows that

$$[v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u''})]\tilde{\alpha}(u'') = [v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'''})]\tilde{\alpha}(u''') =: \delta$$

must hold for any $u'', u''' \in \mathcal{U}_E \setminus u_n$ and, accordingly,

$$\tilde{\alpha}(u') = \frac{\delta}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})} \tag{8}$$

for any $u' \in \mathcal{U}_E \setminus u_n$. Equation (7) can be rearranged to yield

$$\begin{aligned}\rho_n(v_n) &= -\sum_{u' \in \mathcal{U}_E} \left[\sum_{u''' \in \mathcal{U}_E \setminus u'} \alpha(u''', u') - \alpha(u', u''') \right] v_n(\mathbf{u}_{u_n \leftarrow u'}) \\ &= -\sum_{u' \in \mathcal{U}_E} \tilde{\alpha}(u')\ v_n(\mathbf{u}_{u_n \leftarrow u'}) \ .\end{aligned}$$

Using Equation (8) to substitute for every $\tilde{\alpha}(u')$ such that $u' \in \mathcal{U}_E \setminus u_n$ produces

$$\rho_n(v_n) = \ -\tilde{\alpha}(u_n)\ v_n(\mathbf{u}) - \sum_{u' \in \mathcal{U}_E \setminus u_n} \frac{\delta}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})}\ v_n(\mathbf{u}_{u_n \leftarrow u'}) \ . \tag{9}$$

If v_n corresponds to a game such that n is null, then all the indirect utilities included in Equation (9) take the same value. Denoting this value by $\bar{v}$, Axiom A2 then requires

$$\rho_n(v_n) = \ \bar{v} \left(-\tilde{\alpha}(u_n) \ - \sum_{u' \in \mathcal{U}_E \setminus u_n} \frac{\delta}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})} \right) \ = \ 0 \ .$$

Since v_n may be chosen such that $\bar{v} \neq 0$, it follows that

$$-\tilde{\alpha}(u_n) \ = \sum_{u' \in \mathcal{U}_E \setminus u_n} \frac{\delta}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})} \ .$$

Substituting back into Equation (9) and rearranging yields

$$\rho_n(v_n) = \ \delta \sum_{u' \in \mathcal{U}_E \setminus u_n} \frac{v_n(\mathbf{u}) - v_n(\mathbf{u}_{u_n \leftarrow u'})}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})} \ .$$

If v_n instead corresponds to a game such that n is a local dictator,

$$\rho_n(v_n) = \ \delta \sum_{u' \in \mathcal{U}_E \setminus u_n} \frac{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})}{v_n(\mathbf{1}_{u_n}) - v_n(\mathbf{1}_{u'})} \ ,$$

which is only equal to 1 as required by Axiom A3 if

$$\delta = \frac{1}{|\mathcal{U}_E \setminus u_n|} \ .$$

Note that under Assumption 2 it holds that $\mathcal{U}_E \setminus u_n = \mathcal{U}_{\neq n}$. This completes the proof.

B Additional Proofs

Proof of Proposition 1. Pareto efficiency implies that if all players agree that a unique outcome would be optimal, then the equilibrium of the game must produce this outcome with certainty. In a TU-game, under the vector of utility functions $\mathbf{1}_{u_m}$ all players agree that player m should receive everything. Pareto efficiency of the outcomes $\mu^*(\mathbf{1}_{u_n})$ and $\mu^*(\mathbf{1}_{u_m})$ for $m \neq n$ thus implies $v_n(\mathbf{1}_{u_n}) = 1$ and $v_n(\mathbf{1}_{u_m}) = 0$. Furthermore, Pareto efficiency implies $v_n(\mathbf{u}_{u_n \leftarrow u_m}) = 0$ since under the vector of utility functions $\mathbf{u}_{u_n \leftarrow u_m}$ all players other than n prefer more for themselves while player n prefers more for player m. Using all of the above to substitute in Equation 1, it follows that

$$
\begin{aligned}
\rho_n(v_n) &= \frac{1}{|\mathcal{U}_{\neq n}|} \sum_{u' \in \mathcal{U}_{\neq n}} \frac{v_n(\mathbf{u}) - 0}{1 - 0} \\
&= v_n(\mathbf{u}) \ . \qquad \square
\end{aligned}
$$

Proof of Proposition 2. We start by calculating the value of ρ_n for a given vector of endowed utility functions $\mathbf{u} \notin \{\mathbf{1}_{u^0}, \mathbf{1}_{u^1}\}$. It is clear that the agreement outcome under the vector of preferences $\mathbf{1}_{u^0}$ ($\mathbf{1}_{u^1}$) is equal to 0 (1) with certainty. If player n is pivotal, the outcome coincides with that preferred by player n, which implies $v_n(\mathbf{u}) = 1$ and $v_n(\mathbf{u}_{u_n \leftarrow u'}) = 0$ for $u' \in \{u^0, u^1\} \setminus u_n$. It follows that $\rho_n = 1$ if player n is pivotal. If player n is not pivotal, switching the preference of player n has no consequence for the outcome and $\rho_n = 0$. It follows that $\rho_n(v_n^S) = V(S \cup \{n\}) - V(S)$ if $n \notin S$ and $\rho_n(v_n^S) = V(S) - V(S \setminus \{n\})$ if $n \in S$.

Define $F_{PB}(\mathbf{u}) = 1/2^N$ and $F_{SS}(\mathbf{u}) = [|S|! \cdot (N - |S|)!]/(N+1)!$. We can now establish that

$$
\begin{aligned}
\bar{\rho}_n(F_{PB}) &= \sum_{S \subseteq \mathcal{N}} \frac{1}{2^N} \rho_n(v_n^S) \\
&= \sum_{\substack{S \subseteq \mathcal{N} \\ n \notin S}} \frac{1}{2^N} \rho_n(v_n^S) + \sum_{\substack{S \subseteq \mathcal{N} \\ n \in S}} \frac{1}{2^N} \rho_n(v_n^S) \\
&= 2 \sum_{\substack{S \subseteq \mathcal{N} \\ n \notin S}} \frac{1}{2^N} \rho_n(v_n^S) \\
&= \sum_{\substack{S \subseteq \mathcal{N} \\ n \notin S}} \frac{1}{2^{N-1}} \left[V(S \cup \{n\}) - V(S) \right] \ ,
\end{aligned}
$$

where the third equality follows from the fact that for every $S \subseteq \mathcal{N}$ such that $n \notin S$ there exists exactly one $S' \subseteq \mathcal{N}$ such that $n \in S'$ and $S = S' \setminus \{n\}$. Since pivotality of player n only depends on the other players' preferences, it thus holds that $\rho_n(v_n^S) = \rho_n(v_n^{S'})$. Furthermore,

$$\begin{aligned}
\bar{\rho}_n(F_{SS}) &= \sum_{S \in \mathcal{N}} \frac{|S|! \cdot (N - |S|)!}{(N+1)!} \rho_n(v_n^S) \\
&= \sum_{\substack{S \subseteq \mathcal{N} \\ n \in S}} \left[\frac{|S|! \cdot (N - |S|)!}{(N+1)!} \rho_n(v_n^S) \right. \\
&\qquad \left. + \frac{(|S|-1)! \cdot (N - |S| + 1)!}{(N+1)!} \rho_n(v_n^{S \setminus n}) \right] \\
&= \sum_{\substack{S \subseteq \mathcal{N} \\ n \in S}} \left[\frac{|S|! \cdot (N - |S|)!}{(N+1)!} \right. \\
&\qquad \left. + \frac{(|S|-1)! \cdot (N - |S| + 1)!}{(N+1)!} \right] \rho_n(v_n^S) \\
&= \sum_{\substack{S \subseteq \mathcal{N} \\ n \in S}} \frac{(|S|-1)! \cdot (N - |S|)!}{N!} \left[V(S) - V(S \setminus \{n\}) \right] ,
\end{aligned}$$

where the third equality holds since $\rho_n(v_n^S) = \rho_n(v_n^{S \setminus n})$, which follows as the value of ρ_n only depends on whether player n is pivotal, which in turn only depends on the preferences of other players. □