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Sequential Trading with Coarse Contingencies

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Abstract

We consider a dynamic pure exchange economy in which agents have a coarse perception of the future and, in particular, may be unaware of some risks. As awareness of these risks emerges, markets have to re-open to allow the agents to re-optimize and purchase insurance. An inefficiency may nonetheless arise as the cost of insurance is borne at once rather than spread over time. This "savings mistake" is not an issue in two special but important benchmark cases. In those, the ability to re-trade fully negates the ex-ante coarseness of the agents' perceptions. In addition, we discuss the possibility of unexpected default. This arises when agents borrow "too much" and once perceptions change, there is no equilibrium price at which they are able to refinance their debt.

Keywords: coarse perceptions, unforeseen risks, sequential trading, default.

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1 Introduction

Traditional models of dynamic choice exclude the possibility of unforeseen contingencies. The rational agents inhabiting these models are aware of every risk and make fully contingent plans from which there is never a reason to deviate. The implications of such rationality are nowhere more striking than in the study of general equilibrium. If markets are complete ex ante, all trading takes place at a moment in time; there is no need for markets to ever reopen. If markets open gradually over time, economic activity will unfold gradually as well. Yet, like clockwork, everything proceeds in sync with prior expectations. Inevitably, the equilbrium allocations of the dynamic economy mimic those of the static economy with complete markets and once-and-for-all trading.

This paper considers a dynamic economy in which agents are unaware of, or simply neglect, some shocks to their future endowments. We note from the start that we do not aim for generality; rather, we seek to illuminate some conceptual issues through examples and the study of a few benchmark cases. The overarching theme is that in an economy with evolving awareness markets have to reopen. As agents become aware of previously unforeseen risks, they have to re-optimize, e.g. by purchasing insurance. The process of continual re-optimization means that the economy will not evolve according to a single "equilibrium of prices, plans, and price expectations" (Radner [19]). At best, the economy will transition from one such equilibrium to another. Our first contribution is to show that even that is not guaranteed – if perceptions change unexpectedly, there may be no price that simultaneously clears markets and guarantees that all agents *can* repay their debt. In other words, default is unexpected and involuntary.

It is important to emphasize that we stack the deck against default by assuming that at any moment in time the agents perceive correctly all one-step-ahead contingencies and can trade only one-step-ahead securities. Put differently, the agents understand fully every transaction they ever enter.[1] Nonetheless, a problem arises because the agents enter these trades under incorrect expectations of the future. Consider an agent who borrows short in period t, planning to roll over the debt in period

[1] This is how the present analysis differs from Modica, Rustichini, and Tallon [17].

$t + 1$. If in period $t + 1$ the agents become aware of a new (yet unrealized) risk, the value of their future endowments will go down, forcing borrowers to repay any maturing obligations out of current endowments. However, there may be no price which simultaneously clears markets and makes the cost of debt sufficiently low.

The example, which we develop fully in Section 4, is technically simple and can be reduced to a static situation in which the endowment point lies outside the Edgeworth box.[2] The contribution lies in showing how such a situation can arise in a fully fledged dynamic model under a natural and arguably minimal departure from rationality, well-grounded in the literature on awareness.[3] This specificity is important for many reasons. For one, it shows the fragility of Radner equilibria. In turn, one is forced to reconsider the view that models with incomplete markets and fully rational agents can serve as a shortcut for modeling bounded rationality,[4] which, as we see, skirts serious problems pertaining to the smooth functioning of markets. Finally, we believe that the problem of unexpected default, which is at the core of our example, speaks to recent events and deserves a formal framework. Indeed, as Gennaioli and Shleifer [4] argue, many of the bankruptcies that took place during the 2008 financial crisis can only be understood in the context of the unrealistic, overly sanguine beliefs held by market participants.

Our second contribution is to examine the welfare implications of sequential trading in economies in which no default occurs. In particular, we ask if the continual readjustment of plans can overcome an initial misperception of risks and allocate resources efficiently as if the agents were fully rational. Perhaps surprisingly, we show that the answer is yes in two special but canonical cases. The first is when the aggregate endowment is constant across states and time. The second one assumes homothetic utility, but allows for aggregate uncertainty as long as agents make no mistakes about it, which means that only idiosyncratic risks can be unforeseen. Intuition for these positive results is given in the main text. Here, we highlight what may go wrong in general. Consider a risk to consumption in some distant period t. Under our assumptions, the agents will become aware of the risk at least one period before it

[2] See Arrow and Hahn [1, p.119], Green [7], and more recently Ben-Ami and Geanakoplos [2].

[3] Our model is consistent with the analysis in Kochov [14] as well as the Reverse Bayesianism rule of Karni and Vierø [12].

[4] See Magill and Quinzii [16, p.30] for a discussion of this view.

hits, which allows the agents to go ahead and purchase insurance. However, if the risk was unforeseen at earlier dates, an inefficiency may arise because the cost of insurance is borne at once rather than spread over time. Roughly speaking, emerging awareness and the re-opening markets helps the agents smooth consumption across states of the world (by purchasing insurance) but not across time, as the agents cannot undo past savings decisions. Our positive results are therefore examples of economies in which this inefficiency, which we label **a savings mistake**, does not arise.

The paper is organized as follows. Section 2 presents the model and introduces our equilibrium notion. Section 3 starts with an example illustrating the possibility of saving mistakes and provides conditions on the fundamentals under which these mistakes do not arise. Section 4 discusses the possibility of default and Section 5 provides a complete characterization of the market equilibrium for the case of homothetic utility. In Section 6 we discuss related work and some open problems for future research.

2 The Model

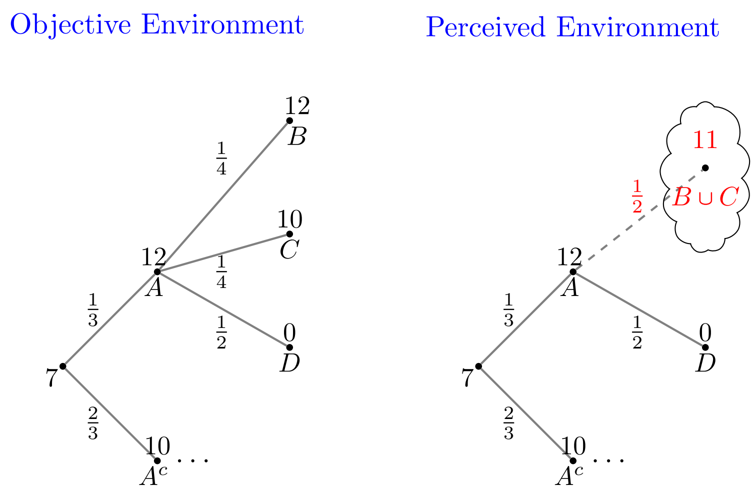
In our general equilibrium model, the agents will have identical perceptions, beliefs, and preferences; they will trade in order to smooth to consumption across states and time. We may therefore begin by describing the perceptions of a single agent.

2.1 Perceptions at a point in time

The driving force of our model is the idea that agents may be unaware of some contingencies and, consequently, of some shocks to their endowments. Figure 1 illustrates how we capture this and the key assumptions we make. As in the figure, note that throughout the paper we assume discrete time, a finite horizon, and a single consumption good.

The left-hand side of Figure 1 depicts the objective environment and the true endowment of some agent i. The former consists of an event tree T, describing which events occur when, and a probability measure μ describing the probabilities of these events. We will refer to the nodes of a tree interchangeably as nodes, events, or

Figure 1



contingencies. Letting 0 be the initial period, one reads the figure as follows. Event A is realized with probability $\frac{1}{3}$ in period 1, and is comprised of three subevents, B, C, and D, one of which will realize in period 2. In the initial period, the agent's endowment is 7; in period 1, it is 12 if A happens, and 10 if A doesn't happen, etc.

The right-hand side of Figure 1 depicts the agent's perception of the environment and of his endowment. We note that the agent is aware of the one-step-ahead contingencies A and A^c. This is the assumption of **one-step-ahead awareness (1A)** mentioned in the introduction. The agent is also aware of the event $B \cup C$ in $t = 2$, but he is unaware of the finer contingencies B and C that comprise the event. This is why $B \cup C$ appears as a node in the agent's subjective tree $\hat{T}$.[5] We acknowledge that Assumption **(1A)** may appear heavy-handed, but we believe it is a good proxy for two important and realistic phenomena. First, it captures the property that contingencies in the distant future are more difficult to foresee than those that lie immediately ahead. Second, it ensures that agents have time to respond to a looming shock, not just do damage control.

[5]Formally, we can identify the true event tree T with a pair $(\Omega, \mathcal{F})$ where Ω is the state space of all complete histories and $\mathcal{F}$ is a filtration describing which events are realized when. The agent's subjective tree $\hat{T}$ is then a filtration coarser than T. These formalities can be found in the online appendix in Kochov [14]. The decision-theoretic setup we adopt is a special case of the model in that paper.

Unaware of the actual contingencies that comprise the event $B \cup C$, we assume that the agent perceives his endowment in that event as being constant and, in fact, equal to the conditional expectation of his actual endowment. Letting y^i denote the agent's actual endowment process and $\hat{y}^i$ his perceived endowment process, we can express this assumption, which we call **Correct in Expectation (CE)**, by the identity[6]

$$\hat{y}^i = \mathbb{E}_\mu[y^i \,|\, \hat{T}]. \tag{2.1}$$

The assumption has several layers. On one hand, it ensures that the agent's perceived endowment is not completely divorced from reality. Obviously, this plays a role in Theorems 1 and 2 which show how sequential trading and the ability to adapt can generate efficient outcomes, overcoming an initial misperception of risks. On the other hand, the assumption reflects an agent who is unaware of his unawareness. As discussed in Kochov [14], if the agent were aware of his unawareness, he may recognize that his endowment conditional on $B \cup C$ is subject to some residual risk and, out of cautiousness, adopt a very low assessment $\hat{y}^i$. In particular, $\hat{y}^i$ could be such that $< \mathbb{E}_\mu[y^i \,|\, \hat{T}]$. Though we do not investigate such cases presently, it is reasonable to conjecture that if agents held such cautious assessments, so as to correct for any unforeseen risks, the chance of unexpected default highlighted in Section 4 may be reduced.

Finally, CE implies the agent knows his endowment at any node he foresees. For example, since the nodes A and A^c are foreseen, the agent knows his endowment at those nodes. We take a similar approach when it comes to specifying the agent's beliefs. Namely, we assume that the agent assess correctly the likelihood of any event he foresees. Formally, the agent's belief $\hat{\mu}$ is the restriction of μ to $\hat{T}$.

As a way of summing up the present discussion, we note that our model is one in which any misperception can be traced to an unawareness of some physical contingencies. Indeed, given the objective environment (T, μ) and agent i's true endowment process y^i, the agent's perceptions thereof, $(\hat{T}, \hat{\mu}, \hat{y}^i)$, are fully determined by $\hat{T}$.

[6] A somewhat similar, but ultimately distinct, assumption is built into Jehiel's [11] notion of analogy-based expectation equilibrium, which captures coarse thinking in the context of games with perfect information.

2.2 Expanding Awareness

Assumption 1A implies that agents become aware of any impending risk before it happens. This ensures that agents can adapt by purchasing insurance. As discussed in the introduction, the assumption also stacks the deck against the possibility of default.

In the context of Figure 1, 1A implies that if the agent finds himself at node A, he will become aware of the contingencies B and C, of his endowment in those contingencies and the respective probabilities. In other words, his perception of the immediate future will coincide with the objective environment depicted on the left-hand side of the figure.

To define the dynamics of the agent's perceptions more formally, let s_t be a period-t node of the tree T. Let T^{s_t} be the corresponding continuation tree and $\mu^{s_t} = \mu(\cdot|s_t)$ the Bayes posterior of μ given s_t. When the agents find themselves at a node s_t, they face a continuation environment (T^{s_t}, μ^{s_t}) just as, at the beginning of time, they face the environment (T, μ). This means that we can define perceptions at s_t analogously to what we did in the previous section. As noted at the end of Section 2.2, all we have to do is specify a tree $\hat{T}^{s_t}$, coarser than T^{s_t}, describing the events foreseen at node s_t. Then, the subjective belief $\hat{\mu}^{s_t}$ at node s_t and agent i's perception $\hat{y}^{i,s_t}$ of the remainder of his endowment process are fully determined by $\hat{T}^{s_t}$.

Finally, we assume that the subjective trees $\hat{T}^{s_t}$ become progressively finer over time, which means that if an event is foreseen at some node s_t, it is foreseen at any node s_{t+k} that succeeds s_t.[7]

2.3 Radner Equilibrium Under Full Rationality

We proceed by defining the dynamic exchange economy under full rationality. The latter will serve as a benchmark in our welfare analysis. Together with the standard notion of Radner equilibrium, it will also lay the groundwork for the study of unawareness.

[7]Formally, let $s_t, s_{t'}, s_{t''}$ be nodes of T lying on the same path and such that $s_{t''}$ succeeds $s_{t'}$ which succeeds s_t. If $s_{t''}$ belongs to $\hat{T}^{s_t}$, then it must also belong to $\hat{T}^{s_{t'}}$.

Definition 1 *An economy with an initial distribution of assets is a tuple* $\mathcal{E}(a) := (N, (a^i), T, \mu, (y^i), u, \beta)$ *where*

1. *N is the set of agents.*

2. *The initial financial savings of agent i is $a^i \in \mathbb{R}$. This can be thought of as the amount of savings or debt which agent i brings to the first period of the economy. We restrict ourselves to the case of $\sum_i a^i = 0$.*

3. *$(T, \mu, (y^i))$ is the objective event tree, beliefs, and endowment processes for each agent.*

4. *Each agent i's utility of consumption process $(c^i_t)_t$ is given by*

$$U^i(c^i_0, c^i_1, ...) = \mathbb{E}_\mu \sum_t \beta^t [u \circ c_t],$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing, strictly concave, differentiable utillity index and $\beta > 0$ is a discount factor.

The financial market is such that at each node a market opens for next-period contingent claims. In a Radner equilibrium of such an economy, the agents execute trades in the available Arrow securities subject to expectations about future prices and their own future behavior. We normalize the price of the consumption good to be 1 at each node. Occasionally, we will speak of dollars as the unit of account, with the understanding that 1 dollar is the current price of 1 unit of the consumption good.

We use $p_{t+1}(s_{t+1})$ to denote the price of the s_{t+1} Arrow security in period t. As Definition 2 below is one of a Radner equilibrium from the perspective of the initial node, $p_1(s_1)$ should be viewed as an actual market clearing price at the initial node, whereas $p_{t+1}(s_{t+1}), t > 0$, is an expected price. Similarly, $a^i_1(s_1)$ is agent i's actual purchases of the s_1 Arrow security, while $a^i(s_{t+1}), t > 0$, are his expected purchases of the s_{t+1} security.

Definition 2 *A Radner Equilibrium in the economy $\mathcal{E}(a)$ consists of a consumption and savings plan for each agent, $\{c^i_t, a^i_{t+1}\}$, along with price expectation $\{p_{t+1}\}$ such that*

1. *All functions* $\{c_t^i, a_t^i, p_{t+1}\}$ *are measurable with respect to the information structure in the economy.*

2. *Each agent's consumption and savings plans are feasible given the price system and initial asset holdings. That is,* $\{c_t^i, a_{t+1}^i\}$ *satisfy*

$$c_0^i + \sum_{s_1|s_0} p_1(s_1)a_1^i(s_1) \leq y_0^i + a^i$$
$$c_t^i(s_t) + \sum_{s_{t+1}|s_t} p_{t+1}(s_{t+1})a_{t+1}^i(s_{t+1}) \leq y_t^i(s_t) + a_t^i(s_t) \quad \textit{for all } t \textit{ and } s_t$$

3. *Each agent takes* $\{p_{t+1}\}$ *as given and selects* $\{c_t^i, a_{t+1}^i\}$ *such that* c^i *maximizes his preferences over the feasible set.*

4. *Markets clear:* $\sum_i c_t^i \leq \sum_i y_t^i$ *and* $\sum_i a_{t+1}^i = 0$.

Under full rationality, the agents' expectations in a Radner equilibrium are self-fulfilling. Expected prices become actual prices, and all consumption and savings plans remain optimal at any future date. In other words, the agents' plans and the expected price system comprise a Radner equilibrium in the continuation economy. To state this formally, let $\overline{a} = (\overline{a}^i)_i$ be a level of savings with which the agents enter node s_τ and define the continuation economy at that node as

$$\mathcal{E}^{s_\tau}(\overline{a}) := (N, (\overline{a}^i), T^{s_\tau}, \mu^{s_\tau}, (y^{i,s_\tau}), u, \beta).$$

Then, any Radner equilibrium $\{p_{t+1}, c_t^i, a_{t+1}^i\}$ of $\mathcal{E}(a)$ induces a Radner equilibrium $\{p_{t+1}, c_t^i, a_{t+1}^i\}_{t\geq\tau}$ of $\mathcal{E}^{s_\tau}(\{a_\tau^i(s_\tau)\})$. We also remark that the economy $\mathcal{E}(a)$ is dynamically complete. Hence, any Radner equilibrium delivers an efficient (first-best) allocation.

2.4 Radner Equilibrium under Coarse Contingencies

We now develop a natural extension of Radner Equilibrium which incorporates unforeseen contingencies. The key observation is that at any point in time, the agents' perceptions constitute a well-defined economy to which we can apply the standard notion of Radner equilibrium. However, since the agents' perceptions of the future can change unexpectedly from one period to the next, so will the agents' plans and price expectations. Thus, the Radner Equilibrium at a future node need not be a continuation of the original equilibrium, but may involve a complete revision of prices and plans.

To formalize these ideas, recall from Section 2.1 that the agent's perceptions at any node s_t are fully determined by subjective tree $\hat{T}^{s_t}$. For example, at the initial node, the true economy $\mathcal{E}(a)$ and the subjective tree $\hat{T}$ give rise to the perceived economy

$$\hat{\mathcal{E}}(a) = (N, (\overline{a}^i), \hat{T}, \hat{\mu}, \hat{y}^i, u, \beta).$$

If there are unforeseen contingencies, the perceived economy $\hat{\mathcal{E}}(a)$ will be distinct from $\mathcal{E}(a)$. Yet, $\hat{\mathcal{E}}(a)$ is a well-defined economy and has a well-defined Radner equilibrium in the sense of Definition 2. This equilbrium gives us the level of savings $a_1^i(s_1)$ with which agent i will enter node s_1. Given these savings, the agents' subjective tree at that node, $\hat{T}^{s_1}$, gives us another perceived, but still well-defined economy,

$$\hat{\mathcal{E}}^{s_1}(a_1^i(s_1)) = (N, (a_1^i(s_1)), \hat{T}^{s_1}, \hat{\mu}^{s_1}, \hat{y}^{i,s_1}, u, \beta),$$

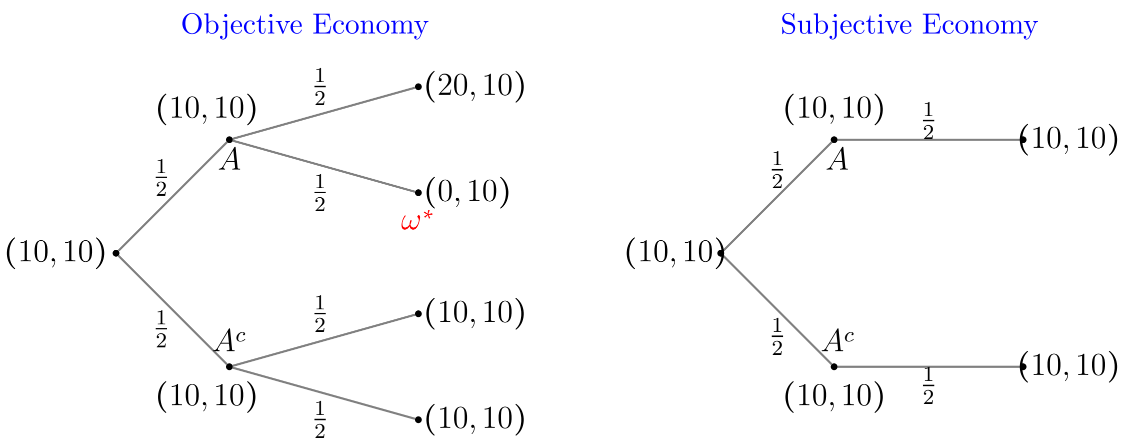
to which we can apply the concept of Radner Equilibrium once again. And so on. Summarizing:

Definition 3 *A Radner Equilibrium in Perceptions (REP) consists of a sequence of plans and price expectations, one at each node $s_t \in T$, such that: $(\hat{p}^{s_t}_{\tau+1}, \hat{c}^{i,s_t}_{\tau}, \hat{a}^{i,s_t}_{\tau+1})_{\tau \geq t}$ is a Radner Equilibrium in*

$$\hat{\mathcal{E}}^{s_t}(\hat{a}_t^{i,s_{t-1}}(s_t)) \coloneqq (N, (\hat{a}_t^{i,s_{t-1}}(s_t)), \hat{T}^{s_t}, \hat{\mu}^{s_t}, (\hat{y}^{i,s_t}), u, \beta)$$

Before we move on, we should highlight an important assumption buried in the above definition. Since the present economy is one without physical storage of the consumption good, the agents save (borrow) by purchasing (selling) Arrow securities, which are promises for delivery. Definition 3 assumes that the agents have to carry

Figure 2: The tree on the left is the environment with perfect awareness, while the tree on the right is the time-0 perception.



out these promises even if perceptions and prices change unexpectedly at the time of delivery.

3 Equilibrium Properties

Under full rationality, Radner equilibria are efficient because agents correctly anticipate the contingencies in which their endowment process leaves them under-insured. This enables them to spread the costs of insurance across all periods. With limited awareness, on the other hand, agents may fail to see the need to save for a distant contingency and therefore fail to make the necessary borrowing or lending decisions. We start this section with an example illustrating the possibility of such mistakes.

3.1 A Savings Mistake

There are two agents, both with log utility and discount factor $\beta = 1$. The tree on the left in Figure 2 depicts the objective environment and the agents' true endowments at each node. Note that agent 2 is perfectly insured while agent 1 faces an endowment risk at the top branch. Since preferences are homothetic, the economy with full awareness has a unique equilibrium in which, moreover, each agent consumes the same fraction of the aggregate endowment at every node.[8] In the example, these fractions are $\frac{17}{36}$ for agent 1 and $\frac{19}{36}$ for agent 2. Agent 1 gives up consumption in each node other than ω^* so as to insure himself against the event in which his endowment is zero.

Suppose now that at time 0, the agents do not foresee all events in the objective tree but have a coarse perception of the endowment in period 2, as depicted in the tree on the right of Figure 2. According to this perception, the agents are perfectly insured and have no motive to trade in period 0. It is only after moving to A that the agents become aware of the endowment uncertainty and trade occurs. At this point, however, it is too late for the agents to efficiently insure: the cost of insurance can only be spread across the nodes following A, rather than all of them. Compared to the equilibrium with perfect awareness, agent 1 ends up consuming a smaller fraction of the aggregate endowment in the nodes following A and a larger fraction in all other nodes.

Limited awareness thus results in a savings mistake; if agent 1 had foreseen his endowment risk, he would have transferred part of his wealth from the bottom branch of the tree to the top branch in order to consume a constant fraction of the aggregate endowment in all states. In the absence of such a transfer, agent 1 can still buy insurance when he becomes aware, but the cost is borne at once rather than spread over time.

3.2 Positive Results

We now show the savings mistake does not arise in two special but important cases. In both, REP exists and is unique, and results in an allocation that is efficient in the economy with full rationality $\mathcal{E}(a)$. First is the case of no aggregate uncertainty.[9]

Theorem 1 *If the aggregate endowment $\sum_i y^i$ is constant across states and time, then REP is efficient.*

[8] See Chapter 8.6.2 in Ljungqvist and Sargent [15].

[9] Since existence and uniqueness are standard, we simplify the statements of our results and do not mention them explicitly.

Proof. Let $\mathcal{E}(0)$ be the economy under full rationality and no initial assets (the latter for simplicity only). Since the aggregate endowment is constant, the Radner equilibrium in $\mathcal{E}(0)$ is described by[10]

$$\begin{aligned} p_t(s_t) &= \beta\mu(s_t|s_{t-1}) \\ c_t^i(s_t) &= (1-\beta)\sum_{t\geq 0}\beta^t\mathbb{E}_\mu[y_t^i] \\ a_1^i(s_1) &= \sum_{t\geq 1}\sum_{s_t|s_1}\beta^{t-1}\mathbb{E}_\mu[c_t^i - y_t^i|s_1] \end{aligned}$$

Let $\hat{\mathcal{E}}(0)$ be the economy with coarse contingencies. Since there is no aggregate uncertainty in $\mathcal{E}(0)$, there is no aggregate uncertainty in $\hat{\mathcal{E}}(0)$ as, by CE, $\sum_i \hat{y}^i = \mathbb{E}_\mu[\sum_i y^i]$. Consequently, the Radner Equilibrium in $\hat{\mathcal{E}}(0)$ has a similar form as that in $\mathcal{E}(0)$, in particular,

$$\begin{aligned} \hat{c}_0^{i,s_0} &= (1-\beta)\sum_{t\geq 0}\beta^t\mathbb{E}_{\hat{\mu}^{s_0}}[\hat{y}_t^i] \\ \hat{a}_1^{i,s_0}(s_1) &= \sum_{t\geq 1}\sum_{s_t|s_1}\beta^{t-1}\mathbb{E}_{\hat{\mu}^{s_0}}[c_t^i - \hat{y}_t^i|s_1] \end{aligned}$$

CE implies that $\mathbb{E}_{\hat{\mu}^{s_0}}[\hat{y}_t^i] = \mathbb{E}_\mu[\hat{y}_t^i]$ and, hence, $\hat{c}_0^{i,s_0} = c_0^i$ and for all s_1, $\hat{a}_1^{i,s_0}(s_1) = a_1^i(s_1)$. The agents' decisions in period 0 under coarse perceptions thus coincide with those in $\mathcal{E}(0)$. Since the initial savings are the same under coarse perceptions and full rationality, we can then use an analogous argument to prove that the s_1 consumption and the s_2 dated savings are identical in $\mathcal{E}^{s_1}(a_1(s_1))$ and $\hat{\mathcal{E}}^{s_1}(a_1(s_1))$. Continuing in this fashion, one can conclude that the resulting consumption profiles are identical. ■

If there are no fluctuations in the aggregate endowment, then the prices for Arrow securities simply reflect the likelihood of the states in which they pay out. Together with the assumption that agents are correct in expectation, this property guarantees that agents accurately foresee the market value of their endowments in the distant future. As a result, they correctly allocate their endowments over time. The assumption that agents foresee all one-step ahead contingencies is then sufficient to imply

[10] See, for example, Ljungqvist and Sargent [15, p.218].

that agents insure efficiently against all events, including those that were not foreseen in period 0.

If preferences are homothetic, we can replace the assumption of no aggregate uncertainty with the assumption that aggregate shocks are foreseen. Formally, we require that the aggregate endowment be measurable with respect to time-0 perceptions **(MAE)**, i.e., for any node s_t^0 in the subjective tree and all nodes s_t in the objective tree that comprise s_t^0 (which we write as $s_t \subset s_t^0$), we have $\sum_i y_t^i(s_t) = \sum_i \hat{y}_t^i(s_t^0)$.

Theorem 2 *If preferences are homothetic and the aggregate endowment is measurable with respect to the initial subjective perceptions, then REP is efficient.*

Proof. The assumption of homethetic preferences means that there is $\alpha > 0$ such that $u(c) = \frac{c^{1-\alpha}}{1-\alpha}$ if $\alpha \neq 1$ and $u(c) = \log c$ if $\alpha = 1$. As remarked in Section 3.1, the equilibrium with perfect awareness involves each agent consuming a constant fraction of the aggregate endowment. Letting $Y_t(s_t) := \sum_i y_t^i(s_t)$, this fraction under perfect foresight is

$$\gamma^i := \frac{\sum_{t\geq 0} \beta^t \sum_{s_t \in T} \mu(s_t) Y_t(s_t)^{-\alpha} y_t^i(s_t)}{\sum_{t\geq 0} \beta^t \sum_{s_t \in T} \mu(s_t) Y_t(s_t)^{1-\alpha}}$$

Analogously, the equilibrium fraction which agent i consumes in the perceived economy $\hat{\mathcal{E}}(0)$ is

$$\hat{\gamma}^{i,s_0} = \frac{\sum_{t\geq 0} \beta^t \sum_{s_t^0 \in \hat{T}^{s_0}} \hat{\mu}(s_t^0) \hat{Y}_t^{s_0}(s_t^0)^{-\alpha} \hat{y}_t^{i,s_0}(s_t^0)}{\sum_{t\geq 0} \beta^t \sum_{s_t^0 \in \hat{T}^{s_0}} \hat{\mu}(s_t^0) \hat{Y}_t^{s_0}(s_t^0)^{1-\alpha}} = \frac{\sum_{t\geq 0} \beta^t \sum_{s_t^0 \in \hat{T}^{s_0}} \mu(s_t^0) \hat{Y}_t^{s_0}(s_t^0)^{-\alpha} \hat{y}_t^{i,s_0}(s_t^0)}{\sum_{t\geq 0} \beta^t \sum_{s_t^0 \in \hat{T}^{s_0}} \mu(s_t^0) \hat{Y}_t^{s_0}(s_t^0)^{1-\alpha}}$$

where the equality proceeds from $\hat{\mu} = \mu$ on the subjective events. We claim that $\hat{\gamma}^{i,s_0} = \gamma^i$. Notice:

$$\sum_{s_t} \mu(s_t) Y_t^{-\alpha}(s_t) y_t^i(s_t) = \sum_{s_t^0 \in \hat{T}^{s_0}} \sum_{s_t \subset s_t^0} \mu(s_t | s_t^0) \mu(s_t^0) Y_t(s_t)^{-\alpha} y_t^i(s_t) \tag{3.1}$$

$$= \sum_{s_t^0 \in \hat{T}^{s_0}} \mu(s_t^0) \hat{Y}_t^{s_0}(s_t^0)^{-\alpha} \sum_{s_t \subset s_t^0} \mu(s_t | s_t^0) y_t^i(s_t) \tag{3.2}$$

$$= \sum_{s_t^0 \in \hat{T}^{s_0}} \mu(s_t^0) \hat{Y}_t^{s_0}(s_t^0)^{-\alpha} \hat{y}_t^{i,s_0}(s_t^0) \tag{3.3}$$

MAE is invoked in (3.2) and CE in (3.3). MAE further implies $\sum_{s_t} \mu(s_t) Y_t(s_t)^{1-\alpha} = \sum_{s_t^0} \mu(s_t^0) \hat{Y}_t^{s_0}(s_t^0)^{1-\alpha}$, which establishes $\hat{\gamma}^{i,s_0} = \gamma^i$. Hence, the consumption fraction for i is the same in $\mathcal{E}(0)$ and $\hat{\mathcal{E}}(0)$. This implies $\hat{c}_0^{i,s_0} = c_0^i$ and $\hat{a}_1^{i,s_0}(s_1) = a_1^i(s_1)$ for all s_1.

Now consider the next-period economies under perfect awareness and coarse contingencies, $\mathcal{E}^{s_1}(a_1(s_1))$ and $\hat{\mathcal{E}}^{s_1}(a_1(s_1))$. These economies are related to each other in the same way as $\mathcal{E}(0)$ and $\hat{\mathcal{E}}(0)$. Since each agent's initial assets are the same across both economies, the current consumption and savings decisions in $\hat{\mathcal{E}}^{s_1}(a_1(s_1))$ are the same as in $\mathcal{E}^{s_1}(a_1(s_1))$. Continuing in this fashion, it follows that the consumption profile from trading under coarse contingencies is equal to the consumption profile under perfect awareness. ■

To gain some intuition for Theorem 2, recall again that, with homothetic preferences, the agents consume a constant fraction of the aggregate endowment in equilibrium. How large this fraction is depends on the value of the agent's endowment process. Endowments in states where the good is relatively scarce are more valuable than those where the good is abundant. The assumption that there are no unforeseen fluctuations in the aggregate endowment implies that agents can foresee the events in which the consumption good is relatively scarce and, therefore, estimate correctly the value of their endowment process at time 0. As a result, agents make the right saving choices and consume a constant fraction of the total resources, even when perceptions change. Idiosyncratic risks are fully insured and the cost of insurance is efficiently spread over time.

We conclude this section by noting some special features of the equilibria in Theorems 1 and 2, and why we believe the results are insightful nonetheless. Thus, under both results, the equilibrium consumption profile in the economy with limited awareness is the same as that under full rationality. This means that when a new contingency arises, agents do not have to adjust their consumption plans but only the implementation of these plans through the purchase of additional insurance. We can also construct examples in which the latter is not true – agents adapt their consumption plans over time – and efficiency continues to hold. Indeed, in Section 5, we will show how to solve for the evolution of an economy with homethetic preferences but without any restrictions on the agents' perceptions or the aggregate endowment.

The calculations will provide a clear roadmap for the construction of such examples. It will also be clear that the instances in which full efficiency obtains are special, just as the assumptions of no aggregate uncertainty and MAE, while important benchmark cases, are highly non-generic. However, we note that full efficiency itself is a rather exacting requirement. Expecting some continuity of the equilibrium allocation, the conceptual lesson we draw from Theorems 1 and 2 is that efficiency losses from unawareness are tied to the agents' perception of aggregate shocks. If unexpected aggregate shocks are small, or there is little correlation among individual endowments, the losses will be small as well. Formalizing this observation is an interesting problem to pursue.

4 Default

The previous section depicted economies which transition successfully from one perceived equilibrium to another as the agents' awareness grows. We now show that this is not guaranteed. The misperceptions that led to insufficient saving in Section 3.1 may lead to excessive debt which the agents cannot repay. The economy can then no longer "self-correct" and transition to a new equilibrium. The possibility of such breakdowns is limited by the assumption of one-steph-ahead awareness, but they can nonetheless occur in economies in which agents plan to roll over high levels of debt. As emerging risks decrease the value of future endowments, the agents in these economies are forced to pay off maturing obligations out of current endowments, but there may be no spot prices that simultaneously clear markets and keep the cost of debt affordable.

To give an example, suppose there are three periods and two agents, both with log utility and a discount factor of 1. The true economy and the period-0 perception are as depicted in Figure 3. In periods 0 and 1, agent 1 is poor, whereas agent 2 is rich. According to time-0 perceptions both agents have the same endowment in the last period.

Given the perceived endowment structure, the optimal period-0 plan for agent 1 is to shift consumption from the last period to the first two. This plan entails agent 1 taking on debt in period 0 and rolling it over in period 1. Figure 4 depicts the

Figure 3

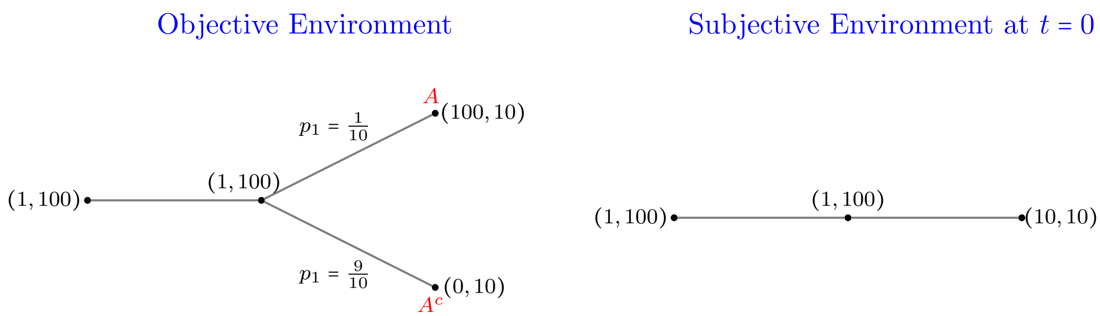
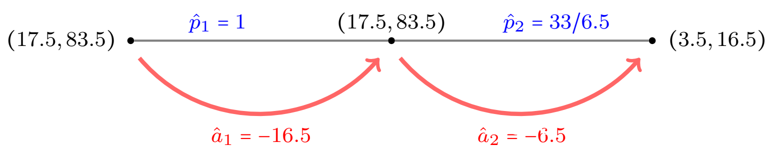


Figure 4

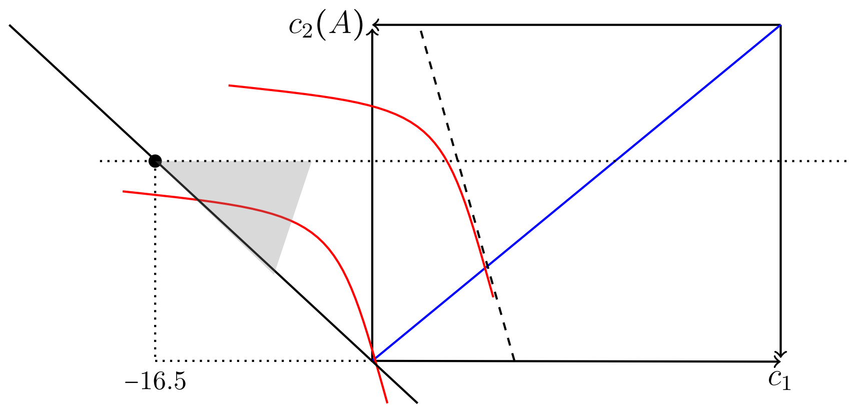


equilibrium. In time 0, agent 1 borrows \$16.5 in order to consume 17.5 units of the consumption good. While this debt exceeds his endowment in period 1, he plans to raise \$33 by selling a claim for 6.5 units of the consumption good in period 2. In this way, the agent can repay his debt and consume 17.5 units of the consumption good in period 1.

Once both agents arrive in period 1, they realize that agent 1's endowment is in fact risky. It is either 100, with probability 1/10, or 0 otherwise. See the left tree in Figure 3. Markets reopen and there are now two relevant Arrow securities, one for each subsequent state. Agent 1 is endowed with 1 unit of the consumption good and owes the other agent 16.5 units. The only way agent 1 can repay this debt is by selling Arrow securities for state A in which agent 1 is rich. Since agent 1's endowment in that state is 100, the price for this asset needs to be weakly greater than $\underline{p} := 0.155$. A calculation shows, however, that at price $\underline{p}$ agent 2's demand for the asset is strictly below 100. Hence, at price $\underline{p}$ agent 1 cannot sell enough units of the asset to repay his debt. Since the same is true for any price above $\underline{p}$, no market clearing price can be found.

The *necessity to default* can be illustrated with an Edgeworth box as in Figure

Figure 5: Period-1 Edgeworth box after updating perceptions



5. Note first that we can abstract from node A^c, in which agent 1's endowment is 0 and cannot be sold. Second, the fact that agent 1 is in debt in period 1 means that the endowment point lies outside the Edgeworth box. The slope of the budget line passing through the endowment point and the origin describes the lowest relative price of consumption in state A at which agent 1 can repay his debt from period 0. Under homothetic preferences, the contract curve is linear and the marginal rates of substitution are constant at each point on the curve. If an equilibrium without default were to exist, its relative price would be equal to that marginal rate of substitution, which is captured by the slope of the dashed line in Figure 5. This line could be made arbitrarily steep by lowering the probability of state A. If the line is steeper than the line passing through the origin and the endowment point, there will be no spot price that simultaneously clears the market and yields non-negative consumption for both agents.

5 Equilibrium Characterization for CRRA Utility

When agents have homothetic utility, the Radner equilibrium has a closed form solution and the evolution of an economy with coarse contingencies can be fully described, as we do presently. We believe that these calculations may help formalize the connection between unforeseen aggregate shocks and any associated loss of efficiency, discussed at the end of Section 3.2, as well as help in the study of default. The equilibrium in $\hat{\mathcal{E}}^{s_\tau}(\overline{a})$ is characterized by

$$\begin{aligned}
\hat{p}_{t+1}^{s_\tau}(s_{t+1}) &= \beta\hat{\mu}^{s_\tau}(s_{t+1}|s_t)\left(\frac{\hat{Y}_{t+1}^{s_\tau}(s_{t+1})}{\hat{Y}_t^{s_\tau}(s_t)}\right)^{-\alpha} && (5.1)\\
\hat{\gamma}^{i,s_\tau} &:= \frac{\sum_{t\geq\tau}\sum_{s_t\in\hat{T}^{s_\tau}}\prod_{k=\tau}^{t}\hat{p}_k^{s_\tau}(s_k)\hat{y}_t^{i,s_\tau}(s_t)+\overline{a}^i}{\sum_{t\geq\tau}\sum_{s_t\in\hat{T}^{s_\tau}}\prod_{k=\tau}^{t}\hat{p}_k^{s_\tau}(s_k)\hat{Y}_t^{s_\tau}(s_t)} && (5.2)\\
\hat{c}_t^{i,s_\tau}(s_t) &= \hat{\gamma}^{i,s_\tau}\hat{Y}_t^{s_\tau}(s_t) && (5.3)\\
\hat{a}_{\tau+1}^{i,s^\tau}(s_{\tau+1}) &= \sum_{t\geq\tau+1}\sum_{s_t\in\hat{T}^{s_\tau}(s_{\tau+1})}\prod_{k=\tau+2}^{t}\hat{p}_k^{s_\tau}(s_k)[\hat{c}_t^{i,s_\tau}(s_t)-\hat{y}_t^{i,s_\tau}(s_t)] && (5.4)
\end{aligned}$$

At each state s_τ the perceived Radner equilibrium involves each agent i consuming a constant fraction $\hat{\gamma}^{i,s_\tau}$ of the aggregate endowment in all future states. The processes $\{\hat{\gamma}^{i,s_\tau}\}_{i,s_\tau}$ thus describe the consumption dynamics of the REP. When agents have perfect foresight, $\hat{\gamma}^{i,s_\tau}$ is time and state independent. Theorem 2 shows that this property is preserved under evolving awareness whenever the aggregate endowment is measurable with respect to the initial subjective perceptions. When MAE is relaxed, $\hat{\gamma}^{i,s_\tau}$ will typically depend on the state s_τ and consumption plans must be adjusted over time.

A change in agent i's planned consumption is due to a change in the agent's perception of his future wealth. For a concrete illustration of this intuition, let us define

$$W^{i,s_\tau}(s_{\tau+k}) = \sum_{t\geq\tau+k}\sum_{s_t\in\hat{T}^{s_\tau}(s_{\tau+k})}\prod_{l=\tau+k+1}^{t}\hat{p}_l^{s_\tau}(s_l)\hat{y}_t^{i,s_\tau}(s_t)$$

as the market value of i's endowment process starting at node $s_{\tau+k}$ in the economy $\hat{\mathcal{E}}^{s_\tau}(a^i(s_\tau))$. The total wealth in this economy is denoted by $T^{s_\tau}(s_{\tau+k}) := \sum_i W^{i,s_\tau}(s_{\tau+k})$. Using these definitions, we can expand (5.4) to write

$$\begin{aligned}
\hat{\gamma}^{i,s_0}T^{s_0}(s_0) &= W^{i,s_0}(s_0) && (5.5)\\
\hat{\gamma}^{i,s_\tau}T^{s_\tau}(s_\tau) &= \hat{\gamma}^{i,s_{\tau-1}}T^{s_{\tau-1}}(s_\tau)+W^{i,s_\tau}(s_\tau)-W^{i,s_{\tau-1}}(s_\tau) && (5.6)
\end{aligned}$$

Agent i's claim to the total resources in $\hat{\mathcal{E}}^{s_\tau}(a^i(s_\tau))$ is $\hat{\gamma}^{i,s_\tau}T^{s_\tau}(s_\tau)$. This might differ from what agent i believed at $s_{\tau-1}$ his consumption would be if s_τ occurred.

As equation (5.6) shows, the perceived claim $\hat{\gamma}^{i,s_{\tau-1}} T^{s_{\tau-1}}(s_\tau)$ is strictly greater than $\gamma^{i,s_\tau} T^{s_\tau}(s_\tau)$ if agent i's perceived wealth $W^{i,s_{\tau-1}}(s_\tau)$ declines after updating perceptions. Agent i is then poorer than previously believed and adjusts consumption accordingly.

When the wealth difference $W^{i,s_{\tau-1}}(s_\tau) - W^{i,s_\tau}(s_\tau)$ is sufficiently large, the parameter $\hat{\gamma}^{i,s_\tau}$ solving (5.6) turns negative. In this case, the transition between equilibria is disrupted and agent i defaults in state s_τ. Hence, default occurs precisely when there is a significant wealth shock due to unforeseen contingencies, causing a large discrepancy between the agent's perceived wealth before and after updating perceptions. For the case of CRRA utility, it is then easy to check whether any of the agents default in a given economy: there is no default in the REP if and only if $\hat{\gamma}^{i,s_\tau} \geq 0$ for all i and s_τ.

6 Discussion

In this section, we discuss our framework, its relationship to the literature, and several open problems.

Modica et al. [17] are the first to study limited awareness and default in general equilibrium. They consider a two-period economy, where agents trade assets whose returns depend on unforeseen contingencies. Teeple [20] studies a similar setting, but unlike Modica et al. [17] restricts agents' beliefs to be correct in expectation, as we do. In contrast to these papers, we study a market where agents fully understand the assets that they trade. We show that default can still occur and focus on the evolution of the market as awareness grows over time. In another recent paper, Guerdjikova and Quiggin [9] propose a notion of competitive equilibrium when agents have asymmetric awareness. They assume that agents are fully aware of their endowment processes and do not consider the dynamics associated with changing awareness over time. Also Gul, Pesendorfer, and Strzalecki [10] consider competitive equilibria with limited cognition. Their model, however, is not one of unawareness but of agents who optimally allocate attention subject to cognitive constraints. The questions they investigate and the resulting analysis are rather different.

Next, the idea that under bounded rationality the economy will transition from one equilibrium to another is shared by the literature on temporary equilibrium. In that literature, however, the source of bounded rationality is not explicitly modeled. Instead, one typically postulates uncertainty about prices that is not pinned down by the physical state of the world. A drawback of this reduced-form approach is that welfare analysis becomes difficult as the concept of welfare itself cannot be dissociated from the market mechanism. By comparison, our framework is explicit about the source of bounded rationality which, as the analysis in Section 3.2 illustrates, makes possible the study of a rich and, in our opinion, interesting array of questions concerning welfare.

We should also mention that the different frameworks deliver complementary lessons about equilibrium existence. Green [8] shows that a temporary equilibrium fails to exist whenever agents' beliefs about future prices are too disparate. Our example shows that REP may fail even when the agents' expectations are in full agreement.

In terms of open problems, one must confront the question as to what happens after default. Paraphrasing Green [7, p.264], there are two issues: settlement and the realization that default may happen again, even as the agents remain unaware of the exact physical contingencies that may cause a recurrence. We conjecture that settlement can be handled in the manner of Green [7], Modica et al. [17], and more recently Ben-Ami and Geanakoplos [2].[11] Incorporating anticipation of future default will be more challenging as it brings about the thorny problem of modeling agents who are aware of their own unawareness. Recent work in decision theory suggests two routes. One route, exemplified by Mukerji [18], Ghirardato [5], Epstein, Marinacci, and Seo [3], and Kochov [14], suggests that agents, being aware that their perception of possible risks is incomplete, focus on the worst case. As explained in Section 2, this may involve modifying the CE assumption. Another route, exemplified by Grant and Quiggin [6] and Karni and Vierø [13], is to introduce a fictitious state or outcome as a stand-in for "the unexpected." This approach may call for some kind of symbiosis with the literature on temporary equilibrium and the work of Green [7] in particular.

[11] In addition to the possibility of default, Ben-Ami and Geanakoplos [2] show that debt, or an endowment outside of the Edgeworth box, may lead to other interesting phenomena, including multiple and unstable equilibria. It will be curious to see if these possibilities can be realized in our framework.

We leave these developments for future work, hoping that the present analysis will not only stir interest but serve as a useful testing ground.

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