A General Framework for Studying Contests

Spencer Bastani   Thomas Giebe   Oliver Gürtler

A General Framework for Studying Contests[*]

Spencer Bastani[†]   Thomas Giebe[‡]   Oliver Gürtler[§]

First version: December 3, 2019
This version: May 5, 2020

Abstract

We develop a general framework for studying contests, including the well-known models of Tullock (1980) and Lazear & Rosen (1981) as special cases. The contest outcome depends on players' efforts and skills, the latter being subject to symmetric uncertainty. The model is tractable, because a symmetric equilibrium exists under general assumptions regarding production technologies and skill distributions. Using a link between our contest model and expected utility theory, we are able to derive new comparative statics results regarding how the size and composition of contests affect equilibrium effort, showing how standard results can be overturned. We also discuss the robustness of our results to changes in the information structure and the implications of our findings for the optimal design of teams.

Keywords: contest theory, symmetric equilibrium, heterogeneity, risk, stochastic dominance

JEL classification: C72, D74, D81, J23, M51

---

1 Introduction

In a contest, two or more players invest effort or other costly resources to win a prize. Many economic interactions can be modeled as a contest. Promotions, for example, represent an important incentive in many firms and organizations. Employees exert effort to perform better than their colleagues and, thus, to be considered for promotion to a more highly paid position within the firm. Litigation can also be understood as a contest, in which the different parties spend time and resources to prevail in court. Procurement is a third example, where different firms invest resources into developing a proposal or lobbying politicians, thereby increasing the odds of being selected, receiving some rent in return.

Players participating in contests are typically heterogeneous in some respect. For instance, employees differ with respect to their skills, the litigant parties differ with respect to the quality of the available evidence, and firms differ with respect to their capabilities of designing a proposal. When accounting for such heterogeneity in contest models, equilibria are often asymmetric, meaning that players choose different levels of effort. Due to this asymmetry, contests between heterogeneous players are typically difficult to analyze, and researchers have often imposed rather strict assumptions to keep the analyses tractable.

In this paper, we provide a novel framework to study contests between (possibly) heterogeneous players. Under general assumptions about the production technology and skill distributions, the class of contests we study has a symmetric equilibrium in which all players exert the same effort. This makes the contest much easier to investigate and allows us to study behavior in situations that proved to be intractable to study in other contest models.

In the class of contests that we focus on, the outcome of the contest depends on players' skills and their efforts. The skill distributions of the competing players (including the expected values) are common knowledge, whereas the exact skill realizations are generally (symmetrically) unknown (as, e.g., in Holmström 1982). In the example of the promotion contest, a player's expected ability may be commonly known (e.g., the education, prior work experience, or CV of a player is known and serves as a signal of ability), whereas the exact ability level for the particular job is unknown (e.g., there might be uncertainty regarding how education translates into workplace performance and job match). Similar arguments apply to the other examples presented earlier. We assume that a player's skill and effort jointly determine the player's "contribution" to the contest and that the player with the highest contribution wins the contest. Heterogeneity among

players is accounted for by allowing the statistical distributions of possible skill realizations to be different for the competing players. Our model is general and contains the well-known models by Tullock (1980) and Lazear & Rosen (1981) as special cases. We make three primary contributions.

First, we show that in a two-player contest, a symmetric (pure-strategy equal-effort) equilibrium exists under general assumptions about the production technology (i.e., the function mapping skill and effort into a player's contribution to the contest) and individual skill distributions. The main requirement is that the production function is such that for any given positive effort level, a player's contribution to the contest is increasing in his or her skill. This is a weak requirement from the perspective of the most commonly used neoclassical production technologies, and also appears to be quite realistic.

Second, we construct a link between our contest model and standard models of decision-making under risk (expected utility theory). Exploiting this link, we revisit important comparative statics results of contest theory and show how these can be overturned. In particular, we analyze how equilibrium effort is affected by making the skill distributions of the competing players more heterogeneous, investigating both the role of differences in expected skill (conceptualized by first-order stochastic dominance) and the role of differences in the uncertainty of the skill distributions of the competing players (conceptualized by second-order stochastic dominance). The general message is that making contest participants more heterogeneous can increase equilibrium effort. To the best of our knowledge, these results have not been found in the contest literature before, and indeed contradict "standard" results (e.g., those from the Tullock contest and the Lazear-Rosen tournament). Thus, the comparative statics results derived from those standard models are not representative of the conclusions derived in the more general model.

Third, in two important special cases, we provide new results on the existence and interpretation of symmetric equilibria in our general setting when the number of players n is greater than two. We show that our solution method and interpretation for the two-player case extends to the n player case when players have identical skill distributions. We also show that, for a specific class of skill distributions, a symmetric equilibrium exists when $n-1$ identical players compete against a player who has a higher expected ability. We investigate the effect of increasing the number of players on equilibrium effort. Exploiting the fact that a contest with $n > 2$ players can be interpreted as a two-player contest in which every player competes against the strongest (i.e., the largest order statistic) of his

or her opponents, we find that increasing the number of contestants can increase equilibrium effort. This result, which contrasts standard results in the literature, can be understood by the fact that as the number of contestants increases, the strongest opponents grow stronger in the sense of first-order stochastic dominance, allowing us to apply our results from the two-player case.

An important aspect of our contribution is that we provide intuition for how the incentive to exert effort depends on the interaction between three factors. The first factor relates to the production technology and is the ratio of the marginal product of effort and the marginal product of skill. The intuition behind this factor is that the purpose of a marginal effort increase for an individual player is to beat marginally more able rivals. The ratio describes how effective a marginal effort increase is to overcome the output advantage of marginally more skilled players. The second and third factors are represented by the product of the densities of the skill distributions of the two competing players, evaluated at the same point. The reason for the presence of this product is that a player only has a marginal incentive to exert effort in cases where the skill realizations of the two players are exactly the same, and the product describes the "likelihood" of this event to happen.

We also investigate the robustness of our results with respect to the assumption of symmetric uncertainty by analyzing the consequences of letting players be privately informed about their skills. In this case, equilibria are in general not symmetric, but focusing on symmetric players, we are able to draw interesting parallels with respect to our baseline case, highlighting the role of our general production technology in influencing the marginal incentive to exert effort.

Throughout the paper we discuss the implications for optimal team composition and certain real-world applications in the context of labor and personnel economics. For instance, our finding that efforts can increase if the skill distribution of one of the competing players becomes more uncertain (in the sense of second-order stochastic dominance) has several interesting managerial implications. It indicates that contest organizers might wish to increase the uncertainty regarding the skills of certain players in order to induce higher effort. In a worker-firm context, employers could achieve this by, for instance, hiring an inexperienced worker for whom little prior information is available, or a minority worker with a skill level drawn from a distribution that generally tends to be more uncertain (as argued, e.g., by Bjerk 2008). This means that having diverse teams might be desirable from the employer's point of view.

The paper is organized as follows. In Section 2 below, we discuss related lit-

erature. Section 3 introduces the contest model and discusses how our model nests the Tullock contest and the Lazear-Rosen tournament as special cases. Section 4 solves the two-player model. In Section 5, we analyze the two-player case in greater detail and provide a set of important comparative statics results. We also discuss implications for organizational design and optimal team composition. Section 6 studies the n-player case and presents novel comparative statics results for this case. Section 7 takes a look at the case of privately known skills. Finally, Section 8 concludes.

2   Related Literature

There are three main approaches to the study of contests, the Tullock or ratio-form contest, the Lazear-Rosen tournament, and the complete-information all-pay auction.[1] In the Tullock contest, a player's winning probability is given by the player's contribution to the contest (which is a function of the player's effort and sometimes also of ability) divided by the total contribution to the contest of all players. The Tullock contest has been introduced to the literature by Tullock (1980).[2] It has been axiomatized in various settings by Skaperdas (1996), Clark & Riis (1998b), and Münster (2009). The Lazear-Rosen tournament assumes that the player with the highest contribution to the contest wins with certainty, and contributions depend on effort, some random factors (e.g., luck), and possibly on abilities. The seminal paper is by Lazear & Rosen (1981) who apply the model in a labor-market context.[3] The all-pay auction, finally, makes the same assumption as the Lazear-Rosen tournament except that contributions to the contest are deterministic and do not depend on random factors; a detailed equilibrium characterization was developed by Baye et al. (1996).[4]

---

[1]The theoretical contest literature has been surveyed in a number of books and papers. See, e.g., Konrad (2009) and Vojnović (2015) for recent textbooks and Chowdhury & Gürtler (2015), Chowdhury et al. (2019), and Fu & Wu (2019) for recent surveys.

[2]It has been further investigated by, e.g., Hillman & Riley (1989), Cornes & Hartley (2005), Fu & Lu (2009a,b), Corchón & Dahm (2010), Schweinzer & Segev (2012), and Chowdhury & Kim (2017).

[3]Further contributions include Green & Stokey (1983), Malcomson (1984, 1986), O'Keeffe et al. (1984), Lazear (1989), Schotter & Weigelt (1992), Zábojník & Bernhardt (2001), Hvide (2002), Grund & Sliwka (2005), Schöttner & Thiele (2010), Gürtler & Gürtler (2015), and Imhof & Kräkel (2016).

[4]The complete-information all-pay auction (with mixed-strategy equilibria) is the most commonly used in contest theory, but a private-values version can be found as well. The all-pay auction has been further studied by, e.g., Clark & Riis (1998a), Barut & Kovenock (1998), Moldovanu & Sela (2001, 2006), Moldovanu et al. (2007), Cohen et al. (2008), Siegel (2009, 2010), Sela (2012), Morath & Münster (2013), Barbieri et al. (2014), Olszewski & Siegel (2016), and Fang et al. (2019).

Most studies analyzing the Tullock contest and the Lazear-Rosen tournament impose assumptions that ensure that equilibria in pure strategies exist. In contrast, only mixed-strategy equilibria exist in the all-pay auction (when players are symmetrically informed about the decision situation). As we indicated in the introduction, and as we explain in more detail in Section 3, the Tullock contest and the Lazear-Rosen tournament are special cases of our model, while the all-pay auction is not.

One important contribution of our paper is to generalize the Tullock contest and the Lazear-Rosen tournament and to show that canonical results arising from these models do not always extend to more general production functions and ability distributions. These important results refer to how player heterogeneity, the extent of risk or uncertainty, and the number of players affect the effort exerted by the competing players. For example, Schotter & Weigelt (1992) have shown that efforts are higher when players have homogeneous skills relative to when they are heterogeneous. The reason in their setting is that disadvantaged players tend to give up and reduce their effort, whereas advantaged players can afford to reduce their effort. Moreover, several studies have shown that greater uncertainty regarding the contest outcome tends to reduce effort (see, e.g., Hvide 2002). Intuitively, if the contest outcome depends to a greater extent on random factors, effort has a lower impact on who becomes the winner and players reduce effort accordingly. Finally, in the seminal work of Tullock (1980), effort decreases in the number of players who participate in the contest, which has been attributed to a discouragement effect. If a player competes against many rivals, his or her chance of winning is relatively low and the player reduces effort in turn. Although all of these results seem highly intuitive, we find that they are sensitive to the choice of production technologies and ability distributions. In our general framework, different comparative statics results may emerge.

Some exceptions to these standard results have already been documented in the literature. Drugov & Ryvkin (2017) study biases in contests between symmetric players. A bias affects the selection of the winner and, in some instances, can be interpreted in the same way as skill heterogeneity. They show that biasing a contest between symmetric players can trigger higher effort under certain conditions. Our paper differs from theirs in several ways. Most importantly, biasing a contest is not always the same as making players heterogeneous by changing their skill distributions. Furthermore, Drugov & Ryvkin (2017) focus on the contest-success function (i.e., the function mapping efforts into winning probabilities) and they provide conditions that the function must fulfil to make it optimal

to bias the contest. In contrast, we determine the contest-success function endogenously, taking the players' production technology and skill distributions into account. We then show under which types of production technologies and skill distribution efforts get higher as the contestants become more heterogeneous.

Our particular set of results regarding the effects of the number of competing players on equilibrium effort are related to Ryvkin & Drugov (2020) who show that effort can be an increasing, a decreasing or a unimodal function of the number of players depending on the distribution of noise (which corresponds to the skill distribution in our model). However, they restrict attention to additive production technologies, whereas our study allows for more general types of production technologies. As we show in the paper, the relation between equilibrium effort and the number of competing players crucially depends on the interaction between the production technology and the skill distributions of the competing players.

Finally, Kirkegaard (2020) is another related paper, which can be seen as complementary to ours. Kirkegaard proposes a general contest model that is similar to our model. His focus, however, is on optimal contest design and on possible microfoundations of biased ratio-form contests. In contrast, our focus is on the effects of (different types of) player heterogeneity on equilibrium effort. Our main contribution is to show that a symmetric equilibrium generally exists in two-player contests and that making players more heterogeneous or adding players to the contest may increase the incentive to exert effort.

3 Model Description

Consider a contest between two risk-neutral players $i \in \{1, 2\}$ who compete for a single prize of value $V > 0$. Both players simultaneously choose effort $e_i \geq 0$, and the cost of effort $c(e_i)$ is described by a continuously differentiable, strictly increasing and strictly convex function satisfying $c(0) = 0$. The ability or skill (type) of player i is denoted by Θ_i. There is uncertainty about skills, which means that Θ_i is a random variable. The realization of Θ_i is denoted by θ_i and it is not known to any of the players (not even player i). It is commonly known, however, that Θ_i is independently and absolutely continuously distributed according to the pdf f_i (with cdf F_i) with finite mean μ_i. For a given density f, we will use $\text{supp}(f) = \{x \in \mathbb{R} : f(x) > 0\}$ to denote its support. We assume that the supports of f_1 and f_2 overlap on a subset of $\mathbb{R}$ with positive measure.

Symmetric uncertainty regarding skills is typically imposed in the career-

concerns literature (e.g., Holmström 1982, Holmström & Ricard I Costa 1986, Dewatripont et al. 1999, and Auriol et al. 2002) and also in the literature on promotion signaling (e.g., Waldman 1984, Bernhardt 1995, Owan 2004, Ghosh & Waldman 2010, DeVaro & Waldman 2012, and Gürtler & Gürtler 2019). This literature refers to firm-worker relationships, and the idea is that both firms and workers are uncertain about how well workers perform when they begin their working careers and that this uncertainty is reduced over time once performance information becomes available. We adopt this idea, referring to Θ_i as a player's skill, but one could also interpret it more broadly as incorporating other factors such as luck, noise, or measurement error.

The production of player i, and hence his or her *contribution* to the contest, is given by the continuously differentiable production function $g(\theta_i, e_i)$. Importantly, we assume that $\frac{\partial g}{\partial \theta_i} > 0$ for all $e_i > 0$ which means (realistically) that each player's contribution to the contest is increasing with respect to his or her skill, for a given level of effort. Player i wins the contest against the opponent player $k \in \{1,2\}$, $k \neq i$, if and only if the contribution of player i is strictly higher than the contribution of player k, namely, $g(\theta_i, e_i) > g(\theta_k, e_k)$.[5] We denote by $P_i(e_i, e_k)$ player i's probability of winning the contest (as a function of the efforts of both players) and we define the expected payoff as $\pi_i(e_i, e_k) := P_i(e_i, e_k)V - c(e_i)$. We also define $\hat{e} := c^{-1}(V)$ and $E := [0, \hat{e}]$. A player's equilibrium effort will always belong to the set E as the probability of winning is bounded above by unity.

We impose the following assumption:

Assumption 1. *The primitives of the model are such that: (i) $\pi_i(e_i, e_k)$ is continuously differentiable, and, (ii) any interior solution of the system of first-order conditions for the players' problems of maximizing $\pi_i(e_i, e_k)$ characterizes a pure-strategy Nash equilibrium.*

The validity of the first-order approach is typically ensured by imposing assumptions on the primitives of the model that guarantee that the objective functions π_i are quasi-concave and increasing at $e_i = 0$. Previous papers in the contest-theory literature, however, have shown that the first-order approach may be valid even when the objective functions are neither quasi-concave nor increasing at $e_i = 0$ (see, e.g., Figure 1 in Schweinzer & Segev 2012). As we do not want to rule out such cases, we simply assume that the Nash-equilibrium efforts are characterized by the players' first-order conditions to their maximization problems

---

[5]Notice that $g(\theta_i, e_i) = g(\theta_k, e_k)$ happens with probability zero. In the following, whenever we refer to two players i and k, we (implicitly) assume that $i, k \in \{1,2\}$, $i \neq k$.

without restricting the shape of π_i too much. Each of the theoretical results we present will be accompanied by at least one example for which we verify that the first-order conditions indeed characterize an equilibrium, by verifying the appropriate second-order conditions. Finally, we assume that there exist $\bar{e}_i, \breve{e}_i \in int\ E$ such that $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=\bar{e}_i} < 0$ and $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=\breve{e}_i} > 0$. This ensures that the first-order condition to player i's maximization problem can be fulfilled in a symmetric equilibrium.

Below we provide some examples of different contest models, skill distributions and production technologies that can be captured in our framework.

Tullock contest   The well-studied rent-seeking contest of Tullock (1980) represents a special case of our model. This is easily illustrated using the results in Jia (2008), who considers a contest with a multiplicative production technology, in which player i wins if and only if $\theta_i e_i$ is highest among all players.[6] It is shown that if Θ_i is distributed according to the pdf

$$f_i(x) = \gamma_i m x^{-(m+1)} \exp\left(-\gamma_i x^{-m}\right) I_{(x>0)},$$

then player i wins the contest with probability

$$P_i(e_i, e_k) = \frac{\gamma_i e_i^m}{\sum_{j=1}^2 \gamma_j e_j^m},$$

where $\gamma_i \geq 0$ for both players i and $m > 0$.[7] Hence, in our model, if $g(\theta_i, e_i) = \theta_i e_i$, and Θ_i is distributed according to the above pdf, then we obtain the above Tullock contest-success function. The literature contains a range of modifications and generalizations of this form of contest-success function, some of which cannot be micro-founded in a similar way. See the recent discussion in Kirkegaard (2020).[8]

Lazear-Rosen tournament   Assuming the production technology $g(\theta_i, e_i) = \theta_i + e_i$, our model includes the standard Lazear-Rosen tournament model (in the original Lazear & Rosen 1981, it is assumed that $\mu_i = 0$). We provide several new results for this well-known setting.

---

[6]See also Clark & Riis (1996) and Fu & Lu (2012).

[7]This contest-success function is slightly more general than the one presented in Tullock (1980). From the contest-theory literature, it is known that m must be sufficiently small for a pure-strategy equilibrium to exist. This is covered by our Assumption 1.

[8]See Fullerton & McAfee (1999) for another example of a micro-foundation for the Tullock contest.

General production technologies   Our framework is general with respect to the set of admissible production technologies $g(\theta_i, e_i)$. For example, feasible technologies include the CES production function $g(\theta_i, e_i) = \left(\alpha \theta_i^\rho + \beta e_i^\rho \right)^{\frac{1}{\rho}}$, with $\alpha, \beta > 0$ (except for the limiting case of perfect complements). Thus, the case of perfect substitutes, $\rho = 1$, is included as well as technologies where effort and ability are complements to different degrees, such as the standard Cobb-Douglas technology $g(\theta_i, e_i) = \theta_i^\alpha e_i^\beta$, with $\alpha, \beta > 0$ (obtained when ρ approaches zero).

Skill distributions   In our framework, standard continuous skill distributions can be employed with both bounded and unbounded supports. Moreover, the distributions can be different for the two players. Examples are the (truncated) Normal distribution, the Exponential distribution, Student's t-distribution, the Gamma distribution, and the Uniform distribution.

4   Model Solution

We focus on pure-strategy Nash equilibria in which both players choose the same level of effort. The following lemma provides a sufficient condition for such a symmetric equilibrium to exist.

Lemma 1. *A sufficient condition for a symmetric equilibrium to exist is that* $\frac{\partial P_i(e_i, e_k)}{\partial e_i}|_{e_i = e_k = e}$ *is the same for $i, k \in \{1, 2\}, i \neq k$, and all $e \in int\ E$.*

Proof. See Appendix A.1. □

We will make use of Lemma 1 to prove the existence of a symmetric equilibrium by checking the sufficient condition. Since this condition depends on the winning probability, we need to specify this probability first. For each $e > 0$, we define the function $g_e : \mathbb{R} \to \mathbb{R}$ by $g_e(x) = g(x, e)$. The function $g_e(x)$ is strictly increasing in x and thus invertible, and we denote the (strictly increasing) inverse by g_e^{-1}. This notation can be motivated by the fact that the event of player i winning over player k can be written as

$$
\begin{aligned}
& g(\theta_k, e_k) < g(\theta_i, e_i) \\
\Leftrightarrow \quad & g_{e_k}(\theta_k) < g_{e_i}(\theta_i) \\
\Leftrightarrow \quad & \theta_k < g_{e_k}^{-1}\left(g_{e_i}(\theta_i) \right).
\end{aligned}
$$

Considering all potential realizations of Θ_i and Θ_k, the winning probability of

player i is

$$P_i(e_i, e_k) = \int_{\mathbb{R}} F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) f_i(x)\,dx.$$

By symmetry, the winning probability of player k is:

$$P_k(e_i, e_k) = \int_{\mathbb{R}} F_i\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) f_k(x)\,dx.$$

The derivative of player i's winning probability with respect to e_i is given by:[9]

$$\frac{\partial P_i(e_i, e_k)}{\partial e_i} = \int_{\mathbb{R}} f_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) \frac{d}{de_i}\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) f_i(x)\,dx. \qquad (1)$$

The derivative of player k's winning probability with respect to e_k is given by:

$$\frac{\partial P_k(e_i, e_k)}{\partial e_k} = \int_{\mathbb{R}} f_i\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) \frac{d}{de_k}\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) f_k(x)\,dx. \qquad (2)$$

It can immediately be seen that expressions (1) and (2) are equal when $e_i = e_k = e \in int\ E$ since, in this case, $g_{e_k}^{-1}(g_{e_i}(x)) = g_{e_i}^{-1}(g_{e_k}(x)) = x$ and $\frac{d}{de_k} g_{e_i}^{-1}\left(g_{e_k}(x)\right) = \frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(x)\right)$. Thus, the sufficient condition for the existence of a symmetric equilibrium in Lemma 1 is satisfied. Hence, we have the following theorem.

Theorem 1. *There exists a symmetric equilibrium in which both players choose the same level of effort.*

Proof. See Appendix A.2. □

The theorem states that, even if the players are asymmetric (i.e., $f_1 \neq f_2$), there always exists a symmetric equilibrium of the contest game. This result is of great importance since it allows a tractable analysis of contests between asymmetric players in a variety of different settings. We define $a_e : \mathbb{R} \to \mathbb{R}$ by

$$a_e(x) = \frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(x)\right)\bigg|_{e_i = e_k = e} = \frac{\partial g(x, e)}{\partial e} \bigg/ \frac{\partial g(x, e)}{\partial x} = MRTS(x, e), \qquad (3)$$

where the equality follows from an application of the inverse function theorem and $MRTS(x, e)$ denotes the marginal rate of technical substitution between skill and effort in a symmetric equilibrium.[10] Recognizing that the two players have the same cost function $c(e)$, we can write the (identical) first-order condition for

---

[9]Notice that F_k is differentiable almost everywhere, since it is the cdf of the absolutely continuous random variable Θ_k with f_k as the corresponding pdf.

[10]To see this, notice that $\frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(x)\right)\big|_{e_i = e_k = e} = \frac{1}{g_{e_k}'(g_{e_k}^{-1}(g_{e_i}(x)))} \frac{d}{de_i} g_{e_i}(x)\big|_{e_i = e_k = e} = \frac{\frac{d}{de} g_e(x)}{g_e'(x)}$.

effort for the two players in a symmetric equilibrium as

$$V \int_{\mathbb{R}} a_{e^*}(x) f_k(x) f_i(x) dx = c'(e^*). \tag{4}$$

The key observation necessary to understand the intuition behind (4) is that a player has a positive marginal incentive to supply effort if and only if $g(\theta_k, e_k) = g(\theta_i, e_i)$. In a symmetric equilibrium where $e_k = e_i$ this implies that $\theta_k = \theta_i$.[11] Accordingly, equation (4) contains the "collision density" $f_k(x) f_i(x)$ that describes how likely it is that the skill realizations of the two competing players are the same. The fact that this term is the same for both players is due to our assumption of symmetric uncertainty. Furthermore, the fact that $a_e(x)$ is the same for both players follows directly from the assumption that the production function $g(\theta, e)$ is the same for both players, and depends only on the level of effort e and the ability θ, both of which are the same for both players in situations where players have a marginal incentive to supply effort in symmetric equilibrium.

The function $a_e(x)$ describes how a marginal increase in effort by a player increases output relative to his or her rivals and is equal to the marginal rate of technical substitution between skill and effort. The purpose of raising effort is to beat players with higher ability. The MRTS determines the range of additional types that the player can win against through a small effort increase. The lower is the sensitivity of output to skill in the production function, the smaller is the advantage of marginally more skilled rivals, and the higher is the marginal incentive to exert effort.

Additional intuition can be provided by considering specific functional forms. For example, if $g(\theta, e) = e + \theta$ we have that $a_e(x) = 1$ since in this case both the numerator and denominator are equal to unity. If instead, $g(\theta, e) = \theta e$, we have that $a_e(x) = x/e$ because of the complementarity between skill and own effort in the production function. The fact that $a_e(x)$ is an increasing function of x reflects that it is in this case more valuable to increase effort the higher is the skill of the player. The fact that $a_e(x)$ is decreasing in e reflects that the marginally more able individual is harder to beat the higher is the baseline (symmetric) level of effort because of the complementarity between skill and effort.

We end this section with an illustrative example. Consider the multiplicative production technology $g(\theta_i, e_i) = \theta_i e_i$ and the cost function $c(e_i) = e_i^2/2$. Assume further that the skill distribution of player 1 follows a Uniform distribution on

---

[11]The reason a player only has a marginal incentive to exert effort when $\theta_k = \theta_i$ is that this is the only situation in which a marginal increase in output would be pivotal to winning the contest.

[1, 2], and the skill distribution of player 2 is given by the Student's t-distribution on support $(-\infty, \infty)$, with one degree of freedom, such that:

$$f_1(s) = \begin{cases} 1 & 1 \le s \le 2 \\ 0 & \text{otherwise} \end{cases}, \qquad f_2(x) = \frac{1}{\pi(1+x^2)}, \; x \in \mathbb{R}.$$

The event of player 1 winning is described by $g(\theta_1, e_1) > g(\theta_2, e_2) \iff \theta_2 < g_{e_2}^{-1}(g_{e_1}(\theta_1)) = \theta_1 e_1 / e_2$. The probability of that event, and its first derivative with respect to e_1, are

$$P_1(e_1, e_2) = \int_{-\infty}^{\infty} F_2\left(x\frac{e_1}{e_2}\right) f_1(x) dx,$$

$$\frac{\partial P_1(e_1, e_2)}{\partial e_1} = \int_{-\infty}^{\infty} f_2\left(x\frac{e_1}{e_2}\right)\left(\frac{x}{e_2}\right) f_1(x) dx.$$

The first-order condition of player 1's maximization problem is

$$\frac{\partial P_1(e_1, e_2)}{\partial e_1} V = e_1.$$

In a symmetric equilibrium with $e_1 = e_2 = e$, this can now be written as

$$V \int_{-\infty}^{\infty} f_2(x) x f_1(x) dx = e^2.$$

For player 2 we obtain the same expression. Using our distributional assumptions, the left-hand side becomes

$$V \int_{-\infty}^{\infty} f_2(x) x f_1(x) dx = V \int_1^2 \frac{x}{\pi(1+x^2)} dx = V \frac{1}{2\pi} \log\left(\frac{5}{2}\right).$$

We thus have a symmetric equilibrium, and the corresponding effort is $e^* = \sqrt{\frac{V \log(\frac{5}{2})}{2\pi}} \approx 0.38\sqrt{V}$.

5   Comparative Statics Results

In this section, we investigate the consequences of player heterogeneity, in terms of the statistical properties of their skill distributions, on the incentive to exert effort. To facilitate the derivation of these results, we define $r_{e,i} : \mathbb{R} \to \mathbb{R}$ given by $r_{e,i}(x) = a_e(x) f_i(x)$. Equation (4) can thus be written as:

$$V \int_{\mathbb{R}} r_{e^*,i}(x) f_k(x) dx = c'(e^*). \tag{5}$$

13

The integral now has the same structure as a decision maker's expected utility in decision theory (e.g., Levy 1992), where the function $r_{e,i}$ corresponds to the decision maker's utility function. As we will see, this link proves useful in deriving several key results. We also need one additional assumption:

Assumption 2. *The primitives of the model are such that $q : E \to \mathbb{R}$, defined by*

$$q(e) = V \int_{\mathbb{R}} r_{e,i}(x) f_k(x) dx - c'(e),$$

is strictly decreasing.

As c is strictly convex, Assumption 2 is not very strong and is always satisfied if $\int_{\mathbb{R}} r_{e,i}(x) f_k(x) dx$ is non-increasing in e. To give a specific example, consider the CES production function $g(\theta_i, e_i) = \left(\alpha \theta_i^\rho + \beta e_i^\rho \right)^{\frac{1}{\rho}}$, with $\alpha, \beta > 0$ and $\rho \leq 1$. Here $a_e(x) = \frac{\beta}{\alpha} \left(\frac{x}{e} \right)^{1-\rho}$, implying that $\int_{\mathbb{R}} a_e(x) f_1(x) f_2(x) dx = e^{\rho-1} \int_{\mathbb{R}} \frac{\beta}{\alpha} x^{1-\rho} f_1(x) f_2(x) dx$. For this specification, Assumption 2 is satisfied in all cases where the worker has an incentive to exert positive effort (i.e., $\int_{\mathbb{R}} \frac{\beta}{\alpha} x^{1-\rho} f_1(x) f_2(x) dx > 0$). Furthermore, the assumption ensures that effort is always increasing in the prize and that the considered equilibrium is unique in the class of symmetric equilibria (the latter result follows from the assumption ensuring that there is a unique e solving equation (5)).

5.1 First-Order Stochastic Dominance

A standard result in contest theory is that heterogeneity among players with respect to their skills reduces the incentive to exert effort (see, e.g., Schotter & Weigelt 1992, or Observation 1 in the survey by Chowdhury et al. 2019). In our framework, this standard result is potentially reversed, as we will now show.

Consider a contest with two players with skills drawn from two distributions with expected values μ_k and μ_i, respectively. If, from the outset, $\mu_k \geq \mu_i$ and the difference $\mu_k - \mu_i$ is increased, then the two players become more heterogeneous in terms of their expected skill. Based on this idea, we proceed by investigating the consequences of making players more heterogeneous in the sense of first-order stochastic dominance, as captured by the following definition.

Definition 1. *Let μ_k and μ_i refer to the expected values of the skill distributions (F_k, F_i) in an initial contest. Players in a contest with skill distributions $(\tilde{F}_k, F_i)$ are said to be more heterogeneous (with respect to their skills) relative to players in the initial contest with skill distributions (F_k, F_i), in a first-order sense, if either of the following conditions hold:*

14

(i) $\mu_k \geq \mu_i$ *and* $\tilde{F}_k$ *dominates* F_k *in the sense of first-order stochastic dominance.*

(ii) $\mu_k \leq \mu_i$ *and* $\tilde{F}_k$ *is dominated by* F_k *in the sense of first-order stochastic dominance.*

We are now in a position to derive our second main result. Due to Assumption 2, equilibrium effort increases if a change in the primitives of the model leads to an increase in $\int_{\mathbb{R}} r_{e,i}(x) f_k(x) dx$. As indicated before, this expression has the same structure as a decision maker's expected utility in decision theory, where the function $r_{e,i}$ is replaced by the decision maker's utility function. Since the structure of the problems is the same, we can make extensive use of results from decision theory in our analysis. We obtain the following theorem.

Theorem 2. *Consider two contests with skill distributions* $(\tilde{F}_k, F_i)$ *and* (F_k, F_i) *where* $\mathrm{supp}(\tilde{f}_k)$ *and* $\mathrm{supp}(f_k)$ *both are subsets of* $\mathrm{supp}(f_i)$. *Let* $\tilde{e}^*$ *and* e^* *denote, respectively, the (symmetric) equilibrium efforts associated with these contests. Then,* $\tilde{e}^* > e^*$ *if either one of the following statements hold:*

(i) $r_{e,i}(x)$ *is strictly increasing for all* $x \in \mathrm{supp}(f_i)$ *and all* $e \geq 0$, *and* $\tilde{F}_k$ *dominates* F_k *in the sense of first-order stochastic dominance.*

(ii) $r_{e,i}(x)$ *is strictly decreasing for all* $x \in \mathrm{supp}(f_i)$ *and all* $e \geq 0$, *and* $\tilde{F}_k$ *is dominated by* F_k *in the sense of first-order stochastic dominance.*

Proof. See Appendix A.3. □

Note that Theorem 2 holds independently of whether $\mu_k \leq \mu_i$ or $\mu_k \geq \mu_i$. Combining Definition 1 with Theorem 2, we have the following corollary.[12]

Corollary 1. *Effort can be higher when contestants are more heterogeneous in a first-order sense.*

We illustrate the intuition behind Theorem 2 and Corollary 1 through two examples. In each example, we start from a situation of equal expected skills, and then introduce a first-order stochastic dominance shift. In the first example, which has a somewhat simpler intuition than the second, $r_{e,i}(x)$ is strictly decreasing and effort gets higher as player k becomes weaker, illustrating part (ii) of Theorem 2. In the second example, $r_{e,i}(x)$ is strictly increasing and effort gets higher as player k becomes stronger, illustrating part (i) of Theorem 2.

---

[12]There is one small caveat to Corollary 1 that we should mention. If equilibrium effort increases as contestants become more heterogeneous, then a symmetric equilibrium in which both players exert positive effort will fail to exist if the heterogeneity between players becomes too large. The reason is that the weaker player would eventually receive a negative payoff, meaning that this player would prefer to choose zero effort.

Example 1. *Suppose that* $g(\theta, e) = \theta + e$, $\Theta_i \sim Exp\left(\frac{4}{3}\right)$, $\Theta_k \sim U\left[\frac{1}{2}, 1\right]$, $\tilde{\Theta}_k \sim U\left[\frac{7}{16}, \frac{15}{16}\right]$, $c(e) = \frac{e^2}{2}$, $V = 1$. *Then* $e^* = \frac{2\left(\exp\left(\frac{2}{3}\right) - 1\right)}{\exp\left(\frac{4}{3}\right)} \approx 0.499$ *and* $\tilde{e}^* = \frac{2\left(\exp\left(\frac{2}{3}\right) - 1\right)}{\exp\left(\frac{5}{4}\right)} \approx 0.543$.

In Example 1, the first thing to notice is that the additive production technology implies that $a_e(x) = 1$. This further implies that $r_{e,i}(x)$ is strictly decreasing for all relevant x, since $f_i(x)$ is the decreasing pdf of the exponential skill distribution. The fact that $a_e(x) = 1$ also implies that the incentive to supply effort, as given by (4), only depends on the collision density $f_k(x)f_i(x)$. Since $f_i(x)$ is decreasing, and $f_k(x)$ is uniform and shifted to the left, the collision density between $\tilde{f}_k$ and f_i is everywhere larger than the collision density between f_k and f_i, see Figure 1 for an illustration. Thus, both players have a higher incentive to exert effort. The simple intuition for the example is that the marginal incentive to supply effort for both players is positive only in situations where they have equal ability, and the considered shift in distributions makes such situations unambiguously "more likely" to happen.
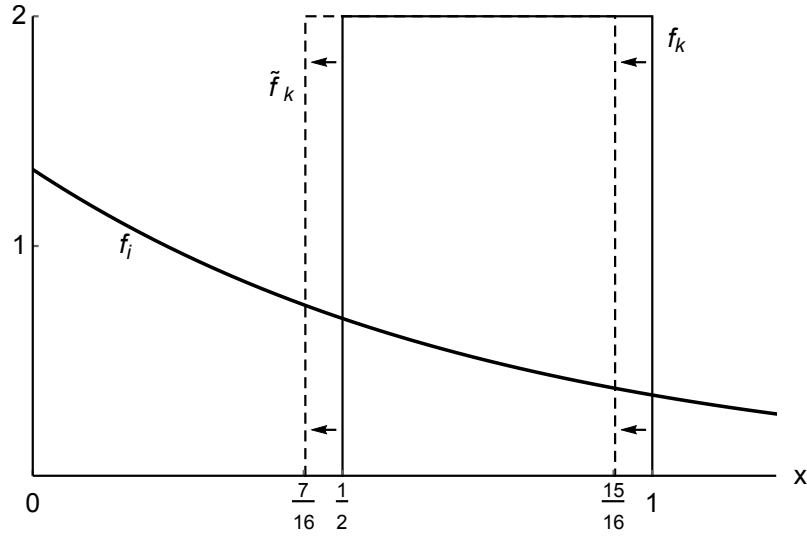

Figure 1: Illustration of Example 1

Example 2. *Suppose that* $g(\theta, e) = \theta \cdot e$, $\Theta_i \sim U[0, 1]$, $\Theta_k \sim U\left[\frac{1}{4}, \frac{3}{4}\right]$, $\tilde{\Theta}_k \sim U\left[\frac{5}{16}, \frac{13}{16}\right]$, $c(e) = \frac{e^2}{2}$, $V = 1$. *Then* $e^* = \frac{1}{\sqrt{2}} \approx 0.707$ *and* $\tilde{e}^* = \frac{3}{4} = 0.75$.

In Example 2, the multiplicative production technology implies that $a_e(x) = x/e$ which is a strictly increasing function of x. This further implies that $r_{e,i}(x)$ is strictly increasing on $[0, 1]$ because f_i is uniform. The shift in the skill distribution of player k from F_k to $\tilde{F}_k$ implies that the expected skill of player k increases. However, the height of the density of player k's skill distribution does not change

$(f_k(x) = 2, x \in \left[\frac{1}{4}, \frac{3}{4}\right]$ and $\tilde{f}_k(x) = 2, x \in \left[\frac{5}{16}, \frac{13}{16}\right])$. Thus, since $f_i(x) = 1$, we have that $f_k(x)f_i(x) = \tilde{f}_k(x)f_i(x) = 2$ at all points where these collision densities are non-zero. However, due to the distributional shift, the subset of $\mathbb{R}$ where the two uniform distributions overlap shifts to the right. Therefore, the two distributions collide at larger values of x (see Figure 2 for an illustration). This would have no effect on the incentive to exert effort if $a_e(x)$ would be constant, as in Example 1. However, in the current example, we have that $a_e(x) = x/e$. Thus, taking into account the three terms in (4), the fact that the two distributions collide at larger values of x increases the incentive to exert effort for both players. Intuitively, given that the only relevant situations (where players have a positive marginal incentive to supply effort) now occur at larger values of skill, the fact that there is a complementarity between skill and effort in the production function implies that the incentive to supply effort is higher for both players.
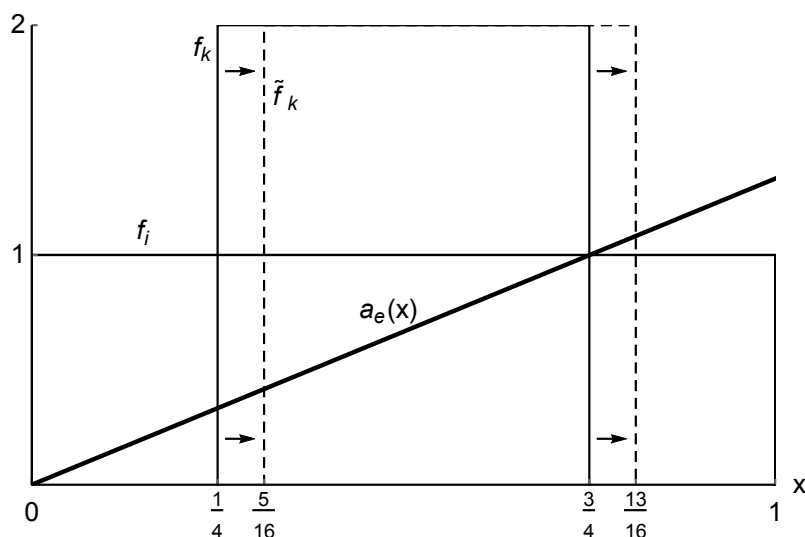

Figure 2: Illustration of Example 2

Concluding this section, we note that the conditions in Theorem 2 are sufficient, but not necessary for the result that effort can be higher when contestants are more heterogeneous. To illustrate this, we present an additional result based on normal distributions where we first determine the marginal winning probability in a situation with symmetric effort.

Proposition 1. *Suppose that* $\Theta_i \sim N(\mu_i, \sigma_i^2)$, $\Theta_k \sim N(\mu_k, \sigma_k^2)$, *and* $g(\theta, e) = \theta \cdot e$.

Then the marginal winning probability when $e_1 = e_2 = e$ is

$$\frac{\partial P_i(e_i, e_k)}{\partial e_i}\Big|_{e_i = e_k = e} = \frac{(\mu_i \sigma_k^2 + \mu_k \sigma_i^2) \exp\left(-\frac{(\mu_i - \mu_k)^2}{2(\sigma_i^2 + \sigma_k^2)}\right)}{e(2\pi)^{\frac{1}{2}}(\sigma_i^2 + \sigma_k^2)^{\frac{3}{2}}}.$$

Proof. See Appendix B.1. □

In the upcoming example, it can be verified that $r_{e,i}(x) = a_e(x) f_i(x)$ is neither always increasing nor always decreasing, by virtue of the multiplicative production technology combined with the bell-shaped normal distribution. Nonetheless, equilibrium effort increases as players become more heterogeneous in the sense of increasing the distance $|\mu_i - \mu_k|$.

Example 3. *Consider Proposition 1 and assume that $(\sigma_i, \sigma_k) = (1, 1)$, $(\mu_i, \mu_k) = \left(\frac{1}{2}, \frac{1}{2}\right)$, $V = 1$, and $c(e) = \frac{e^2}{2}$. Then equilibrium effort is $e^* = \left(2\pi^{\frac{1}{4}}\right)^{-1} \approx 0.38$. If we increase μ_i from $\frac{1}{2}$ to $\frac{3}{2}$, keeping μ_k constant, equilibrium effort increases to $\tilde{e}^* = \left(\sqrt{2} \exp\left(\frac{1}{8}\right) \pi^{\frac{1}{4}}\right)^{-1} \approx 0.47$.*

5.2 Second-Order Stochastic Dominance

The studies by Hvide (2002), Kräkel & Sliwka (2004), Kräkel (2008), Gilpatric (2009), and DeVaro & Kauhanen (2016) investigate how "risk" or "uncertainty" affects players' incentive to exert effort in contests. One result that is common to all of these analyses is that in contests between equally able players, higher risk (as measured by a higher variance of the random variables capturing the uncertainty of the contest outcome) leads to lower efforts. We revisit this result in the context of our model and show that effort may increase as the ability distribution of one of the players becomes more uncertain in the sense of second-order stochastic dominance. In the following definition, we formalize what we mean when we say that one skill distribution is more *uncertain* than another one (see Rothschild & Stiglitz 1970 for details).

Definition 2. *The ability distribution $\tilde{F}_i$ is said to be more uncertain than the distribution F_i if $\tilde{F}_i$ is a mean-preserving spread of F_i. This is equivalent to $\tilde{F}_i$ being dominated by F_i in the sense of second-order stochastic dominance.*

Equipped with this definition, we can use well-known results from decision theory to obtain our next theorem:

Theorem 3. *Consider two contests with skill distributions $(\tilde{F}_k, F_i)$ and (F_k, F_i) where $\mathrm{supp}(\tilde{f}_k)$ and $\mathrm{supp}(f_k)$ both are subsets of $\mathrm{supp}(f_i)$. Let $\tilde{e}^*$ and e^* denote, respectively, the (symmetric) equilibrium efforts associated with these contests. Suppose that $\tilde{F}_k$ is more uncertain than F_k. Then, the following results hold:*

(i) If $r_{e,i}(x)$ is strictly convex on $\mathrm{supp}(f_i)$ for all $e \geq 0$, then $\tilde{e}^* > e^*$.

(ii) If $r_{e,i}(x)$ is linear on $\mathrm{supp}(f_i)$ for all $e \geq 0$, then $\tilde{e}^* = e^*$.

(iii) If $r_{e,i}(x)$ is strictly concave on $x \in \mathrm{supp}(f_i)$ for all $e \geq 0$, then $\tilde{e}^* < e^*$.

Proof. See Appendix A.4. □

The key insight needed to understand Theorem 3 is that applying a mean-preserving spread to the distribution F_k shifts probability mass from the center to the tails of the distribution, and the impact of this change on the incentive to exert effort depends on the curvature of $r_{e,i}(x)$. Notice that Theorem 3 also holds if players have the same expected ability, namely $\mu_i = \mu_k$. This means that, in a contest with two players who are expected to be equally able, higher uncertainty regarding players' abilities may increase the incentive to exert effort, in contrast to what the studies referred to at the beginning of this subsection have shown.

Next, we illustrate and provide intuition for Theorem 3 by presenting an example set in the context of the Lazear-Rosen framework with an additive production technology. The example demonstrates that increasing the uncertainty of the contest while keeping the expected ability of both players unchanged, can increase equilibrium effort.

Example 4. *Consider a contest with the additive production function $g(\theta, e) = \theta + e$, the parameter $V = 1$, and the cost function $c(e) = \frac{e^2}{2}$. Suppose $\Theta_i \sim Exp(1)$ and $\Theta_k \sim U\left[\frac{1}{2}, \frac{3}{2}\right]$ (implying $\mu_i = \mu_k = 1$). Equilibrium effort is then $e^* = \frac{\exp(1)-1}{\exp\left(\frac{3}{2}\right)} \approx 0.38$. Now, consider a mean-preserving spread of the skill distribution of player k, enlarging the support of the uniform distribution, such that $\tilde{\Theta}_k \sim U[0,2]$. Then effort increases to $\tilde{e}^* = \frac{\exp(2)-1}{2\exp(2)} \approx 0.43$.*

In Example 4, we have imposed the additive production technology which implies $a_e(x) = 1$. Thus, the convexity of $r_{e,i}(x)$ referred to in part (i) of Theorem 3 is determined by the convexity of $f_i(x)$. To understand how the shift from f_k to $\tilde{f}_k$ affects the incentive to exert effort, we need to study how the integral in (4) is affected. Similar to Example 1, given that $a_e(x) = 1$, it is sufficient to compare $\int f_i(x) f_k(x) dx$ with $\int f_i(x) \tilde{f}_k(x) dx$. The shift from f_k to $\tilde{f}_k$ entails an enlargement

of the support of the uniform distribution. This implies that the density decreases for intermediate values of x, but increases for low and high values of x (see Figure 3 for an illustration). Given that $f_i(x)$ is strictly decreasing, the part of the skill distribution of player k that is stretched out to the left will collide with relatively large values of f_i, whereas the the part of the skill distribution of player k that is stretched out to the right will collide with relatively small values of f_i, creating a trade-off. The fact that f_i is not only strictly decreasing, but also convex, resolves this trade-off, implying that the overall effect of the shift is to increase the value of the integral expression. Thus, both players have a higher incentive to exert effort as a result of the move from f_k to $\tilde{f}_k$. Intuitively, due to the change in the distribution of player k, situations where the competing players have the same ability become "more likely", implying an increase in equilibrium effort.
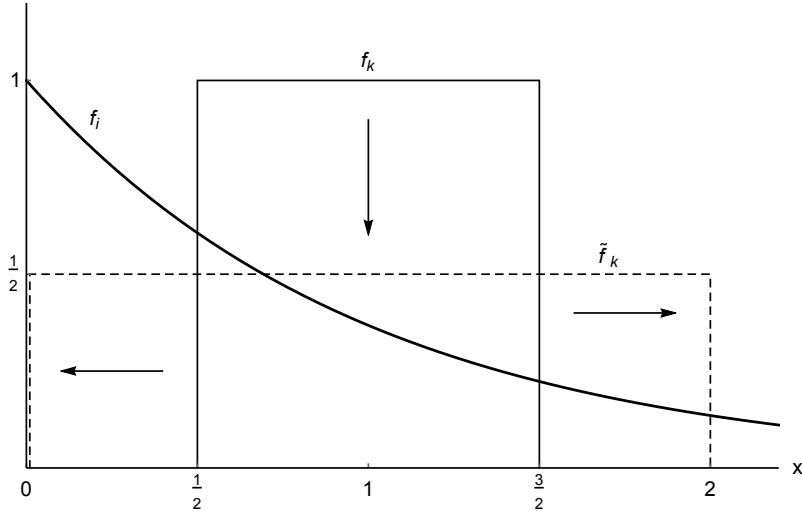

Figure 3: Illustration of Example 4

We conclude this section by defining contestant heterogeneity in a *second-order* sense and we follow the structure of the corresponding definition of heterogeneity in a first-order sense (Definition 1). In Definition 1, we used the ranking of players' mean abilities to characterize the initial situation. In the new definition, we do so through the variances of the skill distributions of the competing players (restricting attention to statistical distributions with finite variance). Notice, however, that variance is not always a good measure of uncertainty or risk (see, e.g, Rothschild & Stiglitz 1970). Therefore one should keep in mind, when applying the definition below, that higher variance entails higher uncertainty only for certain skill distributions (e.g., the normal distribution).

Definition 3. *Let Var_k and Var_i refer to the variances of the skill distributions (F_k, F_i) in an initial contest. Players in a contest with skill distributions $(\tilde{F}_k, F_i)$, are said to be more heterogeneous (with respect to their skills) relative to players in the initial contest with skill distributions (F_k, F_i), in a second-order sense, if either of the following conditions hold:*

(i) *$Var_k \geq Var_i$ and F_k dominates $\tilde{F}_k$ in the sense of second-order stochastic dominance.*

(ii) *$Var_k \leq Var_i$ and F_k is dominated by $\tilde{F}_k$ in the sense of second-order stochastic dominance.*

Combining Theorem 3 with Definition 3, we have the following corollary.

Corollary 2. *Effort can be higher when contestants are more heterogeneous in a second-order sense.*

5.3 Implications for Optimal Team Composition

The results in the preceding two subsections have implications for optimal team composition and organizational design.[13] In particular, our results suggest that employers could find it desirable to employ a more heterogeneous workforce as an instrument to induce higher effort. In Section 5.1, we analyzed the effects of increasing the heterogeneity in players' expected skills, and showed how this can increase equilibrium effort. This means that a firm could benefit (from the perspective of inducing higher effort) by hiring some workers with a high expected ability and some with a low expected ability, based on, for example, signals such as the quality of the institution where a college graduate received his or her degree. In Section 5.2, we showed how increased uncertainty regarding abilities of some players can increase equilibrium effort. Thus, a firm could benefit from hiring a mix of experienced workers (for whom the uncertainty regarding abilities is relatively small) and inexperienced workers (for whom the uncertainty regarding abilities is relatively large).

To see this more formally, suppose a firm already employs a worker with ability distribution F_1 and considers to hire another worker with ability distribution F_2. Moreover, assume that $r_{e,1}(x)$ is strictly decreasing and strictly convex (for example, by assuming that the production function is given by $g(\theta, e) = \theta + e$ and

---

[13]See, e.g., Gershkov et al. (2009, 2016) and Fu et al. (2015).

skills are Exponentially distributed with parameter λ).[14] Then the firm may gain from hiring another worker with a lower expected ability ($\mu_2 < \mu_1$), but where F_2 is more uncertain (meaning that worker 2's skill is drawn from a more uncertain distribution). This finding can be understood from the perspective of Theorem 2, that tells us that effort will be higher due to the lower expected ability of worker 2, combined with Theorem 3, which tells us that effort will be higher due to the larger uncertainty regarding the skill of worker 2. In other words, hiring a worker with a lower expected ability, drawn from a more uncertain distribution, can induce higher effort. Theorem 2 and Theorem 3 also have other managerial implications as they indicate that employers may want to hire workers who have worked on different tasks in the past (or on similar tasks in a different firm or industry), to create uncertainty about workers' abilities. In a similar vein, it might be desirable to implement some kind of job rotation.

6 The Case of More Than Two Players ($n > 2$)

We now turn to the case of $n > 2$ contestants which allows us to address the interesting question of how effort depends on the number of players competing in a contest.[15] In Section 6.1, we show that our solution method generally cannot be extended to the case of $n > 2$ heterogeneous players. In Section 6.2, we consider the case of n homogeneous players. Section 6.3 examines a special case of our model where $n - 1$ homogeneous players compete against a player who is more highly skilled (e.g., as in Brown 2011 and Krumer et al. 2017), which serves to demonstrate that a symmetric equilibrium can exist when players are heterogeneous and the number of players is greater than two. In all these sections, we maintain the generality of the production technology.

6.1 The $n = 2$ Result Does Not Extend to $n > 2$

In the case of $n > 2$ players with different skill distributions, the equilibrium in our model is generally no longer symmetric. A player i will only win the contest

---

[14]An alternative skill distribution that would also be decreasing and convex would be a normal distribution that is truncated to the left at a point to the right of the second inflection point. Such a distribution could be motivated by the observation that abilities are often normally distributed and that, when employing worker 1, the firm tried to hire the most able applicant, meaning that abilities in the higher end of the distribution are most relevant (see, e.g., Aguinis & O'Boyle Jr. 2014).

[15]Contests with more than two players have been studied by, e.g., Tullock (1980), Nalebuff & Stiglitz (1983), Hillman & Riley (1989), Zábojník & Bernhardt (2001), Chen (2003), and Zábojník (2012).

if he or she beats all of his or her opponents. Essentially, each player is thus competing against the best of the other players, that is, the largest order statistic, and therefore faces a different "relevant rival" in the contest. This introduces an asymmetry into the model that was absent in the two-player case, and which generally leads to an asymmetric equilibrium. To see this formally, suppose, for simplicity, that $g(\theta_i, e_i) = \theta_i + e_i$, implying that $a_e(x) = 1$ (the following intuition also holds for general production technologies). Then, using a similar reasoning as in the two-player case (see Section 4), the marginal probability of winning for player i and player k in a symmetric equilibrium can be written, respectively, as:

$$\frac{\partial P_i(e_1, e_2, \ldots, e_n)}{\partial e_i}\Big|_{e_1 = \ldots = e_n = e} = \int_{\mathbb{R}} f_i(x) \frac{d}{dx}\left(F_k(x) \prod_{j \neq i, k} F_j(x)\right) dx$$

and

$$\frac{\partial P_k(e_1, e_2, \ldots, e_2)}{\partial e_k}\Big|_{e_1 = \ldots = e_n = e} = \int_{\mathbb{R}} f_k(x) \frac{d}{dx}\left(F_i(x) \prod_{j \neq i, k} F_j(x)\right) dx.$$

Applying the product differentiation rule on the RHS of the above expressions, we obtain:

$$\int_{\mathbb{R}}\left(f_i(x) f_k(x) \prod_{j \neq i, k} F_j(x) + f_i(x) F_k(x) \frac{d}{dx}\left(\prod_{j \neq i, k} F_j(x)\right)\right) dx \tag{6}$$

and

$$\int_{\mathbb{R}}\left(f_k(x) f_i(x) \prod_{j \neq i, k} F_j(x) + f_k(x) F_i(x) \frac{d}{dx}\left(\prod_{j \neq i, k} F_j(x)\right)\right) dx. \tag{7}$$

The first term in (6) and (7) corresponds to the situation in which all players $j \in \{1, \ldots, n\}, j \neq i, k$ perform worse than players i and k so that the n-player contest collapses to a contest between players i and k. For this subcontest, the marginal winning probabilities are the same as shown in the analysis of the two-player contest. The second term in (6) corresponds to the situation in which player i outperforms his or her rival k, such that the contest boils down to a contest between player i and the strongest of the players $j \in \{1, \ldots, n\}, j \neq i, k$. The interpretation of the second term in (7) is analogous, with the role of i and k interchanged.

Setting expression (6) equal to expression (7), we obtain

$$\int_{\mathbb{R}} f_i(x) F_k(x) \frac{d}{dx} \left(\prod_{j \neq i,k} F_j(x) \right) dx = \int_{\mathbb{R}} f_k(x) F_i(x) \frac{d}{dx} \left(\prod_{j \neq i,k} F_j(x) \right) dx$$

$$\Leftrightarrow \int_{\mathbb{R}} \left(\frac{f_i(x)}{F_i(x)} - \frac{f_k(x)}{F_k(x)} \right) F_i(x) F_k(x) \frac{d}{dx} \left(\prod_{j \neq i,k} F_j(x) \right) dx = 0.$$

Notice that i and k were arbitrarily selected. Hence, in order for a symmetric equilibrium to exist, it must be the case that the above condition holds for all $i, k \in \{1, \ldots, n\}, i \neq k$. We conclude that the condition above is generally violated when the skill distributions of the competing players are distinct, which implies that our solution method cannot be extended to the case of $n > 2$ players.[16]

6.2 The Case of Homogeneous Players

Suppose that all players share the same skill distribution, i.e., $f_1 = f_2 = \ldots = f_n =: f$, and define $r_e(x) := a_e(x) f(x)$.

Proposition 2. *In an n-player contest with homogeneous skill distributions, a symmetric Nash equilibrium with $e_1 = e_2 = \cdots = e_n = e^*$ exists and is characterized by*

$$V \int_{\mathbb{R}} r_{e^*}(x)(n-1)(F(x))^{n-2} f(x) dx = V \int_{\mathbb{R}} r_{e^*}(x) \frac{d}{dx} \left((F(x))^{n-1} \right) dx = c'(e^*). \quad (8)$$

Proof. See Appendix B.2. $\square$

Notice that $(F(x))^{n-1}$ describes the cdf of the largest order statistic out of a group of $n-1$ players. The condition from the proposition therefore illustrates what we claimed before: the n-player contest boils down to a two-player contest, in which every player competes against the strongest of the other players.

A particular focus in the literature has been on the relation between effort and the number of competitors. Early studies of the n-player Tullock contest with $\gamma_1 = \ldots = \gamma_n$, $m = 1$, and linear effort costs found that equilibrium effort is given by $e^* = \frac{n-1}{n^2} V$, so that effort is decreasing in n (e.g., Tullock 1980, Hillman & Riley 1989). With a convex cost function (as in our setting), the condition would change to $e^* c'(e^*) = \frac{n-1}{n^2} V$, but effort would still be decreasing in n. The result can be

---

[16]A sufficient condition for a symmetric equilibrium to exist is that $\frac{f_i(x)}{F_i(x)} = \frac{f_k(x)}{F_k(x)}$ for all $x \in \mathbb{R}$ and all $i, k \in \{1, \ldots, n\}, i \neq k$. However, since $\frac{f_i(x)}{F_i(x)} = \frac{d \log F_i(x)}{dx}$ is the reversed hazard rate, which completely characterizes a statistical distribution, this condition would only hold for all $x \in \mathbb{R}$ if F_i and F_k refer to identical distributions.

explained by a discouragement effect. If a player competes against many rivals, his or her chance of winning is relatively low and the player reduces effort in turn.

In what follows, we study the relationship between effort and the number of competitors in our framework. To do so, we need to extend Assumption 2 to the n-player case.

Assumption 3. *The primitives of the model are such that* $q_n : E \to \mathbb{R}$, *defined by*

$$q_n(e) = V \int_{\mathbb{R}} r_e(x) \frac{d}{dx} \left((F(x))^{n-1} \right) dx - c'(e),$$

is strictly decreasing.

We observe that, in addition to the discouragement effect mentioned before, there is also an encouragement effect, inducing players to increase their effort as they compete against more players. This is reflected by the factor $(n-1)$ in $\int_{\mathbb{R}} r_e(x)(n-1)(F(x))^{n-2} f(x) dx$ in Proposition 2 above. As we will show, the encouragement effect might dominate, opening up for the possibility that effort increases in the number of competitors. In our proof, we make use of the fact that increasing n leads to a distribution of the largest order statistic that first-order stochastically dominates the original distribution. We can thus invoke Theorem 2 to study the effects of an increase in n on equilibrium effort.[17]

Theorem 4. *Consider the n-player contest with homogeneous skill distributions and let e^* denote the symmetric Nash equilibrium effort. Then the following statements hold:*

i) *If $r_e(x)$ is strictly increasing for all $x \in \text{supp}(f)$ and all $e \geq 0$, then e^* increases in n.*

ii) *If $r_e(x)$ is strictly decreasing for all $x \in \text{supp}(f)$ and all $e \geq 0$, then e^* decreases in n.*

iii) *If $r_e(x)$ is constant for all $x \in \text{supp}(f)$ and all $e \geq 0$, then e^* does not depend on n.*

Proof. See Appendix A.5. □

---

[17]Notice that similar to what was mentioned in connection to Corollary 1, there is a small caveat to part (i) of Theorem 4. If e^* is increasing in n, a symmetric equilibrium in which all players exert positive effort will fail to exist if n becomes so large that $V/n < c(e^*)$, as (some) players would prefer to choose an effort of zero.

We conclude this subsection with an example to illustrate the potentially positive relationship between effort and the number of players in the context of the well-known Lazear-Rosen model.

Example 5. *Consider a contest with an additive production function $g(\theta, e) = \theta + e$, $V = 1$, and cost function $c(e) = \frac{e^2}{2}$. Suppose each Θ_i is distributed according to the modified reflected exponential distribution with mean $\mu = 1$ and pdf $f(x) = \frac{1}{2} \exp\left(\frac{1}{2}(x - 3)\right)$ for $x \leq 3$ and zero otherwise (see, e.g., Rinne 2014). With two players, equilibrium effort is $e^* = \frac{1}{4}$. With three players, equilibrium effort increases to $\tilde{e}^* = \frac{1}{3}$.*

6.3 A Contest With One Player Who Is More Highly Skilled

We now turn to a special case of our contest model with $n > 2$ players where we obtain a symmetric equilibrium even when players are asymmetric in the sense of having different expected skills. For this purpose, suppose that $\Theta_i = t_i + \mathcal{E}_i$ for $i = 1, \ldots, n$ where $t_1 > t_2 = \cdots = t_n = t$ and the $\mathcal{E}_i$, $i = 1, \ldots, n$, are i.i.d. according to the *reflected exponential distribution* with cdf $H(x) = e^{\lambda x}$ defined on $(-\infty, 0]$ with $\lambda > 0$ (Rinne 2014). In this case, we have:

$$f_i(x) = \begin{cases} \lambda \exp(\lambda(x - t_i)), & \text{for } x \leq t_i \\ 0, & \text{for } x > t_i \end{cases}$$

and

$$F_i(x) = \begin{cases} \exp(\lambda(x - t_i)), & \text{for } x \leq t_i \\ 1, & \text{for } x > t_i, \end{cases}$$

implying that $\frac{f_i(x)}{F_i(x)} = \lambda$ on the support of f_i which is $(-\infty, t_i]$. Consider the condition

$$\int_{\mathbb{R}} \left(\frac{f_i(x)}{F_i(x)} - \frac{f_k(x)}{F_k(x)} \right) F_i(x) F_k(x) \frac{d}{dx} \left(\prod_{j \neq i,k} F_j(x) \right) dx = 0,$$

that we derived in Subsection 6.1.[18] It is satisfied for all $i, k \in \{2, \ldots, n\}$ since in this case, $\frac{f_i(x)}{F_i(x)} = \frac{f_k(x)}{F_k(x)} = \lambda$ on the common support $(\infty, t]$ of f_i and f_k (since we assumed from the outset that $t_2 = t_3 = \cdots = t_n = t$). Consider now the case where $i = 1$ and $k \in \{2, \ldots, n\}$. In this case, we have that $\frac{f_1(x)}{F_1(x)} = \frac{f_k(x)}{F_k(x)} = \lambda$ for $x \leq t$. For $x > t$, we have $\prod_{j \neq 1,k} F_j(x) = 1 \Rightarrow \frac{d}{dx} \left(\prod_{j \neq 1,k} F_j(x) \right) = 0$. Hence, we conclude that the condition is satisfied for all $i, k \in \{1, \ldots, n\}, i \neq k$ and all $x \in \mathbb{R}$, and that the marginal winning

---

[18]This condition was derived under the assumption of an additive production technology. In Appendix C.1, we provide a proof for the existence of a symmetric equilibrium in the general case.

probabilities are the same. This implies that a symmetric equilibrium exists in which all players choose the same equilibrium effort e^*.

Next, we compute an example with a multiplicative production technology and show that the marginal winning probability is increasing in the number of players.

Proposition 3. *Consider the contest described above. Suppose the production technology takes the form $g(\theta, e) = \theta e$ and assume that t is chosen sufficiently large so that $n\lambda t - 1 > 0$. Then, the marginal winning probability given equal effort e is equal to:*

$$\Psi(n) = \frac{(n-1)}{e} \exp(-\lambda(t_1 - t)) \frac{(n\lambda t - 1)}{n^2}, \quad with \quad \Psi'(n) > 0.$$

Proof. See Appendix B.3. □

7 Privately Known Skills

We now turn to examine how our analysis is affected by assuming that players have private information regarding their own skill. For this purpose, we assume that each player $i \in \{1, \ldots, n\}$ observes his or her own skill realization θ_i before choosing effort e_i. This means that each player chooses a strategy consisting of a function $e_i(\theta_i)$ that specifies the effort level for each value of θ_i. Everything else in our model remains unchanged. In particular, all the opponents' skills Θ_k, $k \in \{1, \ldots, n\}, k \neq i$ remain uncertain, as in the main model, and their distributions are common knowledge.

This private-information assumption effectively implies that player i can, in a deterministic manner, choose output $g(\theta_i, e_i)$ by making the appropriate effort choice e_i. The decision problem of player i can therefore, equivalently, be expressed as the specification of optimal effort $e_i(\theta_i)$ or the choice of optimal output $z_i(\theta_i) := g(\theta_i, e_i(\theta_i))$, as a best response to the opponents' choice of effort or output. Assuming that optimal output is strictly increasing in skill (this will be confirmed in our examples), z_i is invertible with inverse z_i^{-1}.

In the two-player case, where player i competes against another player k, player i wins the contest for given realizations of Θ_i and Θ_k if and only if the following condition holds

$$g(\theta_k, e_k) < g(\theta_i, e_i) \iff z_k(\theta_k) < z_i(\theta_i) \iff \theta_k < z_k^{-1}(z_i(\theta_i)).$$

27

Taking into account that, from the perspective of player i, the uncertainty of the contest only concerns the skill realization of player k, we have that equilibrium efforts $e_i(\theta_i)$ and $e_k(\theta_k)$ satisfy:

$$e_i(\theta_i) \in \underset{e_i}{\arg\max} \left\{ F_k(z_k^{-1}(z_i(\theta_i)))V - c(e_i) \right\},$$

$$e_k(\theta_k) \in \underset{e_k}{\arg\max} \left\{ F_i(z_i^{-1}(z_k(\theta_k)))V - c(e_k) \right\}.$$

It can thus immediately be seen that the first-order condition for player i only involves the skill distribution of the opposing player k, whereas the first-order condition for player k only involves the skill distribution of the opposing player i. Hence, the symmetry that was present in the main model, where the first-order condition for each player involved the product of f_i and f_k (see equation (4)), vanishes when skills are privately known. We thus conclude that the equilibrium effort functions $e_i(\theta_i)$ and $e_k(\theta_k)$ generally are not symmetric.

The n-player case with privately known skills is handled in an almost identical fashion. Instead of competing against player k, player i can be viewed as competing against the strongest of the opponents $j \in \{1, \dots, n\}$, $j \neq i$, in the sense of the highest order statistic. We analyze the n-player case with symmetric skill distributions below.

The case of n homogeneous players. We revisit the setting with n homogeneous players considered in Section 6.2 and introduce the private-information assumption. The exposition also serves to illustrate the two-player case with symmetric players and private information.

In a symmetric setting, we naturally expect symmetric equilibria in which players employ the same effort function, $e(\theta_i)$. Thus, equal types imply equal effort and output even when individuals are privately informed about their own type. As we did in Section 6.2, we analyze the n-player case by analyzing how player i competes against the highest order statistic of his or her opponents. We denote the distribution function of this order statistic by $F^{(n-1)}$ with the associated probability density function $f^{(n-1)}$. Given that players are assumed to have independent skill distributions, $F^{(n-1)}(y) = F(y)^{n-1}$ and $f^{(n-1)}(y) = (n-1)F(y)^{n-2}f(y)$.

To solve for a symmetric equilibrium, we consider the problem of player i maximizing his or her expected payoff when all his or her rivals adopt the common effort function $e(\theta_k)$, or equivalently, the common output function $z(\theta_k) :=$

$g(\theta_k, e(\theta_k)), i \neq k$ with corresponding inverse z^{-1}. The equilibrium effort of player i is thus given by:

$$e_i(\theta_i) \in \underset{e_i}{\arg\max} \left\{ F^{(n-1)}(z^{-1}(g(\theta_i, e_i)))V - c(e_i) \right\}.$$

For each value of θ_i, there is an associated first-order condition:

$$f^{(n-1)}(z^{-1}(z_i(\theta_i))) \frac{1}{z'(z^{-1}(z_i(\theta_i)))} \frac{\partial g(\theta_i, e_i(\theta_i))}{\partial e_i} V = c'(e_i(\theta_i)).$$

where $\frac{\partial g(\theta_i, e_i(\theta_i))}{\partial e_i}$ is the partial derivative of $g(\theta_i, e_i(\theta_i))$ with respect to the second argument. In a symmetric equilibrium, we can drop the subindex i, thus the first-order condition in equilibrium can be written as

$$f^{(n-1)}(\theta) \frac{\partial g / \partial e}{z'(\theta)} V = c'(e(\theta)).$$

The above condition implicitly defines the symmetric equilibrium effort function $e(\theta)$. Note that since

$$z'(\theta) = \frac{dg(\theta, e(\theta))}{d\theta} = \frac{\partial g}{\partial \theta} + \frac{\partial g}{\partial e} \frac{de(\theta)}{d\theta},$$

we have that the first-order condition can be written as:

$$f^{(n-1)}(\theta) \frac{\partial g / \partial e}{\frac{\partial g}{\partial \theta} + \frac{\partial g}{\partial e} e'(\theta)} V = c'(e(\theta)). \tag{9}$$

Condition (9) has an intuitive interpretation. The LHS is the marginal probability of winning times the prize V in a symmetric equilibrium from the perspective of a player who knows that his or her skill is θ. Given that a player only has a marginal incentive to exert effort when the strongest opponent (the largest order statistic) has the same skill, $f^{(n-1)}(\theta)$ is the "likelihood" of this situation. There are two main differences with respect to the corresponding condition for the case of symmetric uncertainty (equation (4)). First, because players know their own skill level, there is no need to integrate over all possible realizations of a considered player's own skill. Second, instead of $a_e(x) = \frac{\partial g(x,e)}{\partial e} \Big/ \frac{\partial g(x,e)}{\partial x}$ (which appeared inside the integral of (4)), we now have the factor $\frac{\partial g / \partial e}{\frac{\partial g}{\partial \theta} + \frac{\partial g}{\partial e} e'(\theta)}$ which includes the new term $\frac{\partial g}{\partial e} e'(\theta)$ in the denominator. This new term arises because effort is a function of skill in the private information case.

Recall that when we discussed the intuition behind $a_e(x)$ in equation (4), we

explained that the purpose of a marginal effort increase is to beat rivals who have marginally higher ability. In the current setting, the output advantage of marginally more able rivals is not only determined by $\frac{\partial g}{\partial \theta}$ (which is positive) but also by the additional term $\frac{\partial g}{\partial e} e'(\theta)$ which generally has an ambiguous sign. If $\frac{\partial g}{\partial e}$ and $e'(\theta)$ are both strictly positive, more highly skilled rivals are harder to beat not only because of their skill advantage, but also because they exert higher effort, reducing the marginal incentive to exert effort by any player.

In the following example, we compute the equilibrium effort for a specific skill distribution and production function.[19]

Example 6. *Consider a contest with n symmetric players with privately known skills independently drawn from the uniform distribution on* $[0,1]$*. The production function is given by* $g(\theta, e) = \theta e$*, and the cost function is* $c(e) = \frac{e^2}{2}$*. Then, the symmetric equilibrium effort is:*

$$e(\theta) = \sqrt{\frac{2(n-1)}{n+1} V \theta^{n-1}}.$$

Notice that for the contest in the above example, $r_e(x)$ is strictly increasing on $[0,1]$ for all $e \geq 0$. Hence, equilibrium effort in the symmetric uncertainty case would be *increasing* in n according to Theorem 4. To obtain an analogue of this result in the case of private information, we can compute the expectation of the equilibrium effort in Example 6 to obtain:

$$E[e(\theta)] = \sqrt{(n-1)\left(\frac{2}{n+1}\right)^3 V}. \tag{10}$$

We immediately see that the expected effort in (10) is *decreasing* in n. Hence, Example 6 serves to demonstrate that the comparative statics results from the baseline case with symmetric uncertainty do not necessarily carry over to the private information case.

8 Concluding Remarks

We have presented a novel framework to study contests between heterogeneous players. Under general assumptions about the production technology and the skill distributions of the competing players, we have shown that the contest has a symmetric equilibrium in which all players exert the same effort. We have

---

[19]The detailed derivations for this example are provided in Appendix C.2.

constructed a link between our contest model and expected utility theory and exploited this link to revisit important comparative statics results of contest theory. We have shown that standard results in the literature are not robust to generalizations of the production technology or skill distributions. In particular, we have found that making skill distributions more heterogeneous (in terms of first and second moments) can increase equilibrium effort. Thus, employers could find it desirable to increase the heterogeneity of the workforce in terms of the statistical properties of the skill distributions of the competing players. We also found that increasing the number of contestants can lead to higher equilibrium effort.

We have also investigated the robustness of our results with respect to the assumption of symmetric uncertainty by analyzing the consequences of letting players be privately informed about their skills. In this case, equilibria are in general not symmetric, but focusing on symmetric players, we are able to draw interesting parallels with respect to our baseline case, highlighting the role of our general production technology in influencing the marginal incentive to exert effort.

A possible next step would be to use our framework to study additional aspects of tournament design. For instance, prior work has investigated strategic information revelation by the tournament designer (e.g., Aoyagi 2010). If the tournament designer possesses some private information about the players' abilities, he or she may decide to reveal some or all of this information to trigger higher effort by the players. Another example would be to allow for different prize structures in the n-player case and investigate how effort depends on the prize structure. For example, one alternative prize structure would be to award the prize V to the $n-1$ best-performing players, and a prize of zero to the worst-performing player. This would change Theorem 4 since every player now would compete against the lowest order statistic associated with the opponent players. The lowest order statistic becomes weaker as the number of players increases, implying that the relationship between effort and the number of players would change. In sum, we believe that our new contest framework opens up many interesting avenues for future research.

References

Aguinis, H., & O'Boyle Jr., E. (2014). Star performers in twenty-first century organizations. *Personnel Psychology*, *67*, 313–350. doi:10.1111/peps.12054.

Aoyagi, M. (2010). Information feedback in a dynamic tournament. *Games and Economic Behavior*, *70*, 242–260. doi:10.1016/j.geb.2010.01.013.

Auriol, E., Friebel, G., & Pechlivanos, L. (2002). Career concerns in teams. *Journal of Labor Economics*, *20*, 289–307. doi:10.1086/338240.

Barbieri, S., Malueg, D. A., & Topolyan, I. (2014). The best-shot all-pay (group) auction with complete information. *Economic Theory*, *57*, 603–640. doi:10.1007/s00199-014-0847-6.

Barut, Y., & Kovenock, D. (1998). The symmetric multiple prize all-pay auction with complete information. *European Journal of Political Economy*, *14*, 627–644. doi:10.1016/S0176-2680(98)00029-9.

Baye, M. R., Kovenock, D., & de Vries, C. G. (1996). The all-pay auction with complete information. *Economic Theory*, *8*, 291–305. doi:10.1007/BF01211819.

Bernhardt, D. (1995). Strategic promotion and compensation. *The Review of Economic Studies*, *62*, 315–339. doi:10.2307/2297807.

Bjerk, D. (2008). Glass ceilings or sticky floors? Statistical discrimination in a dynamic model of hiring and promotion. *The Economic Journal*, *118*, 961–982. doi:10.1111/j.1468-0297.2008.02157.x.

Brown, J. (2011). Quitters never win: The (adverse) incentive effects of competing with superstars. *Journal of Political Economy*, *119*, 982–1013. doi:http://dx.doi.org/10.1086/663306.

Chen, K.-P. (2003). Sabotage in promotion tournaments. *Journal of Law, Economics, and Organization*, *19*, 119–140. doi:10.1093/jleo/19.1.119.

Chowdhury, S., Esteve-Gonzalez, P., & Mukherjee, A. (2019). Heterogeneity, leveling the playing field, and affirmative action in contests. *Working Paper*.

Chowdhury, S. M., & Gürtler, O. (2015). Sabotage in contests: A survey. *Public Choice*, *164*, 135–155. doi:10.1007/s11127-015-0264-9.

Chowdhury, S. M., & Kim, S.-H. (2017). "Small, yet beautiful": Reconsidering the optimal design of multi-winner contests. *Games and Economic Behavior*, *104*, 486 – 493. doi:10.1016/j.geb.2017.05.012.

Clark, D., & Riis, C. (1998a). Competition over more than one prize. *American Economic Review*, *88*, 276–89. URL: https://www.jstor.org/stable/i300820.

Clark, D. J., & Riis, C. (1996). On the win probability in rent-seeking games. *Discussion Paper in Economics E4/96, University of Tromsø, Norway*.

Clark, D. J., & Riis, C. (1998b). Contest success functions: An extension. *Economic Theory*, *11*, 201–204. doi:10.1007/s001990050184.

Cohen, C., Kaplan, T. R., & Sela, A. (2008). Optimal rewards in contests. *The RAND Journal of Economics*, *39*, 434–451. doi:10.1111/j.0741-6261.2008.00021.x.

Corchón, L., & Dahm, M. (2010). Foundations for contest success functions. *Economic Theory*, *43*, 81–98. doi:10.1007/s00199-008-0425-x.

Cornes, R., & Hartley, R. (2005). Asymmetric contests with general technologies. *Economic Theory*, *26*, 923–946. doi:10.1007/s00199-004-0566-5.

DeVaro, J., & Kauhanen, A. (2016). An "opposing responses" test of classic versus market-based promotion tournaments. *Journal of Labor Economics*, *34*, 747–779. doi:10.1086/684911.

DeVaro, J., & Waldman, M. (2012). The signaling role of promotions: Further theory and empirical evidence. *Journal of Labor Economics*, *30*, 91–147. doi:10.1086/662072.

Dewatripont, M., Jewitt, I., & Tirole, J. (1999). The economics of career concerns, part i: Comparing information structures. *The Review of Economic Studies*, *66*, 183–198. doi:10.1111/1467-937X.00084.

Drugov, M., & Ryvkin, D. (2017). Biased contests for symmetric players. *Games and Economic Behavior*, *103*, 116–144. doi:10.1016/j.geb.2016.02.003.

Fang, D., Noe, T., & Strack, P. (2019). Turning up the heat: The discouraging effect of competition in contests. *Journal of Political Economy (forthcoming)*. doi:10.1086/705670.

Fu, Q., & Lu, J. (2009a). The beauty of "bigness": On optimal design of multi-winner contests. *Games and Economic Behavior*, *66*, 146 – 161. doi:10.1016/j.geb.2008.05.006.

Fu, Q., & Lu, J. (2009b). The optimal multi-stage contest. *Economic Theory*, *51*, 351–382. doi:10.1007/s00199-009-0463-z.

Fu, Q., & Lu, J. (2012). Micro foundations of multi-prize lottery contests: a perspective of noisy performance ranking. *Social Choice and Welfare*, *38*, 497–517. doi:10.1007/s00355-011-0542-5.

Fu, Q., Lu, J., & Pan, Y. (2015). Team contests with multiple pairwise battles. *American Economic Review*, *105*, 2120–40. doi:10.1257/aer.20121469.

Fu, Q., & Wu, Z. (2019). Contests: Theory and topics. *Oxford Research Encyclopedia of Economics and Finance*, . doi:10.1093/acrefore/9780190625979.013.440.

Fullerton, R. L., & McAfee, R. P. (1999). Auctioning entry into tournaments. *Journal of Political Economy*, *107*, 573–605. doi:10.1086/250072.

Gershkov, A., Li, J., & Schweinzer, P. (2009). Efficient tournaments within teams. *The RAND Journal of Economics*, *40*, 103–119. doi:10.1111/j.1756-2171.2008.00057.x.

Gershkov, A., Li, J., & Schweinzer, P. (2016). How to share it out: The value of information in teams. *Journal of Economic Theory*, *162*, 261–304. doi:10.1016/j.jet.2015.12.013.

Ghosh, S., & Waldman, M. (2010). Standard promotion practices versus up-or-out contracts. *The RAND Journal of Economics*, *41*, 301–325. doi:10.1111/j.1756-2171.2010.00101.x.

Gilpatric, S. M. (2009). Risk taking in contests and the role of carrots and sticks. *Economic Inquiry*, *47*, 266–277. doi:10.1111/j.1465-7295.2008.00155.x.

Green, J. R., & Stokey, N. L. (1983). A comparison of tournaments and contracts. *Journal of Political Economy*, *91*, 349–364. doi:10.1086/261153.

Grund, C., & Sliwka, D. (2005). Envy and compassion in tournaments. *Journal of Economics & Management Strategy*, *14*, 187–207. doi:10.1111/j.1430-9134.2005.00039.x.

Gürtler, M., & Gürtler, O. (2015). The optimality of heterogeneous tournaments. *Journal of Labor Economics*, *33*, 1007–1042. doi:10.1086/681207.

Gürtler, M., & Gürtler, O. (2019). Promotion signaling, discrimination, and positive discrimination policies. *The RAND Journal of Economics*, *50*, 1004–1027. doi:10.1111/1756-2171.12303.

Hillman, A. L., & Riley, J. G. (1989). Politically contestable rents and transfers. *Economics & Politics*, *1*, 17–39. doi:10.1111/j.1468-0343.1989.tb00003.x.

Holmström, B. (1982). Managerial Incentive Problems: A Dynamic Perspective. In *Essays in Economics and Management in Honor of Lars Wahlbeck, Helsinki 1982.* (pp. 169–182). Reprinted in *The Review of Economic Studies* (1999), 66(1). doi:10.1111/1467-937X.00083.

Holmström, B., & Ricard I Costa, J. (1986). Managerial incentives and capital management. *Quarterly Journal of Economics*, *101*, 835–860. doi:10.2307/1884180.

Hvide, H. K. (2002). Tournament rewards and risk taking. *Journal of Labor Economics*, *20*, 877–898. doi:10.1086/342041.

Imhof, L., & Kräkel, M. (2016). Ex post unbalanced tournaments. *The RAND Journal of Economics*, *47*, 73–98. doi:10.1111/1756-2171.12119.

Jia, H. (2008). A stochastic derivation of the ratio form of contest success functions. *Public Choice*, *135*, 125–130. doi:10.1007/s11127-007-9242-1.

Kirkegaard, R. (2020). Microfounded contest design. *Working Paper*.

Konrad, K. A. (2009). *Strategy and Dynamics in Contests*. Oxford University Press.

Kräkel, M. (2008). Optimal risk taking in an uneven tournament game with risk averse players. *Journal of Mathematical Economics*, *44*, 1219–1231. doi:10.1016/j.jmateco.2008.02.002.

Kräkel, M., & Sliwka, D. (2004). Risk taking in asymmetric tournaments. *German Economic Review*, *5*, 103–116. doi:10.1111/j.1465-6485.2004.00096.x.

Krumer, A., Megidish, R., & Sela, A. (2017). Round-robin tournaments with a dominant player. *The Scandinavian Journal of Economics*, *119*, 1167–1200. doi:10.1111/sjoe.12204.

Lazear, E. P. (1989). Pay equality and industrial politics. *Journal of Political Economy*, *97*, 561–580. doi:10.1086/261616.

Lazear, E. P., & Rosen, S. (1981). Rank-order tournaments as optimum labor contracts. *Journal of Political Economy*, *89*, 841–864. doi:10.1086/261010.

Levy, H. (1992). Stochastic dominance and expected utility: Survey and analysis. *Management Science*, *38*, 555–593. doi:10.1287/mnsc.38.4.555.

Malcomson, J. M. (1984). Work incentives, hierarchy, and internal labor markets. *Journal of Political Economy*, *92*, 486–507. doi:10.1086/261238.

Malcomson, J. M. (1986). Rank-Order Contracts for a Principal with Many Agents. *The Review of Economic Studies*, *53*, 807–817. doi:10.2307/2297720.

Moldovanu, B., & Sela, A. (2001). The optimal allocation of prizes in contests. *American Economic Review*, *91*, 542–558. doi:10.1257/aer.91.3.542.

Moldovanu, B., & Sela, A. (2006). Contest architecture. *Journal of Economic Theory*, *126*, 70–96. doi:10.1016/j.jet.2004.10.004.

Moldovanu, B., Sela, A., & Shi, X. (2007). Contests for status. *Journal of Political Economy*, *115*, 338–363. doi:10.1086/518010.

Morath, F., & Münster, J. (2013). Information acquisition in conflicts. *Economic Theory*, *54*, 99–129. doi:10.1007/s00199-012-0710-6.

Münster, J. (2009). Group contest success functions. *Economic Theory*, *41*, 345–357. doi:10.1007/s00199-009-0479-4.

Nalebuff, B. J., & Stiglitz, J. E. (1983). Prizes and incentives: Towards a general theory of compensation and competition. *The Bell Journal of Economics*, *14*, 21–43. doi:10.2307/3003535.

O'Keeffe, M., Viscusi, W. K., & Zeckhauser, R. J. (1984). Economic contests: Comparative reward schemes. *Journal of Labor Economics*, *2*, 27–56. doi:10.1086/298022.

Olszewski, W., & Siegel, R. (2016). Large contests. *Econometrica*, *84*, 835–854. doi:10.3982/ECTA11696.

Owan, H. (2004). Promotion, turnover, earnings, and firm-sponsored training. *Journal of Labor Economics*, *22*, 955–978. doi:10.1086/423160.

Rinne, H. (2014). *The hazard rate: Theory and inference - With supplementary MATLAB Programs*. Technical Report Justus-Liebig-University, Giessen, Germany.

Rothschild, M., & Stiglitz, J. E. (1970). Increasing risk I: A definition. *Journal of Economic Theory*, *2*, 225–243. doi:10.1016/0022-0531(70)90038-4.

Ryvkin, D., & Drugov, M. (2020). The shape of luck and competition in winner-take-all tournaments. *Theoretical Economics (accepted)*.

Schotter, A., & Weigelt, K. (1992). Asymmetric tournaments, equal opportunity laws, and affirmative action: Some experimental results. *Quarterly Journal of Economics*, *107*, 511–539. doi:10.2307/2118480.

Schöttner, A., & Thiele, V. (2010). Promotion tournaments and individual performance pay. *Journal of Economics & Management Strategy*, *19*, 699–731. doi:10.1111/j.1530-9134.2010.00267.x.

Schweinzer, P., & Segev, E. (2012). The optimal prize structure of symmetric Tullock contests. *Public Choice*, *153*, 69–82. doi:10.1007/s11127-011-9774-2.

Sela, A. (2012). Sequential two-prize contests. *Economic Theory*, *51*, 383–395. doi:10.1007/s00199-010-0547-9.

Siegel, R. (2009). All-pay contests. *Econometrica*, *77*, 71–92. doi:10.3982/ECTA7537.

Siegel, R. (2010). Asymmetric contests with conditional investments. *The American Economic Review*, *100*, 2230–2260. doi:10.1257/aer.100.5.2230.

Skaperdas, S. (1996). Contest success functions. *Economic Theory*, *7*, 283–290. doi:10.1007/BF01213906.

Tullock, G. (1980). Efficient rent seeking. In J. Buchanan, R. Tollison, & G. Tullock (Eds.), *Towards a Theory of the Rent-Seeking Society* (pp. 97–112). College Station, Tx.: Texas A&M University Press.

Vojnović, M. (2015). *Contest theory: Incentive mechanisms and ranking methods*. Cambridge University Press.

Waldman, M. (1984). Job assignments, signalling, and efficiency. *The RAND Journal of Economics*, *15*, 255–267. doi:10.2307/2555679.

Zábojník, J. (2012). Promotion tournaments in market equilibrium. *Economic Theory*, *51*, 213–240. doi:10.1007/s00199-011-0606-x.

Zábojník, J., & Bernhardt, D. (2001). Corporate Tournaments, Human Capital Acquisition, and the Firm Size—Wage Relation. *The Review of Economic Studies*, *68*, 693–716. doi:10.1111/1467-937X.00186.

Appendix

A   Proofs of Lemmas and Theorems

A.1   Proof of Lemma 1

Suppose that $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=e}$ is the same for both $i \in \{1,2\}$ and all $e \in int\ E$. Then we have $\frac{\partial P_1(e_1,e_2)}{\partial e_1}|_{e_1=e_2=e}V - c'(e) = \frac{\partial P_2(e_2,e_1)}{\partial e_2}|_{e_1=e_2=e}V - c'(e)$ for all $e \in int\ E$. Since $\pi_i(e_i,e_k)$ is continuously differentiable, $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=e}V - c'(e)$ is a continuous function of e. Furthermore, recall that there exist $\bar{e}_i, \breve{e}_i \in int\ E$ such that $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=\bar{e}_i} < 0$ and $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=\breve{e}_i} > 0$. Hence, by the Intermediate Value Theorem, there is some $e^* \in int\ E$ such that $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=e^*}V - c'(e^*) = 0$. By Assumption 1, $e_1 = e_2 = e^*$ is a Nash equilibrium.

A.2   Proof of Theorem 1

Since we wish to apply the sufficient condition Lemma 1, we restrict attention to $e_i > 0$. Then, the function $g_e : \mathbb{R} \to \mathbb{R}$ defined by $g_e(x) = g(x,e)$ is strictly increasing and, thus, invertible. The inverse, g_e^{-1}, is strictly increasing as well. For the two (different) players $i, k \in \{1,2\}$, we observe

$$g(\theta_i, e_i) < g(\theta_k, e_k)$$
$$\Leftrightarrow \quad g_{e_i}(\theta_i) < g_{e_k}(\theta_k)$$
$$\Leftrightarrow \quad \theta_i < g_{e_i}^{-1}\big(g_{e_k}(\theta_k)\big).$$

Player k thus wins with probability

$$\int F_i\big(g_{e_i}^{-1}\big(g_{e_k}(x)\big)\big) f_k(x)\,dx.$$

Differentiating with respect to e_k, we obtain

$$\int f_i\big(g_{e_i}^{-1}\big(g_{e_k}(x)\big)\big) \left(\frac{d}{de_k} g_{e_i}^{-1}\big(g_{e_k}(x)\big)\right) f_k(x)\,dx.$$

According to Lemma 1, and noting that $g_{e_k}^{-1}(g_{e_i}(x)) = g_{e_i}^{-1}(g_{e_k}(x)) = x$ if $e_i = e_k$, a sufficient condition for a symmetric equilibrium to exist is that

$$
\int \left(\frac{d}{de_1} g_{e_2}^{-1}\big(g_{e_1}(x)\big) \Big|_{e_1=e_2=e} \right) f_1(x) f_2(x) \, dx
$$
$$
= \int \left(\frac{d}{de_2} g_{e_1}^{-1}\big(g_{e_2}(x)\big) \Big|_{e_1=e_2=e} \right) f_1(x) f_2(x) \, dx,
$$

for all $e \in int \, E$. Since $\frac{d}{de_1} g_{e_2}^{-1}\big(g_{e_1}(x)\big)\big|_{e_1=e_2=e} = \frac{d}{de_2} g_{e_1}^{-1}\big(g_{e_2}(x)\big)\big|_{e_1=e_2=e}$, this condition is always fulfilled.

A.3 Proof of Theorem 2

Suppose that Assumption 2 holds, and consider case (i), i.e., $r_{e,i}(x)$ is monotonically increasing in x, and $\tilde{F}_k$ first-order stochastically dominates F_k. Denote the equilibrium effort levels for the two contests by $\tilde{e}^*$ and e^*, respectively. Our goal is to show that $\tilde{e}^* > e^*$.

The proof proceeds by contradiction, so suppose $\tilde{e}^* \leq e^*$. Now observe that

$$
V \int r_{\tilde{e}^*,i}(x) \tilde{f}_k(x) \, dx - c'\big(\tilde{e}^*\big) \geq
$$
$$
V \int r_{e^*,i}(x) \tilde{f}_k(x) \, dx - c'\big(e^*\big) >
$$
$$
V \int r_{e^*,i}(x) f_k(x) \, dx - c'\big(e^*\big) = 0.
$$

The first inequality follows from $\tilde{e}^* \leq e^*$ together with Assumption 2. The second inequality follows from $r_{e,i}(x)$ being monotonically increasing on $supp(f_i)$, $\tilde{F}_k$ first-order stochastically dominating F_k, and the fact that we have assumed that both $supp(\tilde{f}_k)$ and $supp(f_k)$ are subsets of $supp(f_i)$.[20] The equality follows since e^* is characterized by the first-order condition $V \int r_{e^*,i}(x) f_k(x) \, dx - c'(e^*) = 0$. We conclude that

$$
V \int r_{\tilde{e}^*,i}(x) \tilde{f}_k(x) \, dx - c'\big(\tilde{e}^*\big) > 0.
$$

This shows that the first-order condition for equilibrium effort cannot be fulfilled in the case of the distribution $\tilde{F}_k$, giving us the desired contradiction.

By an analogous argument, we can show that $\tilde{e}^* > e^*$ also in case (ii) where $r_{e,i}$ is monotonically decreasing in x for all $e > 0$ and F_k first-order stochastically

---

[20]See, e.g., Levy 1992, p.557. Notice that, in expected utility theory, the utility function is defined for all possible payoffs and therefore no additional constraints regarding the statistical supports need to be imposed.

dominates $\tilde{F}_k$. In this case, $\int r_{e,i}(x)\tilde{f}_k(x)dx > \int r_{e,i}(x)f_k(x)dx$ for all $e > 0$ (see, e.g., Levy 1992, p.557).

A.4   Proof of Theorem 3

Because of Assumption 2, and the condition characterizing equilibrium effort, we need to show that $\int r_{e,i}(x)\tilde{f}_k(x)dx > (=,<) \int r_{e,i}(x)f_k(x)dx$ if $r_{e,i}$ is convex (linear, concave). The proof is very similar to part a) of the proof of Theorem 2 in Rothschild & Stiglitz (1970, p.237). In the case of convex $r_{e,i}$, the inequality in their proof is reversed, while it is replaced by an equality if $r_{e,i}$ is linear.

A.5   Proof of Theorem 4

Part i)   As explained in the main body of the paper, the equilibrium first-order condition for an n-player contest is equivalent to that of a two-player contest in which the second player's skill distribution is replaced by the strongest rival's skill distribution (the highest order statistic) of the n-player contest. We show that $\int r_e(x)\left(\frac{d}{dx}(F(x))^{n-1}\right)dx$ is increasing in n. If $n_1, n_2 \in \mathbb{N}$, with $n_1 > n_2$, then $(F(x))^{n_1-1}$ first-order stochastically dominates $(F(x))^{n_2-1}$, and the result follows from Theorem 2.

Part ii)   Suppose that $r_e(x)$ is monotonically decreasing in x for all $e \geq 0$, and let $n_1, n_2 \in \mathbb{N}$, with $n_1 > n_2$. It follows that $(F(x))^{n_1-1}$ first-order stochastically dominates $(F(x))^{n_2-1}$, as just mentioned, implying that

$$\int r_e(x)\left(\frac{d}{dx}(F(x))^{n_1-1}\right)dx < \int r_e(x)\left(\frac{d}{dx}(F(x))^{n_2-1}\right)dx.$$

Part iii)   If $r_e(x) = r_e$ is constant in x for all $e \geq 0$, we have

$$\int r_e(x)\left(\frac{d}{dx}(F(x))^{n-1}\right)dx = r_e \int \left(\frac{d}{dx}(F(x))^{n-1}\right)dx = r_e,$$

which is independent of n.

B Proofs of Propositions

B.1 Proof of Proposition 1

Suppose that $g(\theta, e) = \theta e$. This means that

$$a_e(x) = \frac{d}{de_i}\left(g^{-1}_{e_k}\left(g_{e_i}(x)\right)\right)\Bigg|_{e_i=e_k=e} = \frac{d}{de_i}\left(\frac{xe_i}{e_k}\right)\Bigg|_{e_i=e_k=e} = \frac{x}{e}.$$

In the considered situation, the marginal winning probability is

$$\frac{1}{2\pi\sigma_1\sigma_2}\int \frac{x}{e}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\mu_2)^2}{2\sigma_2^2}\right)dx.$$

To prove the proposition, it is sufficient to show that

$$\frac{1}{2\pi\sigma_1\sigma_2}\int x\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\mu_2)^2}{2\sigma_2^2}\right)dx$$

$$= \frac{\left(\mu_1\sigma_2^2 + \mu_2\sigma_1^2\right)\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{(2\pi)^{\frac{1}{2}}\left(\sigma_1^2+\sigma_2^2\right)^{\frac{3}{2}}}.$$

Define

$$Z := \frac{1}{2\pi\sigma_1\sigma_2}\int x\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\mu_2)^2}{2\sigma_2^2}\right)dx$$

and notice that:

$$\frac{(x-\mu_1)^2}{2\sigma_1^2} + \frac{(x-\mu_2)^2}{2\sigma_2^2}$$

$$= \frac{\sigma_2^2(x-\mu_1)^2(\sigma_1^2+\sigma_2^2) + \sigma_1^2(x-\mu_2)^2(\sigma_1^2+\sigma_2^2)}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}$$

$$= \frac{\sigma_2^2(x^2-2x\mu_1+\mu_1^2)(\sigma_1^2+\sigma_2^2) + \sigma_1^2(x^2-2x\mu_2+\mu_2^2)(\sigma_1^2+\sigma_2^2)}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}$$

$$= \frac{x^2(\sigma_1^2+\sigma_2^2)^2 - 2x(\mu_1\sigma_2^2+\mu_2\sigma_1^2)(\sigma_1^2+\sigma_2^2) + (\mu_1^2\sigma_2^2+\mu_2^2\sigma_1^2)(\sigma_1^2+\sigma_2^2)}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}$$

$$= \frac{x^2(\sigma_1^2+\sigma_2^2)^2 - 2x(\mu_1\sigma_2^2+\mu_2\sigma_1^2)(\sigma_1^2+\sigma_2^2) + (\mu_1\sigma_2^2+\mu_2\sigma_1^2)^2}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}$$
$$- \frac{(\mu_1^2\sigma_2^4 + 2\mu_1\sigma_2^2\mu_2\sigma_1^2 + \mu_2^2\sigma_1^4 - \mu_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2) - \mu_2^2\sigma_1^2(\sigma_1^2+\sigma_2^2))}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}$$

$$= \frac{(x(\sigma_1^2+\sigma_2^2) - (\mu_1\sigma_2^2+\mu_2\sigma_1^2))^2 + (\mu_1^2\sigma_1^2\sigma_2^2 - 2\mu_1\sigma_2^2\mu_2\sigma_1^2 + \mu_2^2\sigma_1^2\sigma_2^2)}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}$$

$$= \frac{(x(\sigma_1^2+\sigma_2^2) - (\mu_1\sigma_2^2+\mu_2\sigma_1^2))^2}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)} + \frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}.$$

Using this, we obtain

$$Z = \frac{\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{2\pi\sigma_1\sigma_2} \int x\exp\left(-\frac{(x(\sigma_1^2+\sigma_2^2) - (\mu_1\sigma_2^2+\mu_2\sigma_1^2))^2}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}\right)dx$$

$$= \frac{\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{(2\pi)^{\frac{1}{2}}(\sigma_1^2+\sigma_2^2)^{\frac{3}{2}}}(\sigma_1^2+\sigma_2^2)\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\int x\exp\left(-\frac{\left(x - \frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right)dx.$$

Now notice that

$$\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\int x\exp\left(-\frac{\left(x - \frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right)dx$$

describes the mean of a normally distributed random variable with variance $\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2$ and mean $\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}$, hence

$$\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\int x\exp\left(-\frac{\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right)dx=\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}.$$

We obtain

$$
\begin{aligned}
Z &= \frac{\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{(2\pi)^{0.5}\left(\sigma_1^2+\sigma_2^2\right)^{1.5}}\left(\sigma_1^2+\sigma_2^2\right)\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} \\
&= \frac{\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{(2\pi)^{\frac{1}{2}}\left(\sigma_1^2+\sigma_2^2\right)^{\frac{3}{2}}}.
\end{aligned}
$$

B.2 Proof of Proposition 2

Player i wins the contest with probability

$$\int\prod_{k\neq i}F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)f_i(x)\,dx.$$

Differentiating with respect to e_i, we obtain

$$\int\left(\prod_{k\neq i}F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)\right)\left(\sum_{k\neq i}\frac{f_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)\left(\frac{d}{de_i}g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)}{F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)}\right)f_i(x)\,dx.$$

In a symmetric equilibrium with $e_1^*=...=e_n^*=:e^*$, and symmetric skill distributions, this marginal effect of effort on the probability of winning simplifies to

$$\int\left(\prod_{k\neq i}F(x)\right)\left(\sum_{k\neq i}\left.\left(\frac{d}{de_i}g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)\right|_{e_1^*=...=e_n^*=e^*}\frac{f(x)}{F(x)}\right)f(x)\,dx,$$

and must be identical for all i. We can restate the above expression as

$$\int r_{e^*}(x)(n-1)(F(x))^{n-2}f(x)\,dx=\int r_{e^*}(x)\left(\frac{d}{dx}(F(x))^{n-1}\right)dx,$$

which is identical for all i.

B.3 Proof of Proposition 3

As shown before, if $g(\theta_i, e_i) = \theta_i e_i$, we have

$$\left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_i + x) \right) \right) \Bigg|_{e_1 = \ldots = e_n = e} = \frac{(t_i + x)}{e}.$$

Thus, making use of expression (11), derived in Section C.1, and denoting $\Delta t = t_1 - t > 0$,

$$\lambda^2 \int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\sum_{k \neq i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Bigg|_{e_1 = \ldots = e_n = e} \right) dx$$

$$= \lambda^2 \int^{-\Delta t} (\exp(\lambda x) \cdot \exp((n-1)\lambda(\Delta t + x)))(n-1) \frac{(t_1 + x)}{e} dx$$

$$= \frac{\lambda^2 (n-1)}{e} \int^{-\Delta t} \exp(n\lambda y + (n-1)\lambda \Delta t)(t_1 + y) dy.$$

The map $\phi_2 : \mathbb{R}_x \to \mathbb{R}_y$ given by $x \to y = -\Delta t + x$ is a smooth diffeomorphism with $\det |\phi_2'(x)| = 1$. Applying the associated change of variables to the integral, we obtain

$$\frac{\lambda^2 (n-1)}{e} \int^0 \exp(n\lambda(x - \Delta t) + (n-1)\lambda \Delta t)(t + x) dx$$

$$= \frac{(n-1)}{e} \exp(-\lambda \Delta t) \lambda^2 \left(\int^0 x \exp(n\lambda x) dx + t \int^0 \exp(n\lambda x) dx \right).$$

Notice that

$$n\lambda \int^0 x \exp(n\lambda x) dx$$

is the mean of a random variable that is distributed according to the reflected exponential distribution with parameter $n\lambda$, hence

$$n\lambda \int^0 x \exp(n\lambda x) dx = -\frac{1}{n\lambda}$$

$$\Leftrightarrow \int^0 x \exp(n\lambda x) dx = -\frac{1}{n^2 \lambda^2}.$$

Furthermore,

$$n\lambda \int^0 \exp(n\lambda x) dx = 1$$

$$\Leftrightarrow \int^0 \exp(n\lambda x) dx = \frac{1}{n\lambda}.$$

It follows that

$$\frac{(n-1)}{e}\exp(-\lambda\Delta t)\lambda^2\left(\int^0 x\exp(n\lambda x)dx + t\int^0 \exp(n\lambda x)dx\right)$$

$$= \frac{(n-1)}{e}\exp(-\lambda\Delta t)\lambda^2\frac{(-1+n\lambda t)}{n^2\lambda^2}$$

$$= \frac{(n-1)}{e}\exp(-\lambda\Delta t)\frac{(-1+n\lambda t)}{n^2}.$$

Taking the derivative of the above expression w.r.t. n results in an expression that is positive if $n\lambda t - 1 > 0$.

C   Other Computations and Derivations

C.1   Additional Derivations for Section .

Player i outperforms player k iff

$$g_{e_i}(t_i + \varepsilon_i) > g_{e_k}(t_k + \varepsilon_k)$$

$$\Leftrightarrow \quad \varepsilon_k < g_{e_k}^{-1}\left(g_{e_i}(t_i + \varepsilon_i)\right) - t_k.$$

Recall that the $\mathscr{E}_i$ are i.i.d., following the reflected exponential distribution on $(-\infty, 0]$. The cdf is denoted by H and the pdf by h. Hence, player i wins the contest with probability

$$\int \prod_{k\neq i} H\left(g_{e_k}^{-1}\left(g_{e_i}(t_i + x)\right) - t_k\right) h(x)dx.$$

In a symmetric equilibrium with $e_1^* = \dots = e_n^* =: e^*$, the marginal effect of effort on the probability of winning,

$$\int \left(\prod_{k\neq i} H(t_i + x - t_k)\right)\left(\sum_{k\neq i}\left(\frac{d}{de_i}g_{e_k}^{-1}\left(g_{e_i}(t_i + x)\right)\right)\Big|_{e_1^* = \dots = e_n^* = e^*}\frac{h(t_i + x - t_k)}{H(t_i + x - t_k)}\right)h(x)dx,$$

must be the same for all i. Denote $\Delta t = t_1 - t > 0$. For player 1, we have,

$$\int \left(\prod_{k\neq 1} H(\Delta t + x)\right)\left(\sum_{k\neq 1}\left(\frac{d}{de_1}g_{e_k}^{-1}\left(g_{e_1}(t_1 + x)\right)\right)\Big|_{e_1^* = \dots = e_n^* = e^*}\frac{h(\Delta t + x)}{H(\Delta t + x)}\right)h(x)dx.$$

For any other player $i \in \{2,...,n\}$, we have

$$\int \left(H(-\Delta t + y) \prod_{k \neq 1,i} H(y) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1}(g_{e_i}(t+y)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(-\Delta t + y)}{H(-\Delta t + y)} \right.$$

$$\left. + \sum_{k \neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t+y)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(y)}{H(y)} \right) h(y) dy.$$

The map $\phi_1 : \mathbb{R}_x \to \mathbb{R}_y$ given by $x \to y = \Delta t + x$ is a smooth diffeomorphism with $\det |\phi_1'(x)| = 1$. Applying the associated change of variables to the preceding expression, we obtain

$$\int \left(H(x) \prod_{k \neq 1,i} H(\Delta t + x) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1}(g_{e_i}(t_1 + x)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(x)}{H(x)} \right.$$

$$\left. + \sum_{k \neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t_1 + x)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) h(\Delta t + x) dx.$$

The expressions for the two types of players can be restated as

$$\int \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\sum_{k \neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1}(g_{e_1}(t_1 + x)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) \frac{h(x)}{H(x)} dx,$$

$$\int \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1}(g_{e_i}(t_1 + x)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(x)}{H(x)} \right.$$

$$\left. + \sum_{k \neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t_1 + x)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) \frac{h(\Delta t + x)}{H(\Delta t + x)} dx.$$

Notice that both expressions are equal to zero for $x \geq -\Delta t$. Hence, they can be restated as

$$\int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\sum_{k \neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1}(g_{e_1}(t_1 + x)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) \frac{h(x)}{H(x)} dx,$$

$$\int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1}(g_{e_i}(t_1 + x)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(x)}{H(x)} \right.$$

$$\left. + \sum_{k \neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t_1 + x)) \right) \Big|_{e_1^* = ... = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) \frac{h(\Delta t + x)}{H(\Delta t + x)} dx.$$

For $x < -\Delta t$, we observe $\frac{h(x)}{H(x)} = \frac{h(\Delta t + x)}{H(\Delta t + x)} = \lambda$, and the expressions become

$$\lambda^2 \int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\sum_{k \neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1} \left(g_{e_1}(t_1 + x) \right) \right) \bigg|_{e_1^* = \ldots = e_n^* = e^*} \right) dx,$$

$$\lambda^2 \int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\sum_{k \neq i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \bigg|_{e_1^* = \ldots = e_n^* = e^*} \right) dx \qquad (11)$$

which are identical.

C.2 Computations for Example 6

The first-order condition (9) is equivalent to (we ease notation by writing e instead of $e(\theta)$)

$$c'(e) \frac{\partial g / \partial \theta}{\partial g / \partial e} - f^{(n-1)}(\theta) V + c'(e) \frac{de}{d\theta} = 0, \qquad (12)$$

which can be restated as

$$P(\theta, e) + Q(\theta, e) \frac{de}{d\theta} = 0,$$

with $P(\theta, e) := c'(e) \frac{\partial g / \partial \theta}{\partial g / \partial e} - f^{(n-1)}(\theta) V$ and $Q(\theta, e) := c'(e)$.

Is there an integrating factor $\mu(\theta, e)$ such that $\frac{\partial(\mu P)}{\partial e} = \frac{\partial(\mu Q)}{\partial \theta}$? In other words, is there $\mu(\theta, e)$ such that

$$\frac{\partial \mu}{\partial e} P + \mu \frac{\partial P}{\partial e} = \frac{\partial \mu}{\partial \theta} Q + \mu \frac{\partial Q}{\partial \theta}?$$

The latter equation can be stated as

$$\frac{\partial \mu}{\partial e} \left(c'(e) \frac{\partial g / \partial \theta}{\partial g / \partial e} - f^{(n-1)}(\theta) V \right)$$
$$+ \mu \left(c''(e) \frac{\partial g / \partial \theta}{\partial g / \partial e} + c'(e) \frac{\partial^2 g / \partial \theta \partial e \cdot \partial g / \partial e - \partial g / \partial \theta \cdot \partial^2 g / \partial e^2}{(\partial g / \partial e)^2} \right) = \frac{\partial \mu}{\partial \theta} c'(e).$$

Now, for our example, assume $g(\theta, e) = \theta e$ and $c(e) = 0.5 e^2$, and ignore the argument θ in $e(\theta)$. Then the equation simplifies to

$$\frac{\partial \mu}{\partial e} \left(\frac{e^2}{\theta} - f^{(n-1)}(\theta) V \right) + \mu \left(2 \frac{e}{\theta} \right) = \frac{\partial \mu}{\partial \theta} e.$$

Suppose that $\frac{\partial \mu}{\partial e} = 0$. Then μ needs to satisfy

$$\mu \frac{2}{\theta} = \frac{\partial \mu}{\partial \theta}$$

and a solution is $\mu(\theta, e) = \theta^2$ (confirming $\frac{\partial \mu}{\partial e} = 0$).

Using $g(\theta, e) = \theta e$ and $c(e) = 0.5e^2$, our differential equation (12) can be stated as

$$\frac{e^2}{\theta} - f^{(n-1)}(\theta)V + e\frac{de}{d\theta} = 0,$$

and multiplication with $\mu(\theta, e) = \theta^2$ leads to

$$\theta e^2 - \theta^2 f^{(n-1)}(\theta)V + e\theta^2 \frac{de}{d\theta} = 0.$$

An integral is

$$L(\theta, e(\theta)) = \frac{e(\theta)^2 \theta^2}{2} - V\int_0^\theta x^2 f^{(n-1)}(x)\,dx,$$

which can easily be verified by computing $\frac{dL(\theta, e(\theta))}{d\theta}$.

With a general distribution, effort is given by the solution to

$$\frac{e(\theta)^2 \theta^2}{2} - V\int_0^\theta x^2 f^{(n-1)}(x)\,dx = \tilde{c}$$

$$\Leftrightarrow \quad e(\theta) = \sqrt{\frac{2V}{\theta^2}\int_0^\theta x^2 f^{(n-1)}(x)\,dx + \frac{2\tilde{c}}{\theta^2}},$$

where $\tilde{c}$ is some constant.

Using the assumption that skills are uniformly distributed on $[0,1]$ (implying $f(x) = 1$ and $F^{(n-1)}(t) = t^{n-1} \Rightarrow f^{(n-1)}(t) = (n-1)t^{n-2}$), we can compute effort and expected effort. In particular,

$$\int_0^\theta x^2 f^{(n-1)}(x)\,dx = (n-1)\int_0^\theta x^n\,dx = \frac{n-1}{n+1}\theta^{n+1},$$

meaning that the integral becomes

$$L(\theta, e) = \frac{e^2 \theta^2}{2} - V\frac{n-1}{n+1}\theta^{n+1}.$$

Hence, the solution to the differential equation is given by

$$\frac{1}{2}e^2\theta^2 - V\frac{n-1}{n+1}\theta^{n+1} = \tilde{c},$$

where $\tilde{c}$ is some constant. Solving for e, we obtain

$$e(\theta) = \sqrt{2V\frac{n-1}{n+1}\theta^{n-1} + \frac{2\tilde{c}}{\theta^2}}.$$

Conjecturing $e(0) = 0$, we have $\tilde{c} = 0$ and

$$e(\theta) = \sqrt{2V\frac{n-1}{n+1}\theta^{n-1}}.$$

It follows that expected effort is

$$E[e(\theta)] = \sqrt{2V\frac{n-1}{n+1}}\int_0^1 x^{\frac{n-1}{2}}\,dx = \sqrt{8V\frac{n-1}{(n+1)^3}},$$

which is strictly decreasing in n.

It is straightforward to verify that the equilibrium effort function satisfies $e(0) = 0$ and is strictly increasing in the skill θ. This implies that for any given skill θ, output $g(\theta, e(\theta)) = \theta e(\theta)$ is increasing in skill as well, and the inverse z^{-1} exists.