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Social Choice in Large Populations with Single-Peaked Preferences*

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Abstract

An anonymous social choice function for a large atomless population maps cross-section distributions of preferences into outcomes. Because any one individual is too insignificant to affect these distributions, every anonymous social choice function is individually strategy-proof. However, not every anonymous social choice function is group strategy-proof. If the set of outcomes is linearly ordered and participants have single-peaked preferences, an anonymous social choice function is group strategy-proof if and only if it can be implemented by a mechanism involving binary votes between neighbouring outcomes with nondecreasing thresholds for "moving higher up". Such a mechanism can be interpreted as a version of Moulin's (1980) generalized median-voter mechanism for a large population.

Key Words: Social choice, large populations, strategy proofness, group strategy proofness, single-peaked preferences.

JEL: D60, D70, D82, H41

1 Introduction

In this paper, I study the implementability of anonymous social choice functions when the population of participants is large and no one individual

*This paper originated in a cooperation with Felix Bierbrauer, who has however moved on to other topics. I am grateful to him for this cooperation and for many comments that made this a better paper. I am also grateful to Salvador Barberà and to Matthew Jackson for patiently answering my questions about tie-breaking in their 1994 article.

acting alone has the power to affect the aggregate outcome. I consider constellations where the set of alternatives is linearly ordered and participants have single-peaked preferences over this set. For implementability, I require that social choice functions be *group strategy-proof,* i.e., that, for any group of participants that might coordinate their actions, truthful communication of preferences is a dominant strategy.

For *finitely* many agents with single-peaked preferences over a linearly ordered set of alternatives, Moulin (1980) has shown that an anonymous social choice function mapping vectors of preference peaks to outcomes is individually strategy-proof if and only if it can be implemented by a generalized median-voter mechanism, defined as a median-voter mechanism for a population that is enlarged by a set of dummy voters with fixed and known preference peaks. If a social choice function can be implemented by a generalized median-voter mechanism, then it is group strategy-proof as well as individually strategy-proof, i.e., the two requirements are equivalent.[1] Barberà and Jackson (1994) and Sprumont (1995) extended Moulin's analysis to the full class of anonymous social choice functions that condition on the participants' preference profiles, rather than merely their preference peaks.

This paper provides an analogue of these results for populations with an atomless continuum of agents. In such populations, individual strategy proofness and group strategy proofness are no longer equivalent. Because any one individual is powerless to affect the aggregate outcome, every anonymous social choice function is in fact individually strategy-proof, but not every anonymous social choice functions is group strategy-proof. The characterization of group strategy proofness here can be interpreted as an adaptation of Moulin's (1980) characterization to large populations.

The argument is different from that of Moulin (1980), Barberà and Jackson (1994) and Sprumont (1995), however. In Moulin (1980), Barberà and Jackson (1994), and Sprumont (1995), group strategy proofness only comes in as an afterthought because generalized median-voter mechanisms, which necessary for individual strategy proofness with finitely many participants, happen to be group strategy-proof as well. Since group strategy proofness trivially implies individual strategy proofness, the three property, group strategy proofness, individual strategy proofness, and implementability by generalized median-voter mechanisms are acutally all equivalent. In a large population, this line of argument is not available because the requirement

[1] For finite populations, Barberà et al. (2010) provide a more general systematic analysis of the conditions under which group strategy proofness and individual strategy proofness coincide.

of individual strategy proofness has no bite.

I therefore provide a direct characterization of group strategy proofness in terms of what is needed to avoid collective manipulations. The characterization relies on the fact that, with single-peaked preferences over a linearly ordered set of alternatives, any outcome defines three natural coalitions with locally similar interests: Agents whose preference peaks are "higher", agents whose preference peaks are "lower", and agents whose preference peaks coincide with the given outcome. Agents in the first group all agree that they would like to move "up", agents in the second group all agree that they would like to move "down", and agents in the third group all agree that they would like to stay at the given outcome. The given characterization shows that, if a manipulation of social choice by any one of these groups is to be avoided, the social choice function must not condition on the groups' compositions and can only condition on their sizes.

The sizes of these groups can be found by having participants vote. If all participants have strict preferences over neighbouring outcomes, it suffices to have people indicate for each outcome whether they want to move "up" from that outcome or not. The chosen outcome then is the "lowest" at which the "up" votes fail to meet a specified threshold. The threshold may depend on the outcome considered but *the mapping from outcomes to thresholds must be nondecreasing.*[2] I call this the *monotone binary voting (MBV) property.*

According to a standard criticism, the use of voting for decisions on resource allocation is problematic because voting conveys too little information to enable efficient choices. In particular, in binary voting, the decision taken can only condition on the population shares of the sets of people in favour of one or the other alternative without taking account of preference intensities. A small set of people who care deeply about the decision cannot influence the outcome of the vote even though none of the other people may care very much at all.[3]

This argument is valid as a criticism of simple binary voting but not as a criticism of voting altogether. If there is a continuum of outcome levels, a set of binary votes indicating for each outcome level whether the

[2] If nonnegligible sets of people are indifferent between neighbouring outcomes, one may also need a vote on whether people want to move "down" from a given outcome or not; the combination of votes reveals how many people are indifferent.

[3] Buchanan and Tullock (1962) point to the problem and argue that vote-trading would be desirable because it provides a way to overcome this problem. Similarly, Casella (2005) suggests that intensities could be taken into account if voters had an endowment of votes and could assign more votes to issues that are of greater importance to them. Goeree and Zhang (2017) propose to replace votes by monetary bids. Ledyard (2006) suggests that, once incentive problems are taken into account, voting mechanisms may do relatively well.

voter wants to move higher "up" or not can provide something like full information about the entire preference profile. This information would be sufficient to implement outcomes that maximize aggregate surplus if this is the objective. However, a social choice function that adapts outcomes to reported preference distributions so as to maximize aggregate surplus will not generally be group strategy-proof. The inefficiency of outcomes under a voting mechanism that involves a continuum of such binary votes is not due to the coarseness of the information conveyed through voting but to the constraints imposed by group strategy proofness.

Why Large Populations? The paradigm of a large population where each individual is too insignificant to affect the social outcome has not been much used in social choice theory. In other areas of economics and political science, this paradigm is regularly used to study issues of resource allocation and voting when millions of people are involved. Examples are the theory of competitive equilibrium in markets for private goods, where no one individual is able to affect market prices, the theory of taxation, where no one individual has a noticeable impact on the government budget, and the theory of political decisions through voting, where no one individual expects to be pivotal for the outcome.[4] Underlying these theories is the observation that, in a population with a million people or more, the probability of any one person being able to affect the aggregate outcome is on the order of 10^{-3}, at least if the system treats all individuals alike.[5] While not literally zero, this order of magnitude is so small that, in practice, individuals are unlikely to pay much attention to the effects of their actions on aggregate outcomes.

In social choice, as in private markets or in voting, the impact that an individual in a population of millions can have on the overall outcome is so small that participants do not give it much consideration. The loss in precision that results from studying a continuum model in which the impact of a single person on aggregate outcomes is literally zero is therefore negligible and is outweighed by the gains in insight that can be obtained through the greater simplicity of the continuum model.

[4] Textbook treatments are given by Mas-Colell et al. (1995) for competitive equilibrium in markets for private goods, Ljungqvist and Sargent (2012) for macroeconomics and public finance, and Persson and Tabellini (2000) for political economy models. For additional examples, as well as an abstract treatment, of strategic interdependence in large populations with anonymity, see Hellwig (forthcoming).

[5] In a system with n participants, the probability of being pivotal is on the order of $n^{-\frac{1}{2}}$. See Hellwig (2003) and the references given there.

Motivation from Public-Goods Theory. The motivation for the analysis comes from public economics, in particular the theory of public-goods provision. Much of the theory of public-goods provision is concerned with the incentive problems that may prevent the implementation of efficient social choice when information about preferences is private. These incentive problems are usually analysed in terms of strategy proofness or, more generally, individual incentive compatibility in models with finitely many participants. In these models, each individual can have a noticeable impact on aggregate outcomes. With a focus on achieving individual incentive compatibility, the problem is to calibrate people's payments to their expressions of preferences so that they do not wish either to understate their preferences for the public good (so as to reduce their payments) or to overstate their preferences (so as to get a greater provision level at other people's expense).[6]

The notion that any one individual can have a noticeable effect on the level of public-good provision makes sense if we think about people in a condominium deciding on how much to spend on gardening and maintenance. However, this notion is not very relevant for studying how a society with millions of people decides on how much to spend on defense or on the judicial system. For such choices, the notion that individual agents are too insignificant to have a noticeable influence on aggregate outcomes is as relevant as it is for the allocation of private goods through markets or for elections.

Limiting the theory of public-good provision to models in which each agent has a noticeable influence on aggregate outcomes is akin to limiting the analysis of markets to models of bargaining and oligopoly without ever talking about perfect competition. In the area of public economics, the discrepancy between the small-economy approach to public-good provision and the large-economy approach to taxation is particularly vexing because it stands in the way of an integrated welfare analysis of public spending and taxation.

However, if individuals are too insignificant to have a noticeable influence on aggregate outcomes, the problem of how to calibrate people's payments to their expressions of preferences so that they do not wish either to understate

[6] For implementation in dominant strategies, see Clarke (1971), Groves (1973), Green and Laffont (1979), for (interim) Bayes-Nash implementation, see d'Aspremont and Gérard-Varet (1979). More recently, Bergemann and Morris (2005) have studied interim implementation with a requirement of robustness with respect to the specification of agents' beliefs about the other participants. Hellwig (2021) studies Bayesian implementation in a model with macro shocks to preference parameters, extending d'Aspremont and Gérard-Varet (1979), formulating an impossibility theorem for weakly robust implementation with budget balance and a possibility theorem for robust implementation with budget balance in a large population.

or to overstate their preferences for the public good is trivial. If individuals cannot affect public-good provision levels, their payments must be independent of what they say. Otherwise, they would choose their communications to miminize their payments, regardless of what their preferences might be.[7]

If individual agents are too insignificant to have a noticeable influence on aggregate outcomes, independence of payments from expressed preferences is sufficient as well as necessary for individual incentive compatibility: If agents believe that all aspects of outcomes, payments as well as public-good provision levels, are independent of what they communicate, they will be indifferent as to what they communicate and may as well communicate their true preferences.

If the payment is the same for all agents and equal to the *per-capita* provision costs at whatever level of public-good provision is chosen, the public budget is balanced. In this case, if there is a single public good, the set of relevant outcomes is linearly ordered by the level of public-good provision. Moreover, if utility functions are quasi-concave and the per-capita provision cost function is convex, preference orderings are single-peaked on this set, so the analysis of this paper applies.

There is a small literature on public economics and macroeconomics in large economies that involves public-good provision.[8] In this literature, there is no requirement of equal cost sharing, so public-good provision may contribute to or be supported by the general government budget. The research focuses on how (second-best) public-good provision and taxation interrelate when government funding involves distortionary taxes.[9] This literature assumes that information about individual preferences for public goods is private but *aggregate* benefits from public-good provision can be elicited without problems. Bierbrauer (2009) provides a formal justification on the basis of the above reasoning about individual strategy proofness in a large population.

[7] In a Bayesian setting with correlated values, these sentences assume that one cannot use Crémer-McLean (1988) mechanisms to exploit type dependence of beliefs about other agents. The assumption is justified if Bayesian incentive compatibility must be robust to changes in the specification of beliefs, as postulated by Ledyard (1978) and Bergemann and Morris (2005).

[8] For example Barro (1990), Boadway and Keen (1993), Battaglini and Coate (2008), Gaube (2000), Hellwig (2004), Heathcote et al. (2020). With known benefits from public-good provision, the question becomes how public-good provision rules are affected by the presence of distortions from taxes that are used for public funding.

[9] With finitely many participants, strategy-proof implementation of efficient public-good provision precludes budget balance. With a large population, budget balance is not a problem. See, e.g., Green and Laffont (1979) and Hellwig (2021).

Why Group Strategy Proofness? However, there is a problem. If one thinks of the large population as an idealization of a large finite population, the model with a large population in which no one agent has power to affect aggregates should work essentially "like" a model with a large finite population. This requirement is violated when we move from saying that, in a model with a large population independence of payments from expressed preferences is *necessary* for individual incentive compatibility to saying that this independence is *sufficient* for individual incentive compatibility.

The latter property holds *only* if the influence of any agent on the aggregate outcome is *actually* zero; it need not hold even approximately if the influence of an individual agent on the aggregate outcome is small but not zero. To see this, it suffices to observe that in any finite economy, an individually incentive-compatible social choice function cannot stipulate that the level of public-good provision should be lower if an agent's preferences for the public good are more intense. Similarly, an individually incentive-compatible social choice function cannot stipulate that the level of public-good provision should be lower if *all* agents' preferences for the public good are more intense.

Now consider a social choice function for the large population that requires the level of public-good provision to be *lower* when all agents' preferences for the public good are more intense. With a large population and payments based on equal cost sharing, such a social choice function is individually incentive-compatible, but, with monotonicity going the wrong way, it cannot be approximated by individually incentive-compatible social choice functions for large finite economies.[10]

This difficulty cannot arise if the requirement of individual incentive compatibility is replaced by a requirement of coalition incentive compatibility.[11] Thus, with a focus on dominant-strategy implementation, I postulate that a social choice function must be group strategy-proof, i.e., there is no group of agents and no constellation of strategies of agents outside the group such that agents in the group can benefit by deviating from truthful communication. I do not give a formal result, but it is straightforward to show that a group strategy-proof social choice function for a large population can be approximated by group strategy-proof social choice functions for large finite economies (and, conversely, that any limit of group strategy-proof social choice functions for large finite populations is itself group strategy-proof

[10] Technically, the set of individual incentive compatible social choice functions exhibits a property of upper hemi-continuity but *not* a property of lower hemi-continuity as one goes from large finite populations to the continuum.

[11] See also Bierbrauer (2014) and Bierbrauer and Hellwig (2015).

for the large population).

Group strategy proofness avoids the difficulties that arise because, in a large population, the individual agent feels powerless to affect the overall outcome. A well known example is given by the paradox of voting, i.e., the observation that people participate in elections as if their votes mattered when in fact they know that, individually, their impact is practically zero.[12] Coordination within groups may provide for a sense that participation is not futile and that the vote itself matters.

In principle, the formation of groups and the coordination of behaviours within groups can itself be seen as a problem of social choice. An axiomatic requirement of group strategy proofness assumes this problem away. This is appropriate if one is interested in the constraints that group behaviour can impose on implementation. The implications of frictions that may prevent groups from becoming effective should be treated as a separate subject.[13]

Why Dominant-Strategy Implementation? I also want to avoid the question of what information is available to coalitions when they consider possible deviations. This is why I focus on dominant-strategy implementation rather than Bayesian implementation. In dominant-strategy implementation, the individual or group under consideration is assumed to know everything about the state of the world, including the characteristics of participants who do not belong to the group. This stark assumption puts the focus on the implications of strategic interdependence as such, without worrying about issues of information available to the participants.

An interesting question is what becomes of the analysis if one relies on robust Bayesian, rather than dominant-strategy implementation. In the Bayesian framework, the information that is available to each participant need not be perfect, and this information is explicitly modelled. A requirement of *ex post* coalition proofness in this framework is equivalent to group strategy proofness, so the logic of our analysis applies directly.[14] Preliminary reflections suggest that this logic also applies with a requirement of *interim* coalition proofness, where deviating coalitions know the information

[12] For a political-economy analysis, see Bierbrauer et al. (2021).

[13] Notice, however, that here too the large-population paradigm provides for a significant simplification in that coalitions cannot rely on side payments among the participants. The insensitivity of individual payments to communications about individual preferences that was explained in the text above extends to side payments within a coalition. With voluntary participation as well as budget balance within a coalition, such side payments must all be zero. See Bierbrauer and Hellwig (2015).

[14] See Bierbrauer and Hellwig (2016).

of their own members but are uncertain about the information of others.[15] However, confirmation of this conjecture must be left to future research.

Outline. In the following, Section 2 introduces the formal model of social choice with a large population. It introduces the concepts of individual strategy proofness and group strategy proofness and shows that individual strategy proofness holds trivially for any anonymous social choice function. It also introduces a concept of weak group strategy proofness, which makes it possible to avoid some difficulties that can arise when the range of the social welfare function has a gap and a non-negligible set of people is indifferent between the outcomes that border the gap on either side. Weak group strategy proofness refers only to social choice at preference constellation where sets of people who are indifferent between outcomes on two sides of gaps in the range of the social choice function are negligible.

Section 3 shows that any weakly group strategy-proof social welfare function can be implemented by a mechanism that has the MBV property, asking asks people to indicate for each outcome whether they want to move to a higher outcome or not, with thresholds that are non-decreasing in outcomes, and implementation of an outcome that is "lowest" among those at which the "up" votes fail to meet the specified threshold.

Section 4 extends the analysis to social choice for distributions with nonnegligible sets of people exhibiting indifference between neighbouring outcomes. Section 5 discusses the implications for public-good provision, showing that efficiency is not hampered by a lack of information, but by the monotonicity requirement for the thresholds in binary voting on whether to move "up" or not, which in turn is a consequence of group strategy proofness.

The analysis is developed for social choice functions depending on cross-section distributions of preferences, without reference to the underlying space of agents. Formal proofs are given in Appendix A. Appendix B the cross-section distribution formalism in the main text to a formalism with an atomless measure space of agents.

2 Basic Concepts

Preference Orderings, Type Distributions, and Social Choice. The set of alternatives for social choice is assumed to be well ordered. I represent this set by the real line $\mathbb{R}$. The set of participants is an atomless measure space. Each agent has a preference ordering on $\mathbb{R}$ that is induced by a utility

[15] See Bierbrauer and Hellwig (2015).

function $u(\cdot, t)$, where t is the agent's type. The type t belongs to a complete separable metric space T, and $u(\cdot, \cdot)$ is an upper semi-continuous function on $\mathbb{R} \times T$.

A *social choice function* determines an outcome $x \in \mathbb{R}$ as a function of the profile of the participants' preference orderings. I only consider *anonymous* social choice functions, that is, social choice functions with the property that the chosen outcome depends only on the cross-section distribution of types in the population. This distribution is an element of the set $\mathcal{M}(T)$ of probability measures on T. An *anonymous social choice function* is given by a mapping F from $\mathcal{M}(T)$ to $\mathbb{R}$ with the interpretation that, for any $s \in \mathcal{M}(T)$, $F(s)$ is the outcome chosen if the cross-section distribution of preferences is s.

Because anonymous social choice functions are defined on distributions, I never refer to the underlying measure space of agents. Appendix B shows how the formalism for distributions can be derived from a formalism with an atomless measure space of agents.

Individual Strategy Proofness and Group Strategy Proofness. An anonymous social choice function F is *individually strategy-proof* if, for every $s \in \mathcal{M}(T)$, and every t and t' in T,

$$u(F(s), t) \geq u(F(\hat{s}(s, t, t'), t), \tag{2.1}$$

where $\hat{s}(s, t, t')$ is the distribution of reported types that is obtained if the true type distribution is s, one agent with true type t reports the type t', and all other agents report their types honestly.

Proposition 2.1 *For a population represented by an atomless measure space of agents, every anonymous social choice function is individually strategy-proof.*

I next consider group strategy proofness: A type set $B \subset T$ is said to *block* the anonymous social choice function F at s if there exists $s' \in \mathcal{M}(T)$ such that

$$u(F(s), t) < u(F(s_{T \backslash B} + s(B) \cdot s'), t) \tag{2.2}$$

for all $t \in B$, where $s_{T \backslash B}$ is the restriction of s to the set $T \backslash B$. The anonymous social choice function F is said to be *group strategy-proof* if and only if there are no $B \subset T$ and $s \in \mathcal{M}(T)$ such that B blocks F at s. Group strategy proofness implies, in particular, that, for all s, $F(s)$ is Pareto efficient, i.e., there is no $s \in \mathcal{M}(T)$ such that the grand coalition T blocks F at s.

Single-peakedness of Preferences. The utility function $u(\cdot, t)$ is said to be *single-peaked* if there exists a critical outcome $\pi(t)$ such that $u(\cdot, t)$ is strictly increasing on $(-\infty, \pi(t)]$ and strictly decreasing on $[\pi(t), \infty)$. If all utility functions $u(\cdot, t), t \in T$, are single-peaked, then for any $x \in \mathbb{R}$ there is a natural decomposition of the type set T into the three sets

$$\begin{aligned} P^*(x) := \{t \in T | \pi(t) = x\},\ U(x) := \{t \in T | \pi(t) > x\}, \\ \text{and } D(x) := \{t \in T | \pi(t) < x\}\ . \end{aligned} \tag{2.3}$$

An agent with $t \in P^*(x)$ considers x to be best, an agent with $t \in U(x)$ prefers something higher than x, and would like to move up, an agent with $t \in D(x)$ prefers something lower than x and would like to move down. Agents with types in $U(x)$, do not agree on where they want to end up, but they do have the same attitudes to small changes away from x; the same is true of agents with types in $D(x)$.

The following assumption asserts that single-peakedness is the only restriction on the domain of social choice functions. In the remainder of the paper, this assumption is imposed without further mention.

Richness of T The type set T is rich in the sense that, for any single-peaked function f on $\mathbb{R}$, there exists $t \in T$ such that $f(\cdot) = u(\cdot, t)$.

Weak Group Strategy Proofness. If the range R_F of the social choice function F is a disconnected subset of $\mathbb{R}$, single-peakedness of $u(\cdot, t)$ on $\mathbb{R}$ does not necessarily imply single-peakedness of $u(\cdot, t)$ on R_F, i.e., on the set of outcomes to which the social choice function F is restricted. For suppose that there is gap in R_F so that, for some x_1, x_2 in R_F, $(x_1, x_2) \cap R_F = \emptyset$. For utility functions $u(\cdot, t)$ with $\pi(t) \leq x_1$ or $\pi(t) \geq x_2$, this gap makes no difference. But for utility functions $u(\cdot, t)$ with $\pi(t) \in (x_1, x_2)$, one might have $u(x_1, t) = u(x_2, t)$ so that the restriction of $u(\cdot, t)$ to the set R_F has twin peaks, at x_1 and x_2. Such indifference complicates the analysis of group strategy proofness of F. Because blocking requires *strict* Pareto improvements, the indifferent types will never be part of a blocking coalition.[16] A full characterization of group strategy proofness must somehow deal with this fact.

To avoid dealing with this complication right away, I introduce a notion of *weak group strategy* proofness. For given F, with range R_F, let T_F be

[16] With a weaker definition of blocking that allows for *weak* Pareto improvements, group strategy proofness is unattainable unless the social choice function is constant; see fn. 23 below.

the set of types t such that $u(\cdot, t)$ is single-peaked on R_F and let $\mathcal{M}_F^*(T)$ be the set of distributions $s \in \mathcal{M}(T)$ that are concentrated on T_F. The social choice function F is said to be *weakly group strategy-proof* if and only if the restriction $F|\mathcal{M}_F^*(T)$ of F to the set $\mathcal{M}_F^*(T)$ is group strategy-proof and, moreover, the range of this restriction is the same as the range of F itself, i.e., $R_{F|\mathcal{M}_F^*(T)} = R_F$ and there are no $B \subset T_F$, s, s' in $\mathcal{M}_F^*(T)$ such that (2.2) holds for all $t \in B$.

If F is group strategy-proof, the requirement that $R_{F|\mathcal{M}_F^*(T)} = R_F$ is automatically fulfilled. For suppose that $x \in R_F$ and $s \in \mathcal{M}(T)$ are such that $s(P^*(x)) = 1$. Then also $s \in \mathcal{M}_F^*(T)$ and, since group strategy proofness implies Pareto efficiency of $F(s)$, $F(s) = x$, i.e., $R_{F|\mathcal{M}_F^*(T)} = R_F$. This observation yields:

Remark 2.2 *Any anonymous group strategy-proof social choice function is also weakly group strategy-proof.*

If $R_F \subsetneqq \mathbb{R}$, the sets $P^*(x)$, $U(x)$, and $D(x)$ in (2.3) must be replaced by the sets $P_F^*(x)$, $U_F(x)$, and $D_F(x)$ such that $P_F^*(x)$ is the set of types for which x is the unique preferred outcome in R_F, $U_F(x)$ is the set of types that prefer some higher outcome in R_F to x, and $D_F(x)$ is the set of types that prefer some lower outcome in R_F to x. If x is at the boundary of one or two gaps in R_F, there is an additional set $I_F(x)$ of types that are indifferent between x and (one of) the boundary outcome(s) on the other side of (one of) the gap(s), but prefer x to all other outcomes. For $s \in \mathcal{M}_F^*(T)$, however, $s(I_F(x)) = 0$, and only the sets $P_F^*(x), U_F(x), D_F(x)$ matter.

From the single-crossing property and the upper semi-continuity of u, one obtains the following useful properties of the functions $s(U_F(\cdot))$ and $s(D_F(\cdot))$.[17]

Remark 2.3 *For any F and any $s \in \mathcal{M}_F(T)$, the map $x \longmapsto s(U_F(x))$ is non-increasing and right-continuous, and the map $x \longmapsto s(D_F(x))$ is non-decreasing and left-continuous.*

[17] Monotonicity is implied by single-peakedness. Right-continuity of $s(U_F(\cdot))$ at x is trivial if R_F has a gap immediately above x. If R_F does not have such a gap, single-peakedness implies $U_F(x) = \cup_{x'>x} U_F(x')$, so $\lim_{x' \downarrow x} s(U_F(x')) = s(U_F(x))$ follows by elementary measure theory. Left-continuity of $s(D_F(\cdot))$ follows by a symmetric argument.

3 Weak Group Strategy Proofness

This section provides a complete characterization of weakly group strategy-proof social choice functions. For greater clarity, I proceed in several steps, treating the binary case $R_F = \{x_1, x_2\}$ and the general finite case $R_F = \{x_1, ..., x_n\}$ before giving the result for general R_F.

As an additional piece of notation, for any x and $\hat{x}$, I write

$$P(x, \hat{x}) := \{t \in T | u(x,t) > u(\hat{x}, t)\} \tag{3.1}$$

for the set of types that strictly prefer the outcome x to the outcome $\hat{x}$.

3.1 The Binary Case: Voting

If $R_F = \{x_1, x_2\}$, then for any weakly group strategy-proof social choice function F and any $s \in \mathcal{M}_F^*(T)$, the outcome $F(s)$ can only depend on the population shares of the sets $P(x_2, x_1)$ and $P(x_1, x_2)$. For $s \in \mathcal{M}_F^*(T)$ of course, these shares must add up to one, i.e., $s(P(x_2, x_1)) + s(P(x_1, x_2)) = 1$.

Proposition 3.1 *An anonymous social choice function F with a binary range $R_F = \{x_1, x_2\}$ is weakly group strategy-proof if and only if there exists $\bar{s}_F \in [0, 1]$ such that, for any $s \in \mathcal{M}_F^*(T)$, the following statements hold:*

(i) There exists

$$F(s) = x_1 \quad \textit{implies} \quad s(P(x_2, x_1)) \leq \bar{s}_F \tag{3.2}$$

and

$$F(s) = x_2 \quad \textit{implies} \quad s(P(x_2, x_1)) \geq \bar{s}_F. \tag{3.3}$$

(ii) If $F(s) = x_2$ for some $s \in \mathcal{M}_F^*(T)$ such that $s(P(x_2, x_1)) = \bar{s}_F$, then $F(\hat{s}) = x_2$ for all $\hat{s} \in \mathcal{M}_F^*(T)$ such that $\hat{s}(P(x_2, x_1)) = \bar{s}_F$.

If $R_F = \{x_1, x_2\}$, an anonymous weakly group strategy-proof social choice function F is characterized by a threshold $\bar{s}_F$ such that people are asked whether they prefer x_1 or x_2 and the chosen outcome is x_1 if the population share of the set of people preferring x_2 is below the threshold $\bar{s}_F$ and x_2 if the population share of the set of people preferring x_2 is above the threshold $\bar{s}_F$. If the threshold is exactly met, the chosen outcome can be either x_1 or x_2 but, whichever it is, it must be the same for all type distributions that meet the threshold precisely.

Weak group strategy proofness eliminates any scope for taking account of preference intensities. The social choice function considered in Proposition

3.1 only considers ordinal preferences and asks how many people are on each side of the binary vote.

To understand why this is so, notice that, if $R_F = \{x_1, x_2\}$ and $s \in \mathcal{M}^*_F(T)$, there are exactly two clearly identified interest groups, agents with types in $P(x_1, x_2)$, who prefer x_1 to x_2 and agents with types in $P(x_2, x_1)$, who prefer x_2 to x_1. Suppose that the population shares of these two groups are σ and $1 - \sigma$. Suppose that agents in the first group coordinate their reports to transmit the type distribution $\hat{s}^1$ and agents in the second group coordinate their reports to transmit the type distribution $\hat{s}^2$. Then the overall distribution of reported types is $\sigma\hat{s}^1 + (1 - \sigma)\hat{s}^2$, and the chosen outcome is $F(\sigma\hat{s}^1 + (1 - \sigma)\hat{s}^2)$.

One can think of the situation in terms of a two-player zero-sum game, where the two groups are the players, their reports of type distributions are their strategies and the payoffs are $-F(\sigma\hat{s}^1 + (1 - \sigma)\hat{s}^2)$ for group 1 and $F(\sigma\hat{s}^1 + (1 - \sigma)\hat{s}^2)$ for group 2. Group strategy proofness is equivalent to the condition that, regardless of what the true type distribution s may be, truthtelling is a Nash equilibrium in this game, i.e., if the true overall distribution is $s = \sigma s^1 + (1 - \sigma)s^2$, then the report $\hat{s}^1 = s^1$ is a best response for group 1 to the report $\hat{s}^2 = s^2$ of group 2, and the report $\hat{s}^2 = s^2$ is a best response for group 2 to the report $\hat{s}^1 = s^1$ of group 1.

This non-cooperative game depends on the true type distribution s *only* through the population shares $\sigma = s(P(x_1, x_2))$ and $(1 - \sigma) = s(P(x_2, x_1))$. Therefore, if truthtelling is a Nash equilibrium when the overall type distribution is $s = \sigma s^1 + (1 - \sigma)s^2$, the report pair (s^1, s^2) is also a Nash equilibrium if the overall type distribution is $\bar{s} = \sigma\bar{s}^1 + (1 - \sigma)\bar{s}^2$ (with the same σ). By the saddle-point theorem for zero-sum (strictly competitive) games,[18] it follows that, for given σ, we can write $F(\sigma s^1 + (1 - \sigma)s^2) = F^*(\sigma)$, regardless of s^1 and s^2. The within-group distribution of types has no effect on the chosen outcome.

The choice $F^*(\sigma) = F(\sigma\hat{s}^1 + (1 - \sigma)\hat{s}^2)$ must also be weakly monotonic in σ : For $\sigma' > \sigma$, group 1 can always report a type distribution $\hat{s}^1 = \sigma s^1 + (\sigma' - \sigma)\hat{s}$, with $s^1 \in \mathcal{M}(P(x_1, x_2))$ and $\hat{s} \in \mathcal{M}(P(x_2, x_1))$; this report induces the truthtelling outcome for the type distribution $\sigma s^1 + (\sigma' - \sigma)\hat{s} + (1 - \sigma')s^2$, i.e. $F(\sigma s^1 + (\sigma' - \sigma)\hat{s} + (1 - \sigma')s^2) = F^*(\sigma)$. For truthtelling to be an equilibrium when the population shares are σ' and $1 - \sigma'$, $F(\sigma' s^1 + (1 - \sigma')s^2) = F^*(\sigma')$ must therefore be no greater than $F^*(\sigma)$. A greater population share of group 1 must not lead to a higher outcome level. Since F^* takes only the values x_1 and x_2, monotonicity implies that, for some threshold $\bar{s}_F$, $F^*(\sigma) = x_1$ if

[18] See, e.g., Proposition 22.2, p. 22, in Osborne and Rubinstein (1994).

$\sigma < \bar{s}_F$ and $F^*(\sigma) = x_2$ if $\sigma > \bar{s}_F$.

3.2 The General Finite Case: Monotone Binary Voting over Neighbours

I next consider social choice functions F with finite ranges $R_F = \{x_1, ..., x_n\}$, where, without loss of generality,

$$x_1 \ < \ x_2 \ < \quad ... \quad < \ x_n. \tag{3.4}$$

For convenience of exposition, I also introduce the fictitious outcomes $x_0 = -\infty$ and $x_{n+1} = \infty$, in addition to the outcomes in R_F, with the convention that

$$s(P(x, x_0)) = 1 \quad \text{and} \quad s(P(x_{n+1}, x)) = 0 \ \text{ for all } x,$$

for all s. The following result generalizes Proposition 3.1.

Proposition 3.2 *An anonymous social choice function F with a finite range $R_F = \{x_1, ..., x_n\}$ is weakly group strategy-proof if and only if the following statements hold:*

(i) There exist thresholds $\bar{s}_F^0, ..., \bar{s}_F^n$ such that

$$0 = \bar{s}_F^0 \leq \bar{s}_F^1 ... \leq \bar{s}_F^{n-1} \leq \bar{s}_n = 1, \tag{3.5}$$

and, for any $s \in \mathcal{M}_F^*(T)$ and $j \in \{1, ..., n\}$,

$$F(s) = x_j \ \ \textit{implies} \ \ s(P(x_j, x_{j-1}) \geq \bar{s}_F^{j-1} \ \ \textit{and} \ \ s(P(x_{j+1}, x_j) \leq \bar{s}_F^j. \tag{3.6}$$

(ii) For any s and $\hat{s}$ in $\mathcal{M}_F^*(T)$, any $j \in \{1, ..., n-1\}$ and any $\ell \in \{j+1, ..., n\}$, $F(s) = x_j$ and $F(\hat{s}) = x_\ell$ imply

$$\hat{s}(P(x_{j+1}, x_j)) > s(P(x_{j+1}, x_j)) \quad \textit{or} \quad s(P(x_{j+1}, x_j)) > \hat{s}(P(x_{j+1}, x_j)); \tag{3.7}$$

both these inequalities must hold if $\ell = j + 1$.

Corollary 3.3 *If an anonymous social choice function F with a range $R_F = \{x_1, ..., x_n\}$ that satisfies (3.5) is weakly group strategy-proof, then, for any $s \in \mathcal{M}_F^*(T)$ and $j \in \{1, ..., n\}$, $F(s) = x_j$ if one of the following applies*

$$s(P(x_j, x_{j-1}) > \bar{s}_F^{j-1} \ \ \textit{and} \ \ s(P(x_{j+1}, x_j) < \bar{s}_F^j;$$

$$s(P(x_j, x_{j-1}) > \bar{s}_F^{j-1} \ \ \textit{and} \ \ s(P(x_{j+1}, x_j) = 0;$$

$$s(P(x_j, x_{j-1}) = 1 \ \ \textit{and} \ \ s(P(x_{j+1}, x_j) < \bar{s}_F^j.$$

Proposition 3.2 asserts that weak group strategy proofness requires a reliance on n binary votes over adjacent outcomes. Each participant submits a vector of binary votes expressing preferences for x_1 versus x_2, x_2 versus $x_3, ..., x_{n-1}$ versus x_n. Given these votes, the chosen outcome $F(s)$ must satisfy conditions (i) and (ii). Thus, for outcome j to be chosen, the share $s(P(x_j, x_{j-1}))$ of participants who prefer x_j to x_{j-1} must reach or exceed the threshold $\bar{s}_F^{j-1}$ and the share $s(P(x_{j+1}, x_j))$ of participants who prefer x_{j+1} to x_j must not exceed the threshold $\bar{s}_F^j$.

If one of the thresholds is met precisely, e.g., if $s(P(x_{j+1}, x_j)) = \bar{s}_F^j$, for some j, there is some arbitrariness in the choice of an outcome but, as indicated by condition (ii), social choice is still determined by population shares without regard to preference intensities. For example, if $s(P(x_{j+1}, x_j)) = \bar{s}_F^j$, i.e. if s meets the threshold $\bar{s}_F^j$ precisely, $F(s)$ might be equal to x_j or to x_{j+1}, but, if it is the former, we must have $\hat{s}(P(x_{j+1}, x_j)) > \bar{s}_F^j$ for any distribution $\hat{s}$ for which $F(\hat{s}) = x_{j+1}$. If $\bar{s}_F^j$ is the only threshold that two distributions s and $\hat{s}$ both meet precisely, we must have $F(s) = F(\hat{s})$.

Why Monotonicity of Thresholds? As indicated by (3.5), the thresholds $\bar{s}_F^j$ must be non-decreasing in j. This monotonicity property is an important implication of (weak) group strategy proofness. It ensures that, except for the special cases where thresholds are met precisely, the different binary votes do not interfere with each other, so the logic of the case $R_F = \{x_1, x_2\}$ is directly applicable.

To understand why thresholds must be non-decreasing in outcomes, suppose that, for some j, we had $\bar{s}_F^j > \bar{s}_F^{j+1}$. Then there would exist a type distribution s in $\mathcal{M}_F^*(T)$ such that $s(P(x_j, x_{j-1})) < \bar{s}_F^j$ and $s(P(x_{j+1}, x_j)) > \bar{s}_F^j$, i.e., in the binary vote between x_{j-1} and x_j, the threshold for the higher outcome is not reached but, in the binary vote between x_j and x_{j+1}, the threshold for the higher outcome is surpassed. What could be the outcome $F(s)$ for this type distribution? By the logic of group strategy-proof binary choice, it could not be x_j because this outcome "loses" against both x_{j-1} and x_{j+1} when the type distribution is s.

Could one have $F(s) < x_j$? If so, what is the outcome $F(\hat{s})$ for the type distribution $\hat{s}$ that coincides with s on the set $P(x_{j+1}, x_j)$ and that assigns the remaining mass $1 - s(P(x_{j+1}, x_j))$ to a single type $\hat{t}_j$ that prefers x_j over x_{j-1} .... over x_1 and x_1 over all outcomes above x_j? Under $\hat{s}$, all agents prefer x_j to all lower outcomes, so one cannot have $F(\hat{s}) < x_j$. Because $\hat{s}(P(x_{j+1}, x_j)) = s(P(x_{j+1}, x_j)) > \bar{s}_F^{j+1}$, one also cannot have $F(\hat{s}) = x_j$. The alternative left is $F(\hat{s}) > x_j$. But then the group of agents with type

$\hat{t}_j$ can block F at $\hat{s}$: If these agents coordinate their reports to mimick the behaviour of s on the set $T \backslash P(x_{j+1}, x_j)$, the distribution of reported types will be s, and the outcome will be $F(s) < x_j$, which all group members prefer to $F(\hat{s}) > x_j$.

Alternatively, could one have $F(s) > x_j$? If so, what is the outcome $F(s^*)$ for the type distribution s^* that coincides with s on the set $P(x_{j-1}, x_j)$ and that assigns the remaining mass $1 - s(P(x_{j-1}, x_j))$ to a single type t_j^* that prefers x_j over x_{j+1} .... over x_n and x_n over all outcomes below x_j? By the same kinds of arguments as in the preceding paragraph, $F(s^*) > x_j$ is impossible because all agents prefer x_j over all higher outcomes, $F(s^*) = x_j$ is impossible because $s^*(P(x_j, x_{j-1})) < \bar{s}_F^j$, and $F(s^*) < x_j$ is impossible because the group of agents with type t_j^* could block F at s^* by coordinating their reports to mimick the behaviour of s on the set $T \backslash P(x_{j-1}, x_j)$. The assumption that $\bar{s}_F^j > \bar{s}_F^{j+1}$ for some j is thus incompatible with weak group strategy proofness.

Given the monotonicity of the thresholds $\bar{s}_F^1, ..., \bar{s}_F^n$, social choice is largely governed by the principle that, for any type distribution $s \in \mathcal{M}_F^*(T)$, the outcome $F(s) = x_j$ should be such that the requirement for moving up from x_{j-1} is met and the requirement for moving further up from x_j is not met. Corollary 3.3 shows that, except for the ambiguity whether meeting the requirement for moving up involves a weak or a strict inequality, this principle suffices to determine the outcome $F(s)$ without any concern for the results of binary voting at higher or at lower outcome levels.

Relation to Moulin's (1980) Generalized Median-Voter Mechanism. Condition (i) in Proposition 3.2 can be interpreted as an instance of Moulin's (1980) generalized median-voter rule. The generalized median-voter rule chooses the outcome preferred by the median voter in a population that is augmented by a set of dummy participants with single-peaked preferences.[19] Suppose we have a population of dummy participants of the same size as the "real" population and suppose that the distribution of preference orderings among dummy participants is given by a measure $\bar{s}_F \in \mathcal{M}_F^*(T)$ such that, for any j,

$$\bar{s}_F(P_F^*(x_j)) = \bar{s}_F^j - \bar{s}_F^{j-1}. \tag{3.8}$$

[19] In Moulin's model with n participants having single-peaked preferences, any anonymous strategy-proof social choice function can be implemented by a generalized median voter rule with $n+1$ dummy participants. If the outcomes stipulated by the social choice function always lie between the smallest peak and the largest peak of the "real" participants, it suffices to have $n-1$ dummy participants.

The measure $\bar{s}_F$ obviously satisfies

$$\bar{s}_F(U_F(x_j)) = \sum_{i=j+1}^{n} \bar{s}_F(P_F^*(x_i)) = 1 - \bar{s}_F^j. \tag{3.9}$$

Because single-peakedness implies $s(U_F(x_j)) = s(P(x_{j+1}, x_j))$ for all j, therefore, statement (i) in Proposition 3.2 is equivalent to the condition that $F(s) = x_j$ implies

$$\frac{1}{2}\, s(U_F(x_{j-1})) + \frac{1}{2}\, \bar{s}_F(U_F(x_{j-1})) \quad \geq \quad \frac{1}{2}\,, \tag{3.10}$$

and

$$\frac{1}{2}\, s(U_F(x_j)) + \frac{1}{2}\, \bar{s}_F(U_F(x_j)) \quad \leq \quad \frac{1}{2}\,. \tag{3.11}$$

If both of these inequalities happen to be strict, the type distribution $\frac{1}{2}s + \frac{1}{2}\bar{s}_F$ must assign positive weight to the set $P_F^*(x_j)$, which is equal to the difference between the sets $U_F(x_{j-1})$ and $U_F(x_j)$. Since (3.10) implies

$$\frac{1}{2}\, s(D_F(x_j)) + \frac{1}{2}\, \bar{s}_F(D_F(x_j)) \quad \leq \quad \frac{1}{2}, \tag{3.12}$$

one may think of the set of agents with types in $P_F^*(x_j)$ as a generalized median voter, as less than one half of the population have lower peaks and less than one half of the population have higher peaks. Statement (i) in Proposition 4.2 thus corresponds to Moulin's generalized median-voter rule with a distribution of preference orderings of dummy participants given by $\bar{s}_F \in \mathcal{M}_F^*(T)$. As indicated by (3.8), the monotonicity of the threshold function can be interpreted as an analogue of Moulin's dummy participants.

With a continuum of participants, there also is a possibility that (3.10) and (3.12) both hold as equations. In this case, taking account of the dummies, there is a fifty-fifty split between people who prefer x_j to x_{j-1} and people who prefer x_{j-1} to x_j. In Moulin's (1980) analysis of social choice in a finite population, this case cannot arise because the augmented population has an uneven number of members, so in a vote between the median voter's preferred outcome and any other outcome, the median voter's preferred outcome is always chosen by more than fifty percent of the population. In a large population, however, the median voter as such has no effect on the split of votes, so a fifty-fifty split can occur and the (generalized) median-voter rule must be complemented by a condition indicating what is to happen then. Statement (ii) of the proposition implies that, in this case, the outcome must be the same for all distributions that induce a fifty-fifty split between x_{j-1} and x_j as well as a strict majority for x_j over x_{j+1}.

3.3 Monotone Binary Voting: The General Case

Turning to the general case of social choice functions with arbitrary ranges, I note that, with single-peakedness, the sets $P(x_j, x_{j-1})$ and $P(x_{j+1}, x_j)$ in Proposition 3.2 satisfy

$$P(x_{j+1}, x_j) = U_F(x_j)$$

and, moreover,

$$U_F(x_j) \geq U_F(x_{j+1})$$

for all j, where

$$U_F(x) := \{t \in T_F | \pi(t) > x\} \tag{3.13}$$

is the set of types whose preference peak lies above x.

With this notation, statements (i) and (ii) in Proposition 3.2 are easily seen to be special cases of the following more general property.

MBV Property An anonymous social choice function F with range $R_F \subset \mathbb{R}$ has the *monotone binary voting property* (MBV property) if the following conditions are satisfied:

(i) There exists a non-decreasing right-continuous function $\bar{s}_F(\cdot)$ from R_F to $[0, 1]$ such that, for any $s \in \mathcal{M}_F^*(T)$ and any $x \in R_F$,

$$\begin{array}{lll} F(s) = x & \text{implies} & s(U_F(x')) \geq \bar{s}_F(x'), \text{ for all } x' < x \\ & \text{and} & s(U_F(x')) \leq \bar{s}_F(x'), \text{ for all } x' \geq x\ . \end{array} \tag{3.14}$$

(ii) For any s and $\hat{s}$ in $\mathcal{M}_F^*(T)$ and any x and $\hat{x} > x$ in R_F, $F(s) = x$ and $F(\hat{s}) = \hat{x}$ imply that $\hat{s}(U_F(x)) > s(U_F(x))$ or $s(D_F(\hat{x})) > \hat{s}(D_F(\hat{x}))$.

The threshold function $\bar{s}_F(\cdot)$ plays the same role as the sequence $\{\bar{s}_F^j\}$ in the finite case. Like the sequence $\{\bar{s}_F^j\}$, the function $\bar{s}_F(\cdot)$ must be non-decreasing. It need not be continuous, however. If it has a jump point, its value there is actually indeterminate within the interval defined by the jump. Right-continuity at a jump point is only imposed for specificity.

The following theorem extends the characterizations of Propositions 3.1 and 3.2 to the general case.

Theorem 3.4 *An anonymous social choice function with an arbitrary range is weakly group strategy-proof if and only if it has the MBV property.*

Graphical Illustration. Before discussing the interpretation of this result in terms of voting, I illustrate the MBV property graphically. Suppose that the social choice function F has the range $R_F = \mathbb{R}$ so that $\mathcal{M}_F^*(T) = \mathcal{M}(T)$. For any type distribution $s \in \mathcal{M}_F^*(T)$, consider the function

$$x \to s(U(x)) \tag{3.15}$$

that indicates how many people would prefer the chosen outcome to be greater than x. With $R_F = \mathbb{R}$, obviously $s(U_F(x)) = s(U(x))$, regardless of x. The function (3.15) is non-increasing in x. In the simplest case, it is strictly decreasing and continuous, as shown in Figure 1 for two distributions $s(\cdot)$ and $s(\cdot) = s^*(\cdot)$.

Figure 1: see next page

Whereas the functions $x \to s(U_F(x))$ for $s \in \mathcal{M}_F^*(T)$ are exogenously given, the specification of a *threshold function*

$$x \to \bar{s}_F(x) \tag{3.16}$$

that characterizes a social choice function with the MBV property is a matter of choice. In Figure 1, the threshold is taken to be constant, so with strict monotonicity of the function (3.15), the graphs of (3.15) and (3.16) intersect and, for any s, the intersection point is unique. The MBV property implies that the social choice $F(s)$ for the type distribution s is given by this intersection point. Thus, in Figure 1, $F(s)$ is the unique x for which $s(U_F(x)) = \bar{s}_F(x)$, and $F(\hat{s}) = \hat{x}$ is the unique $\hat{x}$ for which $s^*(U_F(\hat{x})) = \bar{s}_F(\hat{x})$.

Quite generally, if R_F is an interval, if the functions $\bar{s}_F(\cdot)$ and $s(U_F(\cdot))$ are continuous and at least one of them is strictly monotonic, only part (i) of the MBV property is relevant, and is equivalent to the condition:

$$F(s) = x \text{ if and only if } s(U_F(x)) = \bar{s}_F(x), \tag{3.17}$$

which singles out the intersection points in Figure 1.

The functions (3.15) and (3.16) need not be continuous. If they are discontinuous, their graphs need not intersect, i.e., the equation $s(U(x)) = \bar{s}_F(x)$ need not have a solution. However, if at least one of the functions (3.15) and (3.16) is *strictly* monotonic, there is a unique point at which one of the two curves "jumps" over the other. The outcome $F(s)$ then is

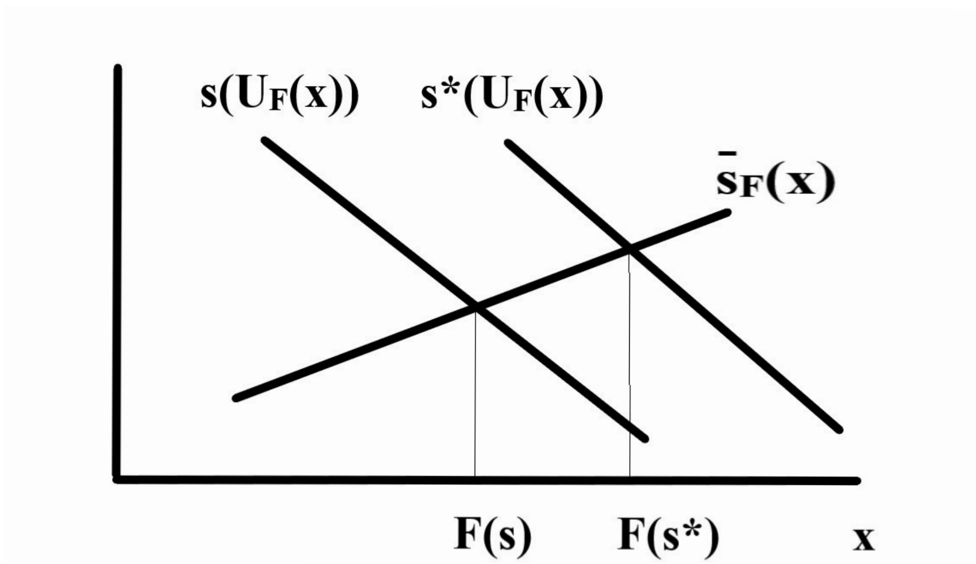

Figure 1: Part (i) of the MBV Property

the abscissa of this point, which we may think of as a "generalized point of intersection" . Part (i) of the MBV property gives a necessary condition for such a point, the following generalization of Corollary 3.3 a suffi cient condition.

Corollary 3.5 *If an anonymous social choice function F with range R_F is weakly group strategy-proof, then, for any $s \in \mathcal{M}^*_F(T)$ and any $x \in R_F$, $F(s) = x$ if one of the following applies:*

$$s(U_F(x')) > \bar{s}_F(x') \text{ for all } x' < x \text{ and } s(U_F(x)) < \bar{s}_F(x);$$

$$s(U_F(x')) > \bar{s}_F(x') \text{ for all } x' < x \text{ and } s(U_F(x)) = 0;$$

$$s(U_F(x')) = 1 \text{ for all } x' < x \text{ and } s(U_F(x)) < \bar{s}_F(x).$$

If neither (3.15) nor (3.16) are strictly monotonic, there may be multiple points of intersection. As shown in Figure 2, both functions (3.15) and (3.16) may be flat over some interval and any outcome x in this interval may be a solution to the equation $s(U_F(x)) = \bar{s}_F(x)$.[20] In this case, $F(s) = x$ can be any outcome in this interval. However, for this constellation, part (ii) of the MBV property implies that, if $F(s) = x$ is also a solution to the equation $\hat{s}(U_F(x)) = \bar{s}_F(x)$ for some distribution $\hat{s} \neq s$, then $F(\hat{s}) = \hat{x} \neq x$ and $F(\hat{s}) = \hat{x}$ implies $s(U_F(\hat{x})) \neq \bar{s}_F(\hat{x})$.

Figure 2: see next page

Implementation by Voting. The social choice function underlying Figure 1 can be implemented by asking people for each $x \in \mathbb{R}$ whether, starting from x, they would like to stay at x or whether they would like to move to a higher outcome. With honest voting in response to this question, people with types in $U_F(x)$ will indicate a preference for moving up and others will indicate that they prefer x to anything higher.

If the actual type distribution is $\hat{s}$, the outcomes of these votes will trace out the graph of the function (3.15) in the figure. The outcome $F(\hat{s}) = \hat{x}$

[20] This constellation might seem highly exceptional. However, a threshold function with the constant value $\bar{s}_F(x) = \frac{1}{2}$ merely embodies the majority rule, and, among the distributions in $\mathcal{M}^*_F(T)$ that must be considered, some will give rise to the constellation in Figure 3.

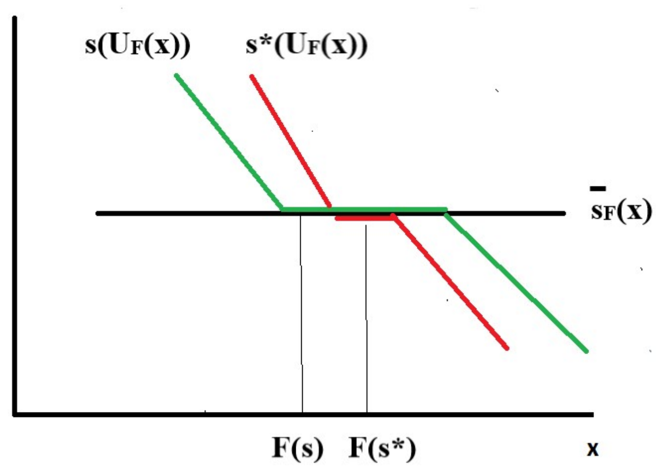

Figure 2: Part (ii) of the MBV Property

in the figure is thus implemented by a rule requiring an outcome x to be chosen, if, for all $x' < x$, the votes in favour of moving up from x' exceed the threshold $\bar{s}_F(x')$ and, at x itself, the votes in favour of moving further up from x do not exceed the threshold $\bar{s}_F(x)$.

This mechanism generalizes the voting mechanisms in Propositions 3.1 and 3.2. In those results, with discrete sets of outcomes, there are binary votes over neighbouring outcomes. In the general case, with a continuum of outcomes, one cannot generally specify "neighbours" any more but one can still have a multiplicity of linearly ordered binary votes on whether to move "up" or not to move "up" from x, one such vote for each $x \in R_F$, with a possibly distinct threshold $\bar{s}_F(x)$ for each x. Essentially the same argument as in the finite case implies that the thresholds must be non-decreasing in x.

A threshold function with the constant value $\bar{s}_F(x) = \frac{1}{2}$ for all x corresponds to *majority voting.* If $s(U_F(x')) > \frac{1}{2}$, a majority of participants wants to move up from x'; and if $s(U_F(x')) < \frac{1}{2}$, a majority of participants wants to move down from x'. The outcome x at which $s(U_F(x)) = \bar{s}_F(x) = \frac{1}{2}$ is exactly the one where there is no majority for moving up, and where there is no majority for moving down.

A Further Equivalence Theorem. I conclude the characterization of weak group strategy proofness with a further equivalence theorem.

Theorem 3.6 *An anonymous social choice function F with arbitrary range R_F is weakly group strategy-proof if and only if, for all s and $\hat{s}$ in $\mathcal{M}^*_F(T)$, $F(s) \neq F(\hat{s})$ implies*

$$s(P(F(s), F(\hat{s}))) > \hat{s}(P(F(s), F(\hat{s}))) \tag{3.18}$$

or

$$s(P(F(\hat{s}), F(s))) < \hat{s}(P(F(\hat{s}), F(s))). \tag{3.19}$$

Theorem 3.6 expresses a simple unifying principle that underlies weak strategy proofness: If two distributions s and $\hat{s}$ in $\mathcal{M}^*_F(T)$ give rise to different outcomes $F(s)$ and $F(\hat{s})$; then either the population share of the set of people who prefer $F(s)$ to $F(\hat{s})$ is strictly larger under s than under $\hat{s}$ or the population share of the set of people who prefer $F(\hat{s})$ to $F(s)$ is strictly larger under $\hat{s}$ than under s (or both). Thus, one might consider binary votes between all alternatives.

The condition in Theorem 3.6 is simpler to state than the condition in Theorem 3.4. However, it is also more abstract, and the MBV property is not apparent from Theorem 3.6.

The proofs of Theorem 3.4 and 3.6 involve showing that weak group strategy proofness implies the MBV property, the MBV property implies the condition given in Theorem 3.6, and the latter condition implies weak group strategy proofness.

4 Group Strategy Proofness

4.1 Group Strategy Proofness with Simple Tie-Breaking

Weak group strategy proofness is only concerned with the behaviour of a social choice function F on the set $\mathcal{M}_F^*(T)$ of distributions that are concentrated on the set T_F of types with preferences that are single-peaked on the range R_F of F. For a discussion of group strategy proofness, one must also consider the behaviour of a social choice function F on the set $\mathcal{M}(T)\backslash\mathcal{M}_F^*(T)$ that are indifferent between different (neighbouring) elements of R_F. In voting between neighbouring alternatives, such type distributions would give rise to substantial numbers of abstentions, so the question is how the social choice function deals with such a constellation.

One way to deal with such distributions is to treat the indifferent types as if they had a strict preference for one of the two alternatives between which they are indifferent. Barberà and Jackson (1994) refer to this operation as *tie-breaking.* Given a social choice function F with range R_F, a *simple tie-breaking function* for F is a mapping g_F from T into T such that, for any $t \in T$, if $u(\cdot, t)$ has twin peaks $\pi_F^1(t)$, $\pi_F^2(t)$ on R_F, then

$$u(\pi_F^i(t), g_F(t)) = u(\pi_F^i(t), t) + 1 \quad \text{for } i = 1 \ \text{ or } \ i = 2, \tag{4.1}$$

and

$$u(x, g_F(t)) = u(x, t) \quad \text{for all} \quad x \neq \pi_F^i(t); \tag{4.2}$$

in particular,

$$u(\pi_F^j(t), g_F(t)) = u(\pi_F^j(t), t) \quad \text{for } j \neq i. \tag{4.3}$$

If $u(\cdot, t)$ is single-peaked on R_F, then $g_F(t) = t$. Thus, regardless of whether $u(\cdot, t)$ is single-peaked or twin-peaked on R_F, $u(\cdot, g_F(t))$ is always single-peaked on R_F; moreover, the peak $\pi_F(g_F(t))$ of $u(\cdot, g_F(t))$ is also a peak of $u(\cdot, t)$. Except for the fact that $u(\cdot, g_F(t))$ induces a strict ordering on the set $\{\pi_F^1(t), \pi_F^2(t)\}$, the utility functions $u(\cdot, g_F(t))$ and $u(\cdot, t)$ induce the same ordering on R_F.

The following result shows that, if the behaviour of F on $\mathcal{M}(T)\backslash\mathcal{M}_F^*(T)$ is given by applying a simple tie-breaking function g_F to the type distributions $s \in \mathcal{M}(T)\backslash\mathcal{M}_F^*(T)$, then the conditions for weak group strategy proofness in Theorems 3.4 and 3.6 are also necessary and sufficient for group strategy proofness.

Theorem 4.1 *Let F be an anonymous social choice function with range R_F and assume that, for some simple tie-breaking function g_F, $F(s) = F(s \circ g_F^{-1})$ for all $s \in \mathcal{M}(T)$. Then the following statements are equivalent:*

(a) F is group strategy-proof.

(b) F has the MBV property.

(c) For all s and $\hat{s}$ in $\mathcal{M}_F^*(T)$, $F(s) \neq F(\hat{s})$ implies

$$s(P(F(s), F(\hat{s}))) \quad > \quad \hat{s}(P(F(s), F(\hat{s}))) \tag{4.4}$$

$$or$$

$$s(P(F(\hat{s}), F(s))) \quad < \quad \hat{s}(P(F(\hat{s}), F(s))). \tag{4.5}$$

The interpretation of this result in terms of implementability by voting is the same as before, with one difference. If $R_F \cap (\bar{x}, x) = \emptyset$, i.e. if R_F has a gap between $\bar{x}$ and x, neither voting on whether to move "down" from x nor voting on whether to move "up" from $\bar{x}$ by itself provides all the information that is needed to choose between x and $\bar{x}$. One needs both votes in order to infer who is indifferent between the two alternatives so that one can apply the tie-breaking operation.

4.2 Group Strategy Proofness with Contingent Tie-Breaking

Simple tie breaking is very rigid. Any type with twin peaks x and $\hat{x}$ in R_F is treated as if the peak was either x or $\hat{x}$. This leaves no room for the notion that the behaviour of such a type involves some arbitrariness.

I therefore consider group strategy-proof social choice without assuming anything about tie-breaking. As it turns out, even if no tie-breaking function is explicitly specified, if the social choice function is group strategy-proof, there is no loss of generality in assuming that some form of tie-breaking occurs. However, the tie-breaking is contingent, rather than simple.

For given F and R_F, a *contingent-tie-breaking function* is a function g_F from $T \times \mathcal{M}(T)$ to T such that, for any given $s \in \mathcal{M}(T)$, the function $g_F(\cdot, s)$

is a simple tie-breaking function. Thus, with contingent tie-breaking, the tie-break itself depends on the overall type distribution s.

Proposition 4.2 *Let F be an anonymous social choice function F with range R_F. Let $\Pi_F(t)$ be the set of peaks of type t on R_F and define a contingent tie-breaking function g_F^*,such that, for all $t \in T$ and all $s \in \mathcal{M}(T)$,*

$$\pi_F(g_F^*(t,s)) = \min \Pi_F(t) \quad \text{if} \quad F(s) \leq \min Pi_F(t) \tag{4.6}$$
and
$$\pi_F(g_F^*(t,s)) = \max \Pi_F(t) \quad \text{if} \quad F(s) \geq \max \Pi_F(t). \tag{4.7}$$

If F is group strategy-proof, then for all $s \in \mathcal{M}(T)$,

$$F(G(s, g_F^*)) = F(s), \tag{4.8}$$

where

$$G(s, g_F^*) = s \circ g_F^*(\cdot, s)^{-1} \in \mathcal{M}_F^*(T) \tag{4.9}$$

is the type distribution that results when the original type distribution is s and each type $t \in T$ is replaced by $g_F^*(t,s)$.

The contingent tie-breaking function in Proposition 4.2 takes a very simple form: For any t and s, if $u(\cdot,t)$ has twin peaks on R_F, the type $g_F^*(t,s)$ that replaces t has a single peak on R_F that is equal to the peak of $u(\cdot,t)$ that is closest to $F(s)$.[21] If this operation were to change the outcome, i.e., if (4.8) were violated somewhere, there must exist a distribution s and a set of types that can block F at s.

With contingent tie breaking, for any gap $(x, \hat{x})$ in R_F, any tie-break between x and $\hat{x}$ depends on whether $F(s) = x$ or $F(s) = \hat{x}$. An analysis of group strategy proofness must therefore take account of the possibility that coalitions of agents with strict preferences between neighbouring alternatives might manipulate social choice by manipulating tie-breaks. To some extent, this can be done by extending the conditions of Theorems 3.4 and 3.6 from $\mathcal{M}_F^*(T)$ to $\mathcal{M}(T)$.

Even then, however, the interdependence of tie breaking and social choice makes room for some arbitrariness. For example, $F(s)$ might depend on the behaviour of s on the set of types that are indifferent between x and $\hat{x}$. For

[21] Barberà and Jackson (1994) also work with this tie-breaking function. However, given their focus on individual strategy proofness, their use of this function is somewhat different from ours.

some type distributions s, s' that differ on this set but are otherwise the same, we might therefore have $F(s) = x$ and $F(s') = \hat{x}$.

Such a dependence is problematic because s and $\hat{s}$ induce exactly the same distributions of prefence orderings on R_F. To avoid this possibility, we restrict our analysis to social choice functions that do not exhibit this kind of dependence.[22] Intuitively, ties are broken so that the individuals' preferences are as closely as possible aligned with the intended social choice.

For anonymous social choice functions exhibiting a certain independence property, one obtains the following equivalence theorem.

Theorem 4.3 *Let F be an anonymous social choice function with range R_F and assume that $F(s) = F(s')$ whenever s and s' induce the same distributions on the space of preference orderings on R_F. Then the following statements are equivalent:*

(a) F is group strategy-proof.

(b) F has the MBV property. Moreover, for any s and $\hat{s}$ in $\mathcal{M}(T)$ and any x and $\hat{x} > x$ in R_F such that $F(s) = x$, $F(\hat{s}) = \hat{x}$, and $R_F \cap (x, \hat{x}) = \emptyset$,

$$\hat{s}(U_F(x)) > s(U_F^+(x)) \quad or \quad s(D_F(\hat{x})) > \hat{s}(D_F(\hat{x})).$$

(c) For all s and $\hat{s}$ in $\mathcal{M}(T)$, $F(s) \neq F(\hat{s})$ implies

$$s(P(F(s), F(\hat{s}))) > \hat{s}(P(F(s), F(\hat{s}))) \tag{4.10}$$

$$or$$

$$s(P(F(\hat{s}), F(s))) < \hat{s}(P(F(\hat{s}), F(s))). \tag{4.11}$$

Theorem 4.3 differs from Theorem 4.1 in having stronger versions of statements (b) and (c). In particular, statement (b) requires that condition (ii) in the definition of the MBV property holds on all of $\mathcal{M}(T)$, not just on $\mathcal{M}_F^*(T)$; statement (c) must also hold on all of $\mathcal{M}(T)$.

[22] The problem would disappear if one defined blocking in terms of *weak* Pareto improvements, so that the collective manipulation makes a positive mass of group members better off without making any group member worse off. However, with such a weakening of the conditions for blocking, group strategy proofness would be unattainable, except for social choice functions that are constant. For example, in the case $n = 2$, with any social choice function F with threshold $\bar{s}_F^2 \in (0,1)$ would be (weakly) blocked at any s satisfying $s(W_F^+(x_1)) < \bar{s}_F^2 < s(W_F^+(x_1)) + s(V_F(x_1, x_2))$: If $F(s)$ were equal to x_1, F would be blocked by the group of agents with types in $W_F^+(x_1) \cup V_F(x_1, x_2)$ all claiming to have types in $W_F^+(x_1)$; if $F(s)$ were equal to x_2, F would be blocked by the group of agents with types in $W_F^-(x_2) \cup V_F(x_1, x_2)$ all claiming to have types in $W_F^-(x_2)$.

The proof of Theorem 4.3 uses Proposition 4.2 to exploit the properties of the specified contingent tie-breaking function. The additional independence property in the statement of the theorem serves to prove that condition (ii) in the definition of the MBV property must hold for measures in $\mathcal{M}(T)\backslash\mathcal{M}_F^*(T)$, as well as measures in $\mathcal{M}_F^*(T)$.[23]

5 Group Strategy Proofness, Voting and Welfare

As mentioned in the introduction, economists tend to be critical of voting as a device for allocating resources. They consider voting to be too coarse a device to allow the transmission of all the information that is needed for welfare maximization. In particular, in any binary decision, the outcome of social choice based on voting can only depend on the numbers of votes in favour of the different alternatives, without any consideration of preference intensities.

The preceding analysis suggests that this criticism must be refined in several respects. First, reliance on voting as a device for allocating resources may be mandated by considerations of group strategy proofness. In this case, the criticism that choices based on voting are inefficient because they neglect preference intensities is misplaced because the presumed inefficiency is merely the consequence of additional constraints imposed by group strategy proofness. Given these constraints, social choice may well be constrained-efficient.

Second, voting mechanisms as such need *not* be too coarse to allow the transmission of all the information that is needed for welfare maximization. Theorems 4.3 and 4.1 allow for a continuum of binary votes. Given these votes, the implementation mechanism is fully informed about the functions $x \to s(U_F(x))$ and $x \to s(D_F(x))$ that indicate for each outcome x the share of people in the population who would prefer an outcome higher than x and the share of people in the population who would prefer an outcome lower than x. The information contained in these functions is usually rich enough to allow for social choice to condition on preference intensities.

[23] As mentioned above, with $R_F \cap (x, \hat{x}) = \emptyset$, without the additional independence property, we might have $F(s) = x$ and $F(s') = \hat{x} > x$ even though or s and s' differ only on the set of types that are indifferent between x and $\hat{x}$. Even then, group strategy proofness imposes additional restrictions, for example, that types in $U_F(x)$ and types in $U_F(\hat{x})$ must not have an incentive to feign indifference. The logic of these restrictions is fairly clear, but they do not seem worth spelling out.

For example, let T be a subset of $\mathbb{R}$ and suppose that the utility function takes the quasi-linear form

$$u(x,t) = t \cdot \hat{u}(x) - k(x), \tag{5.12}$$

where $\hat{u}(\cdot)$ is non-negative-valued, strictly increasing and strictly concave and $k(\cdot)$ is non-negative-valued, strictly increasing and convex. In this case, an agent with type t prefers an outcome higher than x if $t \cdot \hat{u}'(x) - k'(x) > 0$. If $R_F = \mathbb{R}$, the set $U_F(x)$ is given by the interval $(\gamma(x), \infty)$, where

$$\gamma(x) = \frac{k'(x)}{\hat{u}'(x)} > 0. \tag{5.13}$$

Suppose that the share

$$\sigma^+(x) := s(U_F(x))$$

of people preferring an outcome higher than x is known for all $x \in \mathbb{R}$. From this information, the type distribution s is obtained by setting

$$s((-\infty, t]) = 1 - \sigma^+(\gamma^{-1}(t)). \tag{5.14}$$

The voting mechanisms considered in Theorems 4.1 and 4.3 are thus rich enough to convey all the information one would need to maximize a measure of welfare such as aggregate surplus. The coarseness of information transmission that is traditionally criticized is *not* due to the reliance on voting as such but due to the reliance on overly simple forms of voting, e.g. binary voting.

Third, even if voting is sufficiently rich to provide the social planner with complete information about the type distribution, the constraints imposed by group strategy proofness may still prevent the maximization of whatever welfare objective is being pursued. For example, in the quasi-linear specification (5.12), consider the problem of maximizing the aggregate surplus

$$\int_T t \, ds(t) \cdot \hat{u}(x) - k(x) \tag{5.15}$$

for each s. Under the given assumptions, for any s, the social choice $F(s)$ maximizes this objective if and only if

$$\bar{t}(s) = \gamma(F(s)), \tag{5.16}$$

where

$$\bar{t}(s) := \int_T t \, ds(t) \tag{5.17}$$

is the cross-section average of types, and $\gamma(x)$ if given by (5.13), as before. Whereas voting makes social choice depend on s through the population shares of the sets $(\gamma(x), \infty)$ for different x, condition (5.16) makes social choice depend on s through the mean $\bar{t}(s)$. The two approaches are not generally compatible.

The two approaches would be compatible if somehow the social planner "knew" that s is a symmetric distribution so that the median coincides with the mean. In this case, surplus-maximizing outcomes can in fact be implemented by majority voting, i.e., by setting

$$\bar{s}_F(x) = \frac{1}{2} \tag{5.18}$$

for all x. Given the threshold function (5.18), the outcome $F(s)$ that is chosen when the type distribution is s satisfies

$$s(W_F^+(F(s))) = s((\gamma(F(s)), \infty)) = \frac{1}{2}, \tag{5.19}$$

which is true if and only if the median of the type distribution is equal to $\gamma(x)$. If the median is equal to the mean, the surplus maximization condition (5.16) holds automatically.

Without symmetry, the median and the mean of the type distribution can differ so that majority voting is not suitable for implementing surplus-maximizing outcomes. For example, if the type distribution s is concentrated on an interval $[t_1, t_2]$ and if it has a density $\frac{2}{(t_2-t_1)^2}(t-t_1)$ on this interval, the distribution is left-skewed and the mean is $\bar{t}(s) = \frac{1}{3}t_1 + \frac{2}{3}t_2$, which is less than the median, $(1 - \frac{1}{2^{1/2}})t_1 + \frac{1}{2^{1/2}}t_2$. For such distributions, surplus-maximizing outcomes can be implemented by setting $\bar{s}_F(x) = \frac{5}{9}$ for all x, i.e. by requiring more than a majority of votes for going to higher outcomes. If instead the density of s is $\frac{2}{(t_2-t_1)^2}(t_2 - t)$ on the interval $[t_1, t_2]$, the distribution is right-skewed, the mean is $\bar{t}(s) = \frac{2}{3}t_1 + \frac{1}{3}t_2$, which is greater than the median, and surplus-maximizing outcomes can be implemented by setting $\bar{s}_F(x) = \frac{4}{9}$ for all x, requiring less than a majority of votes for going to higher outcomes. More generally, if s is left-skewed with an isoelastic density $cr(t - t_1)^{r-1}$ on $[t_1, t_2]$, surplus-maximizing outcomes can be implemented by setting $\bar{s}_F(x) = 1 - \left(\frac{r}{r+1}\right)^r > \frac{1}{2}$ for all x; if s is right-skewed with an isoelastic density $cr(t_2 - t)^{r-1}$ on $[t_1, t_2]$, surplus-maximizing outcomes can be implemented by setting $\bar{s}_F(x) = \left(\frac{r}{r+1}\right)^r < \frac{1}{2}$ for all x.

These considerations show that, *for some classes of type distributions*, one can use voting, even with a threshold $\bar{s}_F(x)$ that is independent of x, to implement surplus-maximizing outcomes for all distributions in the given class. However, they also show that the requisite thresholds depend on the specified class. No one threshold function $\bar{s}_F(\cdot)$ can be used to implement surplus-maximizing outcomes for all $s \in \mathcal{M}(T)$ or even all $s \in \mathcal{M}^*_F(T)$. *There is therefore no group strategy-proof social choice function that implements surplus-maximizing outcomes for all type distributions.* The problem is *not* that the requisite information cannot be obtained but that the use of this information for surplus maximization is incompatible with group strategy proofness.

Social choice with a view to maximizing aggregate surplus subject to group strategy proofness cannot be addressed pointwise, by maximizing (5.15) for each s. This choice must be considered from an *ex ante* perspective, *before* the type distribution is known. From this *ex ante* perspective, the social planner must assign weights to the different type distributions that can arise, so the objective function might take the form

$$\int_{\mathcal{M}(T)} [\bar{t}(s)\hat{u}(F(s)) - k(F(s))] \, dP(s), \tag{5.20}$$

where P is a measure on $\mathcal{M}(T)$. The choice of the function $F(\cdot)$ is constrained by the conditions listed in Theorem 3.4 - 4.3. Among these conditions is the requirement that, for any $s \in \mathcal{M}^*_F(T)$, the share $s(U_F(F(s)))$ of people who want to move up from $F(s)$ must belong to the interval $[\bar{s}_F(F(s)-)), \bar{s}_F(F(s))]$ for *some non-decreasing threshold function* $\bar{s}_F(\cdot)$.[24]

These considerations point to a problem of commitment in implementation. If group strategy proofness precludes the maximization of aggregate surplus at all type distributions but in the process of implementation, the requisite information about these distribution becomes available, the question is why, *ex post* there should not be a "correction" to shift the chosen outcome from $F(s)$ to a surplus-maximizing outcome. A simple answer to this question might be that the initial commitment to the social choice function $F(\cdot)$ is fully binding. But that merely begs the question why this commitment is binding. If commitment powers are in fact limited, the analysis must move from a normal-form revelation game to an extensive form treatment in which sequential rationality is modelled explicitly.

[24]This monotonicity requirement can be weakened by introducing "gaps" in R_F so that the set on which the threshold function must be nondecreasing is smaller. There may therefore be a tradeoff between the fineness of calibration of outcomes as determined by the "size" of R_F and the stringency of having thresholds be monotone on R_F.

A Appendix: Proofs

A.1 Preliminaries

Throughout this appendix, I assume without further mention that preference orderings on $\mathbb{R}$ are single-peaked, that T is rich, and that F is an anonymous weakly group strategy-proof social choice function with range R_F.

I also impose the following notational conventions. First, for any $s \in \mathcal{M}(T)$ and any Borel set $B \subset T$, s_B is the (non-normalized) measure on T that coincides with s on B and that assigns the measure zero to the set $T \backslash B$. Thus, trivially, $s = s_B + s_{T \backslash B}$. Moreover, for any two measures s and s' in $\mathcal{M}_F^*(T)$ and any Borel set $B \subset T$ such that $s(B) = s'(B)$, the measures $s_B + s'_{T \backslash B}$ and $s'_B + s_{T \backslash B}$ also belong to $\mathcal{M}_F^*(T)$.

Second, for any $t \in T$, I write δ_t for the element of $\mathcal{M}(T)$ that assigns all mass to the singleton $\{t\}$. Notice that δ_t is a probability measure. Thus, if the true type distribution is s and all agents with types in a set B claim to have the type t, the distribution of reported types will be $s(B) \cdot \delta_t + s_{T \backslash B}$.

I begin the formal analysis with two preliminary lemmas.

Lemma A.1 *For any $s \in \mathcal{M}_F^*(T)$ and any $x \in R_F$, $s(D_F(x)) = 0$ implies $F(s) \geq x$, and $s(U_F(x)) = 0$ implies $F(s) \leq x$. In particular, $s(P_F^*(x)) = 1$ implies $F(s) = x$.*

Proof. Suppose that, contrary to the first statement of the lemma, there exist $s \in \mathcal{M}_F^*(T)$ and $x \in R_F$ such that $s(D_F(x)) = 0$ and $F(s) < x$. Since $x \in R_F$ and, by the definition of weak group strategy proofness, $R_{F|\mathcal{M}_F^*(T)} = R_F$, there exists $s' \in \mathcal{M}_F^*(T)$ such that $F(s') = x$. Since agents with types in $T \backslash D_F(x)$ all prefer x to $F(s)$, a coalition of these agents can block F at s by announcing s' rather than s, thus inducing the outcome $F(s') = x$ rather than $F(s) < x$.

The proof of the second statement is completely symmetric and is left to the reader. The third statement follows immediately. ■

Lemma A.2 *Given F and given any x and $\bar{x}$ in R_F with $x \leq \bar{x}$, there exists an anonymous weakly group strategy-proof social choice function $\hat{F}$ taking values in $R_F \cap [x, \bar{x}]$ such that, for any $s \in \mathcal{M}_F^*(T)$, if $F(s) \in [x, \bar{x}]$, then $\hat{F}(s) = F(s)$.*

Proof. I first prove that, for any $x \in R_F$, there exists an anonymous group strategy-proof social choice function F' taking values in $R_F \cap [x, \infty)$ such that, for any $s \in \mathcal{M}^*_F(T)$, if $F(s) \geq x$, then $F'(s) = F(s)$. For this purpose, I define a mapping $s'(\cdot)$ by setting

$$s'(s) = s \text{ if } F(s) \geq x \tag{A.1}$$

and

$$s'(s) = s_{U_F(x)} + s_{P^*_F(x)} + s(D_F(x)) \cdot \delta_{t'} \text{ if } F(s) < x \tag{A.2}$$

where t' is an arbitrary fixed element of $P^*_F(x)$. Notice that $s'(s) \in \mathcal{M}^*_F(T)$ whenever $s \in \mathcal{M}^*_F(T)$, i.e., the mapping s' maps $\mathcal{M}^*_F(T)$ into itself.

Given the mapping $s'(\cdot)$, I define a new social choice function F' by setting

$$F'(s) = F(s'(s)) \tag{A.3}$$

for any s. Trivially then, $F(s) \geq x$ implies $F'(s) = F(s) \in R_F \cap [x, \infty)$.

If $F(s) < x$, then by (A.2) $s'(D_F(x)|s) = 0$, so Lemma 1 and the weak group strategy proofness of F yield $F(s'(s)) \geq x$. By (A.3), it follows that $F(s) \in R_F \cap [x, \infty)$ when $F(s) < x$, as well as $F(s) \geq x$.

I claim that F' inherits weak group strategy proofness from F. If this claim is false, there exist a Borel set $B \subset T$ and a measure $s \in \mathcal{M}^*_F(T)$ such that B blocks F' at s by B. Let $\hat{s} \in \mathcal{M}^*_F(T)$, with $\hat{s}_{T \setminus B} = s_{T \setminus B}$, be the distribution of types that is induced by the collective deviation of agents with types in B. Then $F'(\hat{s}) \neq F'(s)$, and all types in B prefer $F'(\hat{s})$ to $F'(s)$, i.e. we have

$$u(F'(\hat{s}), t) > u(F'(s), t), \tag{A.4}$$

and therefore

$$u(F(s'(\hat{s})), t) > u(F(s'(s)), t) \tag{A.5}$$

for all $t \in B$.

If $F'(\hat{s}) > F'(s)$, B must be a subset of $U_F(F'(s))$, which in turn is a subset of $U_F(x)$, so (A.1) and (A.2) imply $s'(s)_B = s_B$. Therefore, if the true distribution of types is $s'(s)$ and the group of agents with types in B coordinates their reports so as to make their types appear to be distributed as $\hat{s}_B$ rather than $s'(s)_B = s_B$, the overall distribution of reported types will be

$$\hat{s}_B + s'(s)_{T \setminus B} = s'(\hat{s}_B + s_{T \setminus B}) = s'(\hat{s}),$$

so that the outcome under F is $F(s'(\hat{s})) = F'(\hat{s})$, which all agents with types in B prefer to the outcome $F(s'(s)) = \hat{F}(s)$. In this case, F is blocked by B at $s'(s)$. This conclusion is incompatible with the assumption that F is weakly group strategy-proof.

Alternatively, if $F'(\hat{s}) < F'(s)$, let $\hat{B} \supset B$ be the set of types that prefer $F'(\hat{s})$ to $F'(s)$. Since $F'(\hat{s}) \geq x$, we have $\hat{B} \supset P_F^*(x) \cup D_F(x)$ and $T \backslash \hat{B} \subset U_F(x)$, so (A.2) implies that $s'(s)_{T \backslash \hat{B}} = s_{T \backslash \hat{B}}$. If the true distribution of types is $s'(s)$ and the group of agents with types in $\hat{B}$ coordinates reports so as to make their types appear to be distributed as $s'(\hat{s})_{\hat{B}}$ rather than $s'(s)_{\hat{B}}$, the overall distribution of reported types will be

$$s_{T \backslash \hat{B}} + s'(\hat{s})_{\hat{B}} = \hat{s}_{T \backslash \hat{B}} + s'(\hat{s})_{\hat{B}} = s'(\hat{s})_{T \backslash \hat{B}} + s'(\hat{s})_{\hat{B}} = s'(\hat{s}),$$

so that the outcome under F is $F(s'(\hat{s})) = F'(\hat{s})$, which all agents with types in $\hat{B}$ prefer to the outcome $F(s'(s)) = F'(s)$. In this case, F is blocked by $\hat{B}$ at $s'(s)$. This conclusion is again incompatible with the assumption that F is weakly group strategy-proof.

By a precisely symmetric argument, which is left to the reader, one can also show that, for any $\bar{x} \geq x$, there also exists an anonymous, weakly group strategy-proof social choice function $\hat{F}$ that takes values in $R_{F'} \cap (-\infty, \bar{x}]$ and satisfies $\hat{F}(s) = F'(s)$ whenever $F'(s) \leq \bar{x}$. Since $R_{F'} = R_F \cap [x, \infty)$, $\hat{F}$ actually takes values in $R_F \cap [x, \bar{x}]$ and satisfies $\hat{F}(s) = F(s)$ whenever $F(s) \in [x, \bar{x}]$, as claimed in the lemma. ∎

A.2 Key Lemmas

The next few lemmas contain the core of the arguments for the MBV Property. I recall that, for any two outcomes x_1, x_2,

$$P(x_1, x_2) = \{t \in T | u(x_1, t) > u(x_2, t)\}$$

is the set of types with a strict preference for x_1 over x_2. I also write $I(x_1, x_2)$ for the set of types that are indifferent between x_1 and x_2.

Lemma A.3 *Let F, let $F(s) = x$ and $F(\hat{s}) = \hat{x}$ for some s and $\hat{s}$ in $\mathcal{M}_F^*(T)$ and some x and $\hat{x} \neq x$ in R_F. If $s(I(x, \hat{x})) = \hat{s}(I(x, \hat{x})) = 0$, then $s(P(\hat{x}, x)) < \hat{s}(P(\hat{x}, x))$ and $s(P(x, \hat{x})) > \hat{s}(P(x, \hat{x}))$.*

Proof. Since $s(I(x, \hat{x})) = \hat{s}(I(x, \hat{x})) = 0$, the two claims in the lemma are actually equivalent. Thus it suffices to prove that, under the indicated conditions, $s(P(\hat{x}, x)) < \hat{s}(P(\hat{x}, x))$.

If this claim is false, there exist s and $\hat{s}$ in $\mathcal{M}_F^*(T)$ and x and $\hat{x}$ in R_F such that $s(I(x, \hat{x})) = \hat{s}(I(x, \hat{x})) = 0$, $F(s) = x$, $F(\hat{s}) = \hat{x}$, and $s(P(\hat{x}, x)) \geq \hat{s}(P(\hat{x}, x))$. There is no loss of generality in assuming that $x < \hat{x}$. By Lemma

A.2, there is also no loss of generality in assuming that $R_F \subset [x, \hat{x}]$, i.e. that F admits no outcomes smaller than x or larger than $\hat{x}$.

I first consider the case $s(P(\hat{x}, x)) = \hat{s}(P(\hat{x}, x))$. Given that $s(I(x, \hat{x})) = \hat{s}(I(x, \hat{x})) = 0$, in this case, the formula

$$s^* = s_{P(x,\hat{x})} + \hat{s}_{P(\hat{x},x)}$$

defines a further element of $\mathcal{M}_F^*(T)$. Consider the value of F at s^*. If $F(s^*) \in [x, \hat{x})$, then $P(x, \hat{x})$ blocks F at $\hat{s}$: If the true type distribution is $\hat{s}$, the group of agents with types in $P(x, \hat{x})$ can coordinate their reports so as to make it appear as if their types were distributed as $s_{P(x,\hat{x})}$. The distribution of reported types then is s^*, inducing the outcome $F(s^*) \in [x, \hat{x})$, which all group members prefer to $F(\hat{s}) = \hat{x}$. Alternatively, if $F(s^*) = \hat{x}$, then $P(\hat{x}, x)$ blocks F at s: If the true type distribution is s, the group of agents with types in $P(\hat{x}, x)$ can coordinate their reports so as to make it appear as if their types were distributed as $\hat{s}_{P(\hat{x},x)}$. The distribution of reported types then is s^*, inducing the outcome $F(s^*) = \hat{x}$, which all group members prefer to $F(s) = x$. In either case, if $F(s^*) \in [x, \hat{x})$ and if $F(s^*) = \hat{x}$, the assumption that $s(I(x, \hat{x})) = \hat{s}(I(x, \hat{x})) = 0$, $F(s) = x$, $F(\hat{s}) = \hat{x}$, and $s(P(\hat{x}, x)) = \hat{s}(P(\hat{x}, x))$ thus leads to a contradiction and must be false.

I next consider the case $s(P(\hat{x}, x)) > \hat{s}(P(\hat{x}, x))$. In this case, for any $\hat{t} \in T$, the formula

$$s^* = s_{P(x,\hat{x})} + \hat{s}_{P(\hat{x},x)} + (s(P(\hat{x}, x)) - \hat{s}(P(\hat{x}, x)) \cdot \delta_{\hat{t}}$$

defines a further element of $\mathcal{M}_F^*(T)$. If $\hat{t} \notin P(\hat{x}, x)$, one has $s^*(P(\hat{x}, x)) = \hat{s}(P(\hat{x}, x))$ so, by the conclusion of the preceding paragraph, $F(s^*) = F(\hat{s}) = \hat{x}$. But then $P(\hat{x}, x)$ blocks F at s: If the true type distribution is s, the group of agents with types in $P(\hat{x}, x)$ can coordinate their reports so as to make it appear as if their types were distributed as $\hat{s}_{P(\hat{x},x)} + (s(P(\hat{x}, x)) - \hat{s}(P(\hat{x}, x)) \cdot \delta_{\hat{t}}$. The distribution of reported types then is s^*, inducing the outcome $F(s^*) = \hat{x}$, which all group members prefer to $F(s) = x$. The assumption that $s(I(x, \hat{x})) = \hat{s}(I(x, \hat{x})) = 0$, $F(s) = x$, $F(\hat{s}) = \hat{x}$, and $s(P(\hat{x}, x)) > \hat{s}(P(\hat{x}, x))$ thus leads to a contradiction and must be false. ■

Lemma A.4 *Given F, let $F(s) = x$ and $F(\hat{s}) = \hat{x}$ for some s and $\hat{s}$ in $\mathcal{M}_F^*(T)$ and some x and $\hat{x} \neq x$ in R_F. Then $s(P(\hat{x}, x)) < \hat{s}(P(\hat{x}, x)) + \hat{s}(I(\hat{x}, x))$ and $\hat{s}(P(x, \hat{x})) < s(P(x, \hat{x})) + s(I(x, \hat{x}))$.*

Proof. If $\hat{s}(I(\hat{x}, x)) = s(I(x, \hat{x})) = 0$, the lemma is trivially implied by Lemma A.3. Suppose therefore that $\hat{s}(I(\hat{x}, x)) > 0$ and/or $s(I(x, \hat{x})) > 0$.

As in the proof of the preceding lemma, there is no loss of generality in assuming that $x < \hat{x}$ and $R_F \subset [x, \hat{x}]$, i.e., that F admits no outcomes smaller than x or larger than $\hat{x}$. Let s^* and $\hat{s}^*$ be given as

$$s^* = s_{T \setminus I(x,\hat{x})} + s(I(x,\hat{x})) \cdot \delta_t$$

and

$$\hat{s}^* = \hat{s}_{T \setminus I(\hat{x},x)} + \hat{s}(I(\hat{x},x)) \cdot \delta_{\hat{t}},$$

where t and $\hat{t}$ are arbitrary elements of $P^*(x)$ and $P^*(\hat{x})$. I claim that $F(s^*) = x$ and $F(\hat{s}^*) = \hat{x}$ so that Lemma A.3 implies $s^*(P(\hat{x},x)) < \hat{s}^*(P(\hat{x},x))$ and $s^*(P(x,\hat{x})) > \hat{s}^*(P(x,\hat{x}))$. Lemma A.4 then follows from the observation that, by construction,

$$s^*(P(\hat{x},x)) = s(P(\hat{x},x)) \;\text{ and }\; \hat{s}^*(P(\hat{x},x)) = \hat{s}(P(\hat{x},x)) + \hat{s}(I(\hat{x},x)),$$

as well as

$$s^*(P(x,\hat{x})) = s(P(x,\hat{x})) + s(I(x,\hat{x})) \;\text{ and }\; \hat{s}^*(P(x,\hat{x})) = \hat{s}(P(x,\hat{x})).$$

To prove that $F(\hat{s}^*) = \hat{x}$, I note that, with $R_F \subset [x, \hat{x}]$, $F(\hat{s}^*) \neq \hat{x}$ would imply that $\hat{s}(I(\hat{x},x)) > 0$ and $F(\hat{s}^*) < \hat{x}$. But then the singleton set $\{\hat{t}\}$ can block F at $\hat{s}^*$: If the true type distribution is $\hat{s}^*$ and the agents with type $\hat{t}$ coordinate their reports to make it appear as if their types were distributed as $\hat{s}_{I(\hat{x},x)} + \hat{s}_{\{\hat{t}\}}$, the overall distribution of reported types will be $\hat{s}$, inducing the outcome $F(\hat{s}) = \hat{x}$, which they all prefer to $F(\hat{s}^*) < \hat{x}$. The assumption that $F(\hat{s}^*) \neq \hat{x}$ is thus incompatible with the weak group strategy proofness of F, which proves that $F(\hat{s}^*) = \hat{x}$. The proof that $F(s^*) = x$ uses a precisely symmetric argument, which is left to the reader. ■

Lemma A.5 *Given F, for any $x \in R_F$, there exists $\bar{s}_F(x) \in [0,1]$ such that, for any $s \in \mathcal{M}_F^*(T)$,*

$$F(s) = x \;\textit{ implies }\; s(U_F(x)) \leq \bar{s}_F(x) \tag{A.6}$$

and

$$F(s) > x \;\textit{ implies }\; s(U_F(x)) \geq \bar{s}_F(x). \tag{A.7}$$

Proof. By Lemma A.4, for any $x \in R_F$ and any $s \in \mathcal{M}_F^*(T)$, $F(s) = x$ implies

$$s(P(\hat{x},x)) < \hat{s}(P(\hat{x},x)) + \hat{s}(I(\hat{x},x)) \tag{A.8}$$

for all $\hat{x} \in R_F \backslash \{x\}$ and all $\hat{s} \in \mathcal{M}_F^*(T)$ such that $F(\hat{s}) = \hat{x}$. If $\hat{x} > x$ and $R_F \cap (x, \hat{x}) = \emptyset$, we have $U_F(x) = P(\hat{x}, x)$ and, for $\hat{s} \in \mathcal{M}_F^*(T)$, $\hat{s}(I(\hat{x}, x)) = 0$, so (A.8) implies $s(P(\hat{x}, x)) < \hat{s}(U_F(x))$. Alternatively, if $\hat{x} > x$ and $R_F \cap (x, \hat{x}) \neq \emptyset$, i.e., if there exists $x' \in R_F \cap (x, \hat{x})$, we have $P(\hat{x}, x) \cup I(\hat{x}, x) \subset P(x', x) \subset U_F(x)$, so (A.8) implies

$$s(P(\hat{x}, x)) < \hat{s}(P(x', x)) \leq \hat{s}(U_F(x)).$$

In either case, if $\hat{x} > x$ and $R_F \cap (x, \hat{x}) = \emptyset$ and if $\hat{x} > x$ and $R_F \cap (x, \hat{x}) \neq \emptyset$, we find that

$$s(P(\hat{x}, x)) < \hat{s}(U_F(x)) \tag{A.9}$$

whenever $\hat{s} \in \mathcal{M}_F^*(T)$ is such that $F(\hat{s}) = \hat{x}$.

By the single-peakedness of preferences, for any $s \in \mathcal{M}_F^*(T)$, one also has

$$s(P(\hat{x}, x)) \leq s(U_F(x)) \tag{A.10}$$

for all $\hat{x} \in R_F \cap (x, \infty)$ and

$$s(U_F(x)) = \sup_{\hat{x} \in R_F \cap (x, \infty)} s(P(\hat{x}, x)). \tag{A.11}$$

Upon combining (A.9) - (A.11), one obtains

$$s(U_F(x)) \leq \hat{s}(U_F(x)) \tag{A.12}$$

for all $s \in \mathcal{M}_F^*(T)$ and all $\hat{s} \in \mathcal{M}_F^*(T)$ such that $F(s) = x$ and $F(\hat{s}) = \hat{x} > x$.

To complete the proof of the lemma, define $\bar{s}_F(x)$ as the supremum of $s(U_F(x))$ over the set of measures $s \in \mathcal{M}_F^*(T)$ such that $F(s) = x$. With this definition, trivially, $s(U_F(x)) \leq \bar{s}_F(x)$ for all $s \in \mathcal{M}_F^*(T)$ such that $F(s) = x$. Moreover, if $F(\hat{s}) = \hat{x}$ for some $\hat{s} \in \mathcal{M}_F^*(T)$ and some $\hat{x} > x$, one cannot have $\hat{s}(U_F(x)) < \bar{s}_F(x)$ since otherwise there would be a contradiction to (A.12) for any $s \in \mathcal{M}_F^*(T)$ satisfying $F(s) = x$ and $s(U_F(x)) \in (\hat{s}(U_F(x)), \bar{s}_F(x))$. The lemma follows immediately. ■

Lemma A.6 *Given F, the function $\bar{s}_F(\cdot)$ in Lemma A.5 is non-decreasing on R_F.*

Proof. Proceeding indirectly, suppose that the lemma is false. Then there exist x and $\bar{x} > x$ in R_F such that $\bar{s}_F(x) > \bar{s}_F(\bar{x})$. Let $s \in \mathcal{M}_F^*(T)$ be such that

$$s(D_F(x)) = 0, \quad s(U_F(x)) < \bar{s}_F(x), \quad \text{and} \quad s(U_F(\bar{x})) \in (\bar{s}_F(\bar{x}), 1).$$

Then, by Lemma A.1, $F(s) \geq x$. By Lemma A.5 and the assumption that $s(U_F(x)) < \bar{s}_F(x)$, one also has $F(s) \not> x$. Thus, $F(s) = x$.

Next, consider a distribution $\hat{s}(\bar{t}) \in \mathcal{M}_F^*(T)$ such that

$$\hat{s}(\bar{t}) = s(D_F(\bar{x}) \cup P_F^*(x)) \cdot \delta_{\bar{t}} + s_{U_F(\bar{x})},$$

where $\bar{t}$ is some fixed element of $P_F^*(\bar{x})$. By Lemma A.1, $F(\hat{s}(\bar{t})) \geq \bar{x}$. By Lemma A.5 and the assumption that $s(U_F(\bar{x})) > \bar{s}_F(\bar{x})$, one also has $F(\hat{s}(\bar{t})) \neq \bar{x}$. Thus, there exists $\hat{x}(\bar{t}) > \bar{x}$ such that $F(\hat{s}(\bar{t})) = \hat{x}(\bar{t})$.

For any $\bar{t} \in P_F^*(\bar{x})$ and any $\hat{x}, \hat{x}'$ such that $\hat{x}' > \hat{x} > \bar{x}$,

$$\begin{array}{rcl} \hat{s}(P(\hat{x}, \hat{x}')|\bar{t}) & = & s(D_F(\bar{x}) \cup P_F^*(x)) + s_{U_F(\bar{x})}(P(\hat{x}, \hat{x}')) \\ & & \text{and} \\ \hat{s}(I(\hat{x}, \hat{x}')|\bar{t}) & = & s_{U_F(\bar{x})}(I(\hat{x}, \hat{x}')), \end{array}$$

so both $\hat{s}(P(\hat{x}, \hat{x}')|\bar{t})$ and $\hat{s}(I(\hat{x}, \hat{x}')|\bar{t})$ are independent of $\bar{t}$. By Lemma A.4, it follows that $F(\hat{s}(\bar{t}))$ is independent of $\bar{t}$, i.e., there exists $\hat{x} > \bar{x}$ such that $\hat{x}(\bar{t}) = \hat{x}$ for all $\bar{t} \in P_F^*(\bar{x})$.

Because T is rich, there exists $\bar{t}^* \in P_F^*(\bar{x})$ such that

$$u(\hat{x}, \bar{t}^*) < u(x, \bar{t}^*) < u(\bar{x}, \bar{t}^*).$$

I claim that F is blocked at $\hat{s}(\bar{t}^*)$ by $\{\bar{t}^*\}$. For suppose that the true cross-section distribution of types is $\hat{s}(\bar{t}^*)$, with stipulated outcome $F(\hat{s}(\bar{t}^*)) = \hat{x}$. Suppose also that the group of all agents with type $\bar{t}^*$ coordinate their reports so as to mimick $s_{D_F(\bar{x}) \cup P_F^*(x)}$, which they can because, by construction they have the total mass $s(D_F(\bar{x}) \cup P_F^*(x))$. Then the distribution of reported types is $s_{D_F(\bar{x}) \cup P_F^*(x)} + s_{U_F(\bar{x})} = s$ and the induced outcome is $F(s) = x$, which all members of the group prefer to the outcome $F(\hat{s}(\bar{t}^*)) = \hat{x}$.

This finding contradicts the assumption that F is weakly group strategy-proof. The assumption that $\bar{s}_F(\cdot)$ is not everywhere non-decreasing has thus led to a contradiction and must be false. ■

Lemma A.7 *Given F, the function $\bar{s}_F(\cdot)$ in Lemma A.5 is right-continuous on R_F.*

Proof. If the lemma is false, there exists $x \in R_F$ and there exists a sequence $\{x_k\}_{k=1}^{\infty}$ of elements of R_F that converges to x from above such that the associated threshold sequence $\{\bar{s}_F(x_k)\}_{k=1}^{\infty}$ does not converge to $\bar{s}_F(x)$. Without loss of generality, one may assume that the sequence $\{x_k\}_{k=1}^{\infty}$ decreases monotonically to x. By Lemma A.6, it follows that $\bar{s}_F(x_k) \geq \bar{s}_F(x)$ for all k and that the sequence $\{\bar{s}_F(x_k)\}_{k=1}^{\infty}$ is (weakly) monotonically decreasing. Therefore, this sequence converges to a limit $\bar{s}_F(x+)$. The assumption that the lemma is false implies that $\bar{s}_F(x+) > \bar{s}_F(x)$.

If $\bar{s}_F(x+) > \bar{s}_F(x)$, there exists $s \in \mathcal{M}_F^*(T)$ such that $s(U_F(x)) \in (\bar{s}_F(x), \bar{s}_F(x+))$. Consider the outcome $F(s)$. If $F(s) \leq x$, then $s(U_F(F(s))) \geq s(U_F(x))$ and, by Lemma A.6, $\bar{s}_F(F(s)) \leq \bar{s}_F(x)$. Since $s(U_F(x)) \in (\bar{s}_F(x), \bar{s}_F(x+))$, it follows that

$$s(U_F(F(s))) \geq s(U_F(x)) > \bar{s}_F(x) \geq \bar{s}_F(F(s)).$$

By (A.6), however, $s(U_F(F(s))) \leq \bar{s}_F(F(s))$. The assumption that $F(s) \leq x$ thus leads to a contradiction and must be false.

Alternatively, if $F(s) = x' > x$, then, for any $x'' \in (x, x')$, $s(U_F(x'')) \leq s(U_F(x))$ and, by Lemma A.6, $\bar{s}_F(x) \leq \bar{s}_F(x'')$. Since $s(U_F(x)) \in (\bar{s}_F(x), \bar{s}_F(x+))$, it follows that

$$s(U_F(x'')) \leq s(U_F(x)) < \bar{s}_F(x) \leq \bar{s}_F(x'').$$

By (A.7), however, $F(s) > x''$ also implies $s(U_F(x'')) \geq \bar{s}_F(x'')$. The assumption that $F(s) > x$ thus also leads to a contradiction and must be false.

Since both alternatives, $F(s) \leq x$ and $F(s) > x$, lead to contradictions, the assumption that $\bar{s}_F(x+)$ exceeds $\bar{s}_F(x)$ must be false, which proves the lemma. ■

Given the monotonicity and continuity properties of the functions $s(U_F(\cdot))$, $s(D_F(\cdot))$, and $\bar{s}_F(\cdot)$, the following lemma is trivial.

Lemma A.8 *For all $s \in \mathcal{M}_F^*(T)$ and all $x \in R_F$,*

$$s(U_F(x') \leq \bar{s}_F(x') \quad \textit{for all } x' \geq x \quad \textit{if and only if} \quad s(U_F(x) \leq \bar{s}_F(x)$$

and

$$s(D_F(x') \leq 1-\bar{s}_F(x') \quad \textit{for all } x' < x \quad \textit{if and only if} \quad s(D_F(x) \leq 1-\bar{s}_F(x-),$$

where $\bar{s}_F(x-) := \lim_{x' \uparrow x} \bar{s}_F(x')$.

The following lemma is less trivial.

Lemma A.9 *For all $s \in \mathcal{M}_F^*(T)$ and all $x \in R_F$,*

$$s(U_F(x')) \geq \bar{s}_F(x') \text{ for all } x' < x \text{ if and only if } s(D_F(x)) \leq 1 - \bar{s}_F(x-) \tag{A.13}$$

and

$$s(D_F(x')) \geq 1 - \bar{s}_F(x'-) \text{ for all } x' > x \text{ if and only if } s(U_F(x) \leq \bar{s}_F(x). \tag{A.14}$$

Proof. I only give a proof for (A.13) The proof for (A.14) is completely symmetric and is left to the reader. In proving (A.13), I distinguish three cases, depending on the behaviour of R_F just below x. I first consider the case where R_F has a gap just below x.

(a) For some $\bar{x} \in R_F$ with $\bar{x} < x$, $R_F \cap (\bar{x}, x) = \emptyset$, i.e., x is the upper bound of a gap in R_F. In this case, $s \in \mathcal{M}_F^*(T)$ implies that

$$s(D_F(x)) = 1 - s(U_F(\bar{x})).$$

One also has $\bar{s}_F(x-) = \bar{s}_F(\bar{x})$. Therefore,

$$s(U_F(\bar{x})) \geq \bar{s}_F(\bar{x}) \text{ if and only if } s(D_F(x)) \leq 1 - \bar{s}_F(x-),$$

hence, by Lemma A.8,

$$s(U_F(x')) \geq \bar{s}_F(x') \text{ for all } x' < x \text{ if and only if } s(D_F(x)) \leq 1 - \bar{s}_F(x-),$$

which is just (A.13).

(b) For some sequence $\{\bar{x}_k\}$ in R_F, there exists an associated sequence $\{\hat{x}_F(\bar{x}_k)\}$ in R_F such that, for each k, $R_F \cap (\bar{x}_k, \hat{x}_F(x_k)) = \emptyset$, and, moreover, both sequences, $\{\bar{x}_k\}$ and $\{\hat{x}_F(\bar{x}_k)\}$, converge to x from below. In this case, $s \in \mathcal{M}_F^*(T)$ implies that

$$s(D_F(\hat{x}_F(\bar{x}_k))) = 1 - s(U_F(\bar{x}_k))$$

for all k and therefore

$$s(U_F(\bar{x}_k)) \geq \bar{s}_F(\bar{x}_k) \text{ for all } k \text{ if and only if } s(D_F(\hat{x}_F(\bar{x}_k))) \leq 1 - \bar{s}_F(\bar{x}_k) \text{ for all } k. \tag{A.15}$$

Because the function $x' \to s(U_F(x'))$ is nonincreasing and the function $x' \longmapsto \bar{s}_F(x')$ is nondecreasing, one also has

$$s(U_F(\bar{x}_k)) \geq \bar{s}_F(\bar{x}_k) \text{ for all } k \text{ if and only if } s(U_F(x')) \geq \bar{s}_F(x') \text{ for all } x' < x.$$

Because, by Remark 2.3, the function $x' \longmapsto s(D_F(x'))$ is nondecreasing and left-continuous, one obtains

$$s(D_F(x)) = \lim_{k\to\infty} s(D_F(\hat{x}_F(\bar{x}_k)))$$

and

$$s(D_F(\hat{x}_F(\bar{x}_k))) \leq 1-\bar{s}_F(\bar{x}_k) \;\text{ for all } k \;\text{ if and only if }\; s(D_F(x)) \leq 1-\bar{s}_F(x-).$$

Again (A.13) follows.

(c) For some $\bar{x} < x$, $[\bar{x}, x] \subset R_F$, i.e., R_F has no gap below x and near x. In this case, $s \in \mathcal{M}_F^*(T)$ implies that $s(D_F(x') + s(U_F(x')) + s(P_F^*(x')) = 1$ for all x'. Because for different $x'' \in R_F$, the sets $P_F^*(x'')$ are disjoint, $s(P_F^*(x'')) = 0$ for all but at most countably many x''. For some sequence $\{x_k\}$ that converges to x from below, it follows that $s(P_F^*(x_k)) = 0$ for all k and hence $s(D_F(x_k) + s(U_F(x_k)) = 1$ for all k. Thus,

$$s(U_F(x_k)) \geq \bar{s}_F(x_k) \;\text{ for all } k \;\text{ if and only if }\; s(D_F(x_k)) \leq 1-\bar{s}_F(x_k) \;\text{ for all } k. \tag{A.16}$$

Upon using the monotonicity properties of the functions $x' \to s(D_F(x'))$, $x' \to s(U_F(x'))$, and $x' \to \bar{s}_F(x')$, as well as the left-continuity of the function $x' \to s(D_F(x'))$, as before, one also obtains the equivalences

$$s(U_F(x_k)) \geq \bar{s}_F(x_k) \;\text{ for all } k \text{ if and only if }\; s(U_F(x')) \geq \bar{s}_F(x') \;\text{ for all }\; x' < x$$

and

$$s(D_F(x_k) \leq 1 - \bar{s}_F(x_k) \;\text{ for all } k \;\text{ if and only if }\; s(D_F(x)) \leq 1 - \bar{s}_F(x-).$$

Again (A.13) follows. ■

A.3 Proofs for Section 3

Among the results of Section 3, Propositions 3.1 and 3.2 are special cases of Theorem 3.4, and Corollary 3.3 is a special case of Corollary 3.5. Therefore, it suffices to prove Theorem 3.4, Corollary 3.5, and Theorem 3.6. As indicated in the text, the equivalences in Theorems 3.4 and 3.6 will be proved jointly.

A.3.1 Weak Group Strategy Proofness Implies the MBV Property

The following lemma shows that, if F is group strategy-proof, then, for the threshold function given by Lemma A.5, part (i) of the MBV Property must hold.

Lemma A.10 *If F is weakly group strategy-proof, then, for all $s \in \mathcal{M}_F^*(T)$ and all $x \in R_F$,*

$$\begin{aligned} F(s) &= x \quad \textit{implies} \\ s(U_F(x')) &\geq \bar{s}_F(x') \quad \textit{for all} \quad x' < x \textit{ and} \\ s(U_F(x') &\leq \bar{s}_F(x') \quad \textit{for all} \quad x' \geq x. \end{aligned} \tag{A.17}$$

Proof. Trivially, $F(s) = x$ implies $F(s) > x'$ for any $x' < x$, so Lemma A.5, with x replaced by x', yields $s(U_F(x')) \geq \bar{s}_F(x')$ for any $x' < x$. By Lemma A.5, $F(s) = x$ also implies $s(U_F(x) \leq \bar{s}_F(x)$ and therefore, $s(U_F(x') \leq \bar{s}_F(x')$ for all $x' \geq x$. ■

The next lemma shows that, if F is weakly group strategy-proof, then part (ii) of the MBV Property must also hold.

Lemma A.11 *If F is weakly group strategy-proof, then, for any s and $\hat{s}$ in $\mathcal{M}_F^*(T)$ and any x and $\hat{x} > x$ in R_F, $F(s) = x$ and $F(\hat{s}) = \hat{x}$ imply that $\hat{s}(U_F(x)) > s(U_F(x))$ or $s(D_F(\hat{x})) > \hat{s}(D_F(\hat{x}))$.*

Proof. The proof is indirect. Suppose that the lemma is false. Then there exist s and $\hat{s}$ in $\mathcal{M}_F^*(T)$ and any x and $\hat{x} > x$ in R_F such that $F(s) = x$ and $F(\hat{s}) = \hat{x}$ and moreover,

$$\hat{s}(U_F(x)) \leq s(U_F(x)) \quad \text{and} \quad s(D_F(\hat{x})) \leq \hat{s}(D_F(\hat{x})). \tag{A.18}$$

Suppose first that $R_F \cap (x, \hat{x}) = \emptyset$. In this case, the fact that s and $\hat{s}$ belong to $\mathcal{M}_F^*(T)$ implies $s(I(\hat{x}, x)) = \hat{s}(I(\hat{x}, x)) = 0$. By Lemma A.3, therefore,

$$s(P(\hat{x}, x)) < \hat{s}(P(\hat{x}, x)) \quad \text{and} \quad s(P(x, \hat{x})) > \hat{s}(P(x, \hat{x})). \tag{A.19}$$

By the single-peakedness of preferences and the fact that $R_F \cap (x, \hat{x}) = \emptyset$, one also has $U_F(x) = P(\hat{x}, x)$ and $D_F(\hat{x}) = P(x, \hat{x})$. Thus the inequalities in (A.19) imply

$$s(U_F(x)) = s(P(\hat{x}, x)) < \hat{s}(P(\hat{x}, x)) = \hat{s}(U_F(x))$$

and

$$s(D_F(\hat{x})) = s(P(x,\hat{x})) > \hat{s}(P(x,\hat{x})) = \hat{s}(D_F(\hat{x})),$$

contrary to (A.18). For the case $R_F \cap (x,\hat{x}) = \emptyset$, the assumption that the lemma fails for s and $\hat{s}$ in $\mathcal{M}_F^*(T)$ with $F(s) = x$ and $F(\hat{s}) = \hat{x}$ has thus led to a contradiction and must be false.

Alternatively, suppose that $R_F \cap (x,\hat{x}) \neq \emptyset$. I will show that (A.18) implies

$$\hat{s}(U_F(x)) = \hat{s}(P(\hat{x},x)) = s(U_F(x)) \tag{A.20}$$

and

$$\hat{s}(D_F(\hat{x})) = s(P(x,\hat{x})) = s(D_F(\hat{x})). \tag{A.21}$$

To prove (A.20), let $x^* \in R_F \cap (x,\hat{x})$. Since $F(s) = x$ and $F(\hat{s}) = \hat{x} > x^* > x$, one obtains

$$\hat{s}(U_F(x)) \geq \hat{s}(U_F(x^*)) \geq \bar{s}_F(x^*) \geq \bar{s}_F(x) \geq s(U_F(x));$$

the first inequality follow from the monotonicity of the functions $\hat{s}(U_F(\cdot))$ and $s(U_F(\cdot))$, the second and fourth inequalities from Lemma A.10, and the third inequality from the monotonicity of the function $\bar{s}_F(\cdot)$. If the first inequality in (A.18) holds, all these inequalities must hold as equations. Then

$$\hat{s}(U_F(x)) = \hat{s}(U_F(x^*)) = s(U_F(x)) \tag{A.22}$$

for any $x^* \in R_F \cap (x,\hat{x})$.

If $R_F \cap (x^*,\hat{x}) = \emptyset$, we have $U_F(x^*) = P(\hat{x},x^*)$, so (A.22) implies

$$\hat{s}(U_F(x)) = \hat{s}(P(\hat{x},x^*)) = s(U_F(x)), \tag{A.23}$$

and (A.20) follows because, by elementary set theory, $P(\hat{x},x^*) \subset P(\hat{x},x) \subset U_F(x)$.

If, instead, $R_F \cap (x^*,\hat{x}) \neq \emptyset$ for all $x^* \in R_F \cap (x,\hat{x})$, there exists a non-decreasing sequence $\{x_k\}$ of elements of $R_F \cap (x,\hat{x})$ that converges to $\hat{x}$ from below. For any k, (A.22) must hold with $x^* = x_k$, i.e. one must have

$$\hat{s}(U_F(x)) = \hat{s}(U_F(x_k)) = s(U_F(x)) \tag{A.24}$$

for all k. By the single-peakedness of preferences and the monotonicity and convergence of the sequence $\{x_k\}$, one also has $U_F(x_{k+1}) \subset U_F(x_k)$ for all k and

$$\cap_{k=1}^{\infty} U_F(x_k) = P_F^*(\hat{x}) \cup U_F(\hat{x}).$$

Therefore, (A.24) implies

$$\hat{s}(U_F(x)) = \hat{s}(P_F^*(\hat{x}) \cup U_F(\hat{x})) = s(U_F(x)), \tag{A.25}$$

and again (A.20) follows because, by elementary set theory, $P_F^*(\hat{x}) \cup U_F(\hat{x}) \subset P(\hat{x}, x) \subset U_F(x)$.

By the monotonicity of the functions $\hat{s}(D_F(\cdot))$, $\bar{s}_F(\cdot)$, and $s(D_F(\cdot))$ and by Lemma A.10, $F(s) = x$, $F(\hat{s}) = \hat{x} > x$, and $x^* \in R_F \cap (x, \hat{x})$ also imply

$$s(D_F(\hat{x})) \geq s(D_F(x^*)) \geq 1 - \bar{s}_F(x^*-) \geq 1 - \bar{s}_F(\hat{x}-) \geq \hat{s}(D_F(\hat{x})).$$

If the second inequality in (A.18) holds, all these inequalities must hold as equations. Then

$$s(D_F(\hat{x})) = s(D_F(x^*)) = \hat{s}(D_F(\hat{x})) \tag{A.26}$$

for any $x^* \in R_F \cap (x, \hat{x})$.

If $R_F \cap (x, x^*) = \emptyset$, one has $D_F(x^*) = P(x, x^*)$, so (A.26) implies

$$s(D_F(\hat{x})) = s(P(x, x^*)) = \hat{s}(D_F(\hat{x})), \tag{A.27}$$

and (A.21) follows because, by elementary set theory, $P(x, x^*) \subset P(x, \hat{x}) \subset D_F(\hat{x})$.

If, instead, $R_F \cap (x^*, \hat{x}) \neq \emptyset$ for all $x^* \in R_F \cap (x, \hat{x})$, there exists a non-decreasing sequence $\{x_k\}$ of elements of $R_F \cap (x, \hat{x})$ that converges to x from above. For any k, (A.26) must hold with $x^* = x_k$, i.e. one has

$$s(D_F(\hat{x})) = s(D_F(x_k)) = \hat{s}(D_F(\hat{x})) \tag{A.28}$$

for all k. By the single-peakedness of preferences and the monotonicity and convergence of the sequence $\{x_k\}$, we also have $D_F(x_{k+1}) \subset D_F(x_k)$ for all k and

$$\cap_{k=1}^{\infty} D_F(x_k) = P_F^*(x) \cup D_F(x).$$

Therefore, (A.28) implies

$$s(D_F(\hat{x})) = s(P_F^*(x) \cup D_F(x)) = \hat{s}(D_F(\hat{x})), \tag{A.29}$$

and again (A.21) follows because, by elementary set theory, $P_F^*(x) \cup D_F(x) \subset P(x, \hat{x}) \subset D_F(\hat{x})$.

To complete the proof of the lemma, I note that, by single-peakedness, $I(\hat{x}, x) \subset U_F(x) \backslash P(\hat{x}, x)$ and $I(x, \hat{x}) \subset D_F(\hat{x}) \backslash P(x, \hat{x})$. Since (A.20) implies $\hat{s}(U_F(x) \backslash P(\hat{x}, x)) = 0$ and (A.21) implies $s(D_F(\hat{x}) \backslash P(x, \hat{x})) = 0$, it follows that $\hat{s}(I(\hat{x}, x)) = s(I(x, \hat{x})) = 0$. Since $I(\hat{x}, x) = I(x, \hat{x})$, Lemma 3 is applicable, but (A.20) and (A.21) jointly are incompatible with Lemma A.3. ■

Upon combining Lemmas A.10 and A.11, one obtains the "only if" part of Theorem 3.4.

Proof of Corollary 3.5. If the corollary is false, there exist an anonymous weakly group strategy-proof social choice function F, a type distribution $s \in \mathcal{M}_F^*(T)$, and an outcome $x \in R_F$ such that one of the conditions in the corollary is satisfied and yet $F(s) \neq x$. Suppose first that $F(s) = x^* < x$. By the "only if" part of Theorem 3.4 and part (i) of the MBV Property, it follows that $s(U_F(x')) \leq \bar{s}_F(x')$ for all $x' \in (x^*, x)$, which is incompatible with the first two conditions in the corollary. Since one of the three conditions in the corollary has been assumed to hold, it must be the case that $s(U_F(x')) = 1$ for all $x' < x$. Hence $s(D_F(x)) = 0$. By Lemma A.1 it follows that $F(s) \geq x$. The assumption that $F(s) = x^* < x$ has thus led to a contradiction and must be false.

A precisely symmetric argument, which is left to the reader, also shows that one cannot have $F(s) = x^* > x$.

The assumption that the corollary is false thus leads to a contradiction, which proves the corollary. ■

A.3.2 The MBV Property Implies the Condition in Theorem 3.6

Lemma A.12 *If F has the MBV Property, then for all s and $\hat{s}$ in $\mathcal{M}_F^*(T)$ such that $F(s) \neq F(\hat{s})$,*

$$s(P(F(s), F(\hat{s}))) > \hat{s}(P(F(s), F(\hat{s}))) \qquad \text{(A.30)}$$

$$\textit{or}$$

$$s(P(F(\hat{s}), F(s))) < \hat{s}(P(F(\hat{s}), F(s))). \qquad \text{(A.31)}$$

Proof. Suppose that the lemma is false, let F have the MBV Property and let s and $\hat{s}$ in $\mathcal{M}_F^*(T)$ be such that $F(s) \neq F(\hat{s})$ and

$$s(P(F(s), F(\hat{s}))) \leq \hat{s}(P(F(s), F(\hat{s}))) \qquad \text{(A.32)}$$

$$\text{as well as}$$

$$s(P(F(\hat{s}), F(s))) \geq \hat{s}(P(F(\hat{s}), F(s))). \qquad \text{(A.33)}$$

Set $F(s) = x$ and $F(\hat{s}) = \hat{x}$ and assume without loss of generality that $\hat{x} > x$. If $R_F \cap (x, \hat{x}) = \emptyset$, one has $U_F(x) = P(\hat{x}, x)$ and $D_F(\hat{x}) = P(x, \hat{x})$, so (A.32) and (A.33) stand in contradiction to part (ii) of the MBV Property.

Suppose therefore that $R_F \cap (x, \hat{x}) \neq \emptyset$. Since s and $\hat{s}$ belong to $\mathcal{M}_F^*(T)$, part (i) of the MBV Property implies that for any $x' \in R_F \cap [x, \hat{x})$,

$$s(U_F(x)) \leq \bar{s}_F(x) \leq \hat{s}(U_F(x')). \tag{A.34}$$

Because the function $x' \to s(U_F(x'))$ is non-increasing, $\inf_{x'<\hat{x}} \hat{s}(U_F(x'))$ is well defined, and

$$\inf_{x'<\hat{x}} \hat{s}(U_F(x')) = \hat{s}(\cap_{x'<\hat{x}} U_F(x')).$$

Moreover, $\cap_{x'<\hat{x}} U_F(x')$ is the union of the sets $P_F^*(\hat{x}), U_F(\hat{x})$, and, if $\inf R_F \cap (\hat{x}, \infty) > \hat{x}$, the set $I(\hat{x}, \inf R_F \cap (\hat{x}, \infty))$. Because $\hat{s} \in \mathcal{M}_F^*(T)$, $\hat{s}(I(\hat{x}, \inf R_F \cap (\hat{x}, \infty))) = 0$. Therefore,

$$\inf_{x'<\hat{x}} \hat{s}(U_F(x')) = \hat{s}(P_F^*(\hat{x}) \cup U_F(\hat{x})). \tag{A.35}$$

By elementary set theory,

$$P_F^*(\hat{x}) \cup U_F(\hat{x}) \subset P(\hat{x}, x) \subset U_F(x), \tag{A.36}$$

so (A.34) - (A.36) imply

$$s(P(\hat{x}, x)) \leq s(U_F(x)) \leq \hat{s}(P_F^*(\hat{x}) \cup U_F(\hat{x})) \leq \hat{s}(P(\hat{x}, x)). \tag{A.37}$$

Upon combining (A.37) and (A.33), one obtains

$$s(P(\hat{x}, x)) = \hat{s}(P(\hat{x}, x)). \tag{A.38}$$

Next, observe that

$$\hat{s}(D_F(\hat{x})) = \sup_{x'<\hat{x}} (1 - \hat{s}(U_F(x')).$$

By part (i) of the MBV Property, therefore,

$$\hat{s}(D_F(\hat{x})) \leq \sup_{x'<\hat{x}} (1 - \bar{s}_F(x')) \leq 1 - \bar{s}_F(x'') \leq 1 - s(U_F(x'')) \tag{A.39}$$

for all $x'' \in R_F \cap (x, \hat{x})$. Because the function $s(U_F(\cdot))$ is non-decreasing and right-continuous, it follows that

$$\hat{s}(D_F(\hat{x})) \leq 1 - s(U_F(x)). \tag{A.40}$$

The right-hand side of (A.40) is the measure (under s) of the complement of $U_F(x)$. This complement is the union of the sets $P^*_F(x), D_F(x)$, and, if $\sup R_F \cap (-\infty, x) < x$, the set $I(x, \sup R_F \cap (-\infty, x))$. Since $s \in \mathcal{M}^*_F(T)$, $s(I(x, \sup R_F \cap (-\infty, x)) = 0$. Therefore, (A.40) implies

$$\hat{s}(D_F(\hat{x})) \leq s(P^*_F(x) \cup D_F(x)). \tag{A.41}$$

By elementary set theory, one also has

$$P^*_F(x) \cup D_F(x) \subset P(x, \hat{x}) \subset D_F(\hat{x}). \tag{A.42}$$

From (A.39) - (A.42), therefore,

$$\hat{s}(P(x, \hat{x})) \leq \hat{s}(D_F(\hat{x})) \leq s(P^*_F(x) \cup D_F(x)) \leq s(P(x, \hat{x})). \tag{A.43}$$

Upon combining (A.43) with (A.32), one further obtains

$$s(P(x, \hat{x})) = \hat{s}(P(x, \hat{x})). \tag{A.44}$$

From (A.38) and (A.44), I infer that all the inequalities in (A.34) - (A.43) hold as equations. For (A.34) with $x' = x$, this implies

$$s(U_F(x)) = \hat{s}(U_F(x)). \tag{A.45}$$

For (A.39), replacing the inequalities by equations yields

$$\hat{s}(D_F(\hat{x})) = 1 - s(U_F(x'))$$

for all $x' \in R_F \cap (x, \hat{x})$, hence, taking limits as $x' \uparrow \hat{x}$,

$$\hat{s}(D_F(\hat{x})) = s(D_F(\hat{x})). \tag{A.46}$$

However, equations (A.45) and (A.46) holding jointly is contrary to part (ii) of the MBV Property. The assumption that F satisfies the MBV Property but not the condition in Theorem 3.6 has thus led to a contradiction and must be false. ∎

A.3.3 The Condition in Theorem 3.6 Implies Weak Group Strategy Proofness

Lemma A.13 *Let F be such that for all s and $\hat{s}$ in $\mathcal{M}_F^*(T)$, $F(s) \neq F(\hat{s})$ implies (A.30) or (A.31). Then F is weakly group strategy-proof.*

Proof. Suppose that the lemma is false. Then there exists a type set B and a type distribution $s \in \mathcal{M}_F^*(T)$ such that B blocks F at s, i.e., there exist distributions $\hat{s}, s^* \in \mathcal{M}_F^*(T)$ such that

$$\hat{s} = s_{T \setminus B} + s(B) \cdot s^* \tag{A.47}$$

is the distribution of overall reports that is induced when the true type distribution is s and agents with types in B coordinate their reports to have the distribution s^*, $F(\hat{s}) \neq F(s)$ and, moreover, agents with types in B prefer $F(\hat{s})$ to $F(s)$. Since types in B prefer $F(\hat{s})$ to $F(s)$, $B \subset P(F(\hat{s}), F(s))$. Therefore

$$\begin{aligned} \hat{s}(P(F(\hat{s}), F(s))) &\leq \hat{s}(P(F(\hat{s}), F(s)) \setminus B) + s(B) \cdot s^*(P(F(\hat{s}), F(s))) \\ &\leq s(P(F(\hat{s}), F(s))). \end{aligned} \tag{A.48}$$

Since $B \subset P(F(\hat{s}), F(s))$, one also has $P(F(s), F(\hat{s})) \subset T \setminus B$. By (A.47), it follows that

$$\begin{aligned} \hat{s}(P(F(s), F(\hat{s}))) &= \hat{s}(P(F(s), F(\hat{s})) \setminus B) + s(B) \cdot s^*(P(F(s), F(\hat{s}))) \\ &\geq s(P(F(s), F(\hat{s}))). \end{aligned} \tag{A.49}$$

The inequalities (A.48) and (A.49), however, are incompatible with (A.30) and (A.31). The assumption that F is not group strategy-proof has thus led to a contradiction and must be false. ■

Theorems 3.4 and 3.6 follow from Lemmas A.10 - A.13.

A.4 Proofs for Section 4

I first recall the definition of a simple tie-breaking function g_F for a social choice function F. For any $t \in T$, let $\Pi_F(t)$ be the set of peaks of $u(\cdot, t)$ on R_F, and let

$$\pi_F^1(t) = \inf \Pi_F(t) \text{ and } \pi_F^2(t) = \sup \Pi_F(t).$$

Then

$$\Pi_F(t) = \{\pi_F^1(t), \pi_F^2(t)\},$$

so $\Pi_F(t)$ is a singleton set if $\pi_F^1(t) = \pi_F^2(t)$ and a doubleton set if $\pi_F^1(t) \neq \pi_F^2(t)$. In the latter case, $R_F \cap (\pi_F^1(t), \pi_F^2(t)) = \emptyset$.

A *simple tie-breaking function* is a function $g_F : T \to T$ such that, for any $t \in T$, if $u(\cdot, t)$ has twin peaks $\pi_F^1(t), \pi_F^2(t)$ on R_F, then

$$u(\pi_F^i(t), g_F(t)) = u(\pi_F^i(t), t) + 1 \tag{A.50}$$

for some i, which is independent of t, and

$$u(x, g_F(t)) = u(x, t) \;\; \text{for all} \;\; x \neq \pi_F^i(t); \tag{A.51}$$

in particular,

$$u(\pi_F^j(t), g_F(t)) = u(\pi_F^j(t), t) \;\; \text{for } j \neq i. \tag{A.52}$$

If $u(\cdot, t)$ has a single peak $\pi_F(t)$ on R_F, then

$$u(x, g_F(t)) = u(x, t) \tag{A.53}$$

for all x.

As mentioned in the text, if g_F is a simple tie-breaking function, then the functions $u(\cdot, g_F(t))$ and $u(\cdot, t)$ induce the same ordering on R_F *except for the fact* that, if $u(\cdot, t)$ has twin peaks, then $u(\cdot, g_F(t))$ induces a strict ordering on $\Pi_F(t)$. Formally, one obtains:

Lemma A.14 *Given a social choice function F with range R_F and a simple tie-breaking function g_F for F, for any x and $\bar{x} \neq x$ in R_F,*

$$t \in P(x, \bar{x}) \;\; \textit{implies} \;\; g_F(t) \in P(x, \bar{x}) \tag{A.54}$$

and

$$g_F(t) \in P(x, \bar{x}) \;\; \textit{implies} \;\; t \in P(x, \bar{x}) \cup I(x, \bar{x}); \tag{A.55}$$

if $x < \bar{x}$ and $R_F \cap (x, \bar{x}) \neq \emptyset$ or $x > \bar{x}$ and $R_F \cap (\bar{x}, x) \neq \emptyset$,

$$g_F(t) \in P(x, \bar{x}) \;\; \textit{implies} \;\; t \in P(x, \bar{x}). \tag{A.56}$$

Proof. By (A.50) - (A.53), $u(x, g_F(t)) \geq u(x, t)$, and the inequality is an equation unless $u(\cdot, t)$ has twin peaks and $x = \pi_F(g_F(t))$. Since $t \in P(x, \bar{x})$ if and only if $u(x, t) > u(\bar{x}, t)$, it follows that $t \in P(x, \bar{x})$ implies

$$u(x, g_F(t)) \geq u(x, t) > u(\bar{x}, t) = u(\bar{x}, g_F(t),$$

which yields (A.54). Similarly, if $g_F(t) \in P(x, \bar{x})$ and $x \neq \pi(g_F(t))$, then

$$u(x, t) = u(x, g_F(t)) > u(\bar{x}, g_F(t)) = u(\bar{x}, t),$$

i.e. $t \in P(x, \bar{x})$. If instead $x = \pi_F(g_F(t, s))$, then also $x = \pi_F^i(t)$ for $i = 1$ or $i = 2$ and $u(x, t) \geq u(\bar{x}, t)$, which implies $t \in P(x, \bar{x}) \cup I(x, \bar{x})$. In either case, (A.55) follows.

To complete the proof, it suffices to note that, in (A.55), $t \in I(x, \bar{x})$ is only possible if $\bar{x}(t) = \pi_F^j(t)$ for $j \neq i$ and the intersection of R_F with the interval between x and $\bar{x}$ is empty. ■

I also recall that a *contingent tie-breaking function* is a function $g_F : T \times \mathcal{M}(T) \to T$ such that, for any $s \in \mathcal{M}(T)$, the section $g_F(\cdot, s)$ of g_F that is determined by s is a simple tie-breaking function.

Given these definitions, Theorems 4.1 and 4.3 can be seen as special cases of the following result.

Theorem A.15 *Let F be an anonymous social choice function with range R_F such that $F(s) = F(s')$ whenever s and s' induce the same distributions on the space of preference orderings on R_F. If there exists a simple or contingent tie-breaking function g_F^* such that, for all $s \in \mathcal{M}(T)$, $F(s) = F(s \circ g_F^*(\cdot, s)^{-1})$, then the following statements are equivalent:*

(a) F is group strategy-proof.

(b) F has the MBV property. Moreover, for any s and $\hat{s}$ in $\mathcal{M}(T)$ and any x and $\hat{x} > x$ in R_F such that $F(s) = x$, $F(\hat{s}) = \hat{x}$, and $R_F \cap (x, \hat{x}) = \emptyset$,

$$\hat{s}(U_F(x)) > s(U_F(x)) \quad \textit{or} \quad s(D_F(\hat{x})) > \hat{s}(D_F(\hat{x})).$$

(c) For all s and $\hat{s}$ in $\mathcal{M}(T)$, $F(s) \neq F(\hat{s})$ implies

$$\begin{array}{rcl} s(P(F(s), F(\hat{s}))) & > & \hat{s}(P(F(s), F(\hat{s}))) \\ & \textit{or} & \\ s(P(F(\hat{s}), F(s))) & < & \hat{s}(P(F(\hat{s}), F(s))). \end{array}$$

To prove Theorem A.15, I will establish the implications (a) $\Longrightarrow$ (b), (b) $\Longrightarrow$ (c), and (c) $\Longrightarrow$ (a). For brevity, the property that $F(s) = F(s')$ whenever s and s' induce the same distributions on the space of preference orderings on R_F will be referred to as the *Independence Property.*

Lemma A.16 *Under the specified assumptions, Statement (a) in Theorem A.15 implies Statement (b) in Theorem A.15.*

Proof. If F is group strategy-proof, Theorem 3.4 implies that F can be implemented on $\mathcal{M}_F^*(T)$ by monotone binary voting over neighbours. It remains to be shown that, for any s and $\hat{s}$ in $\mathcal{M}(T)$ and any x and $\hat{x} > x$ in R_F such that $F(s) = x$, $F(\hat{s}) = \hat{x}$, and $R_F \cap (x, \hat{x}) = \emptyset$,

$$\hat{s}(U_F(x)) > s(U_F(x)) \text{ or } s(D_F(\hat{x})) > \hat{s}(D_F(\hat{x})).$$

Proceeding indirectly, suppose that the claim is false. Then there exist s and $\hat{s}$ in $\mathcal{M}(T)$ and x and $\hat{x} > x$ in R_F with $R_F \cap (x, \hat{x}) = \emptyset$, such that $F(s) = x$ and $F(\hat{s}) = \hat{x}$ and, moreover,

$$\hat{s}(U_F(x)) \leq s(U_F(x)) \text{ and } s(D_F(\hat{x})) \leq \hat{s}(D_F(\hat{x})). \tag{A.57}$$

Because F has the Independence Property, there is no loss of generality in assuming that s and $\hat{s}$ take the form

$$s' = s'_{U_F(x)} + s'_{D_F(\hat{x})} + \sigma(s') \cdot \delta_{\tilde{t}}, \tag{A.58}$$

where

$$\sigma(s') = 1 - s'(U_F(x)) - s'(D_F(\hat{x})) \tag{A.59}$$

and $\tilde{t}$ is an arbitrary fixed element of $I_F(x, \hat{x})$.

Consider the distribution

$$s^* = s_{D_F(\hat{x})} + \hat{s}_{U_F(x)} + (1 - \hat{s}(U_F(x)) - s(D_F(\hat{x}))) \cdot \delta_{\tilde{t}}. \tag{A.60}$$

Notice that (A.58) implies $\hat{s}(U_F(x)) + s(D_F(\hat{x})) \leq 1$, so the last term in (A.60) is non-negative, and s^* is a well-defined element of $\mathcal{M}(T)$.

I claim that, no matter how the outcome $F(s^*)$ is specified, some group of agents can block F at $s, \hat{s}$, or s^*, i.e. (A.58) is incompatible with group strategy proofness. I distinguish four cases.

Case 1: $F(s^*) > \hat{x}$.

In this case, F is blocked at s^* by the group of agents with types in $T \backslash U_F(x) \subset D_F(x^*)$. This group comprises agents with types in $D_F(\hat{x}) \cup I_F(x, \hat{x})$. If they coordinate their reports, they can make it appear as if their types were distributed as $\hat{s}(D_F(\hat{x})) + \sigma(\hat{s}) \cdot \delta_{\tilde{t}}$, so the overall distribution of reported types is $\hat{s}$ and induces the outcome $F(\hat{s}) = \hat{x}$, which they all prefer to $F(s^*) > \hat{x}$.

Case 2: $F(s^*) = \hat{x}$.

In this case, F is blocked at s by the group of agents with types in $U_F(x)$. The first inequality in (A.57) implies that these agents can coordinate their reports so as to make it appear as if their types were distributed as $\hat{s}_{U_F(x)} + (s(U_F(x)) - \hat{s}(U_F(x))) \cdot \delta_{\tilde{t}}$. The overall type distribution of reported types is then given as

$$\hat{s}_{U_F(x)} + (s(U_F(x)) - \hat{s}(U_F(x))) \cdot \delta_{\tilde{t}} + s_{D_F(\hat{x})} + \sigma(s) \cdot \delta_{\tilde{t}},$$

which is equal to s^* because

$$s(U_F(x)) - \hat{s}(U_F(x)) + \sigma(s) = 1 - \hat{s}(U_F(x)) - s(D_F(\hat{x})).$$

The induced outcome then is $F(s^*) = \hat{x}$, which they all prefer to $F(s) = x$.

Case 3: $F(s^*) = x$.

In this case, F is blocked at $\hat{s}$ by the group of agents with types in $D_F(\hat{x})$. The second inequality in (A.57) implies that these agents can coordinate their reports so as to make it appear as if their types were distributed as $s_{D_F(\hat{x})} + (\hat{s}(D_F(\hat{x})) - s(D_F(\hat{x}))) \cdot \delta_{\tilde{t}}$. The overall type distribution of reported types is then given as

$$s_{U_F(x)} + s_{D_F(\hat{x})} + (\hat{s}(D_F(\hat{x})) - s(D_F(\hat{x}))) \cdot \delta_{\tilde{t}} + \sigma(\hat{s}) \cdot \delta_{\tilde{t}},$$

which is equal to s^* because

$$\hat{s}(D_F(\hat{x})) - s(D_F(\hat{x})) + \sigma(\hat{s}) = 1 - \hat{s}(U_F(x)) - s(D_F(\hat{x})).$$

The induced outcome then is $F(s^*) = x$, which they all prefer to $F(\hat{s}) = \hat{x}$.

Case 4: $F(s^*) < x$.

In this case, F is blocked at s^* by the group of agents with types in $T \backslash D_F(x) \subset U_F(x^*)$. This group comprises agents with types in $U_F(x) \cup I_F(x, \hat{x})$. If they coordinate their reports, they can make it appear as if their types were distributed as $s(U_F(x)) + \sigma(s) \cdot \delta_{\tilde{t}}$, so the overall distribution of reported types is s and induces the outcome $F(s) = x$, which they all prefer to $F(s^*) < x$.

In each case, one obtains a contradiction to the assumption that F is group strategy-proof. The assumption that s and $\hat{s}$ satisfy (A.57) must therefore be false. ■

Lemma A.17 *Under the specified assumptions, Statement (b) in Theorem A.15 implies Statement (c) in Theorem A.15.*

Proof. Assume that F satisfies statement (b) in Theorem A.15. The claim is that, for all s and $\hat{s}$ in $\mathcal{M}(T)$, $F(s) \neq F(\hat{s})$ implies

$$\begin{array}{rcl} s(P(F(s), F(\hat{s}))) & > & \hat{s}(P(F(s), F(\hat{s}))) \\ & \text{or} & \\ s(P(F(\hat{s}), F(s))) & < & \hat{s}(P(F(\hat{s}), F(s))). \end{array}$$

Let s and $\hat{s}$ in $\mathcal{M}(T)$ be such that $F(s) \neq F(\hat{s})$. Set $F(s) = x$ and $F(\hat{s}) = \hat{x}$ and without loss of generality assume that $\hat{x} > x$. If $R_F \cap (x, \hat{x}) = \emptyset$, one has $U_F(x) = P(\hat{x}, x)$ and $D_F(\hat{x}) = P(x, \hat{x})$, so the claim in the lemma follows trivially from the second part of statement (b) in Theorem A.15.

If $R_F \cap (x, \hat{x}) \neq \emptyset$, Lemma A.14 implies that

$$P(\hat{x}, x) = \{t \in T | g_F^*(t, s') \in P(\hat{x}, x)\} \quad \text{and} \quad P(x, \hat{x}) = \{t \in T | g_F^*(t, s') \in P(x, \hat{x})\}, \tag{A.61}$$

where s' is an arbitrary element of $\mathcal{M}(T)$ and g_F^* is the specified tie-breaking function. Upon using these equations once with $s' = s$ and once with $s' = \hat{s}$, we obtain

$$s(P(x, \hat{x})) = (s \circ (g_F^*(\cdot, s)^{-1})(P(x, \hat{x})) \quad \text{and} \quad s(P(\hat{x}, x)) = (s \circ (g_F^*(\cdot, s)^{-1})(P(\hat{x}, x)), \tag{A.62}$$

as well as

$$\hat{s}(P(x, \hat{x})) = (\hat{s} \circ (g_F^*(\cdot, \hat{s})^{-1})(P(x, \hat{x})) \quad \text{and} \quad \hat{s}(P(\hat{x}, x)) = (\hat{s} \circ (g_F^*(\cdot, \hat{s})^{-1})(P(\hat{x}, x)). \tag{A.63}$$

By the specified assumption about g_F^*, we also have $F(s \circ (g_F^*(\cdot, s)^{-1}) = F(s) = x$ and $F(\hat{s} \circ (g_F^*(\cdot, \hat{s})^{-1}) = F(\hat{s}) = \hat{x}$. Since $s \circ (g_F^*(\cdot, s)^{-1}$ and $\hat{s} \circ (g_F^*(\cdot, \hat{s})^{-1}$ both belong to $\mathcal{M}_F^*(t)$, therefore, Theorems 3.4 and 3.6 imply that

$$(s \circ (g_F^*(\cdot, s)^{-1})(P(x, \hat{x})) > (\hat{s} \circ (g_F^*(\cdot, \hat{s})^{-1})(P(x, \hat{x}))$$

or

$$(s \circ (g_F^*(\cdot, s)^{-1})(P(\hat{x}, x)) < (\hat{s} \circ (g_F^*(\cdot, \hat{s})^{-1})(P(\hat{x}, x)).$$

By (A.62), it follows that $s(P(x, \hat{x})) > \hat{s}(P(x, \hat{x}))$ or $s(P(\hat{x}, x)) < \hat{s}(P(\hat{x}, x))$ as claimed in the lemma. ■

Lemma A.18 *Under the specified assumptions, Statement (c) in Theorem A.15 implies Statement (a) in Theorem A.15.*

Except for the fact that the space of type distributions is now $\mathcal{M}(T)$ rather than $\mathcal{M}_F^*(T)$, the proof of this lemma is step for step the same as the proof of Lemma A.13 and is left to the reader.

Theorem A.15 follows from Lemmas A.16 - A.18.

Theorem 4.1 follows from Theorem A.15 and the observation that, if the tie-breaking function is simple, then F has the Independence Property. Theorem 4.3 follows from Theorem A.15 and Proposition 4.2, the proof of which follows.

A.5 Proof of Proposition 4.2

Proposition 4.2 asserts that, if F is a regular anonymous group strategy-proof social choice function, then, for the contingent tie-breaking function g_F^* such that

$$\begin{aligned} \pi_F(g_F^*(t,s)) &= \pi_F^1(t) = \min \Pi_F(t) \;\text{ if }\; F(s) \leq \pi_F^1(t) \\ &\quad \text{and} \\ \pi_F(g_F^*(t,s)) &= \pi_F^2(t) = \max \Pi_F(t) \;\text{ if }\; F(s) \geq \pi_F^2(t) \end{aligned} \tag{A.64}$$

for any $t \in T$ and $s \in \mathcal{M}(T)$, we have $F(s) = F(G_F^*(s))$ for all $s \in \mathcal{M}(T)$, where $G_F^*(s)$ is short-hand for $s \circ g_F^*(\cdot, s)^{-1}$. The equation $F(s) = F(G_F^*(s))$ results from the two inequalities $F(s) \geq F(G_F^*(s))$ and $F(s) \leq F(G_F^*(s))$. The following lemma establishes the first of these two inequalities.

Lemma A.19 *Under the given assumptions on F, $F(s) \geq F(G_F^*(s))$.*

Proof. Before proceeding with the argument, I note that, by Remark 2.2 and Theorem 3.4, F has the MBV property.

Suppose that, contrary to the lemma, there exists $s \in \mathcal{M}(T)$ such that $F(s) < F(G_F^*(s))$. Let $F(s) = x$ and $F(G_F^*(s)) = \bar{x} > x$.

For any $x' \in R_F$ and $t \in U_F(x')$, we have $\pi_F^2(t) \geq \pi_F^1(t) > x'$, so (A.64) implies $g_F^*(t,s) \in U_F(x')$. Moreover, if $x' \geq x = F(s)$ and $g_F^*(t,s) \in U_F(x')$, we have $\pi_F(g_F^*(t,s)) > x'$ and, by (A.64), $\pi_F(g_F^*(t,s)) = \pi_F^1(t)$, hence also $t \in U_F(x')$. Therefore,

$$G_F^*(U_F(x')|s) = s(U_F(x')) \;\text{ for all } x' \in R_F \cap [x, \infty). \tag{A.65}$$

By Theorem 3.4, we also have $G_F^*(U_F(x')|s) \geq \bar{s}_F(x')$ and hence

$$s(U_F(x')) \geq \bar{s}_F(x') \;\text{ for all }\; x' < \bar{x}.$$

Because the population shares $s(U_F(x'))$ are non-increasing and right-continuous and the thresholds $\bar{s}_F(x')$ are non-decreasing in x', it follows that either

$$s(U_F(x')) = \bar{s}_F(x') \text{ for all } x' \in R_F \cap (x, \bar{x}), \tag{A.66}$$

or

$$s(U_F(x^*)) > \bar{s}_F(x^*) \text{ for some } x^* \in R_F \cap (x, \bar{x}). \tag{A.67}$$

Distinguishing two cases, according to whether (A.66) or (A.67) is true, I will show that the restriction of F to $\mathcal{M}_F^*(T)$ violates part (ii) of the MBV property if (A.66) is true and part (i) of the MBV property if (A.67) is true.

Case 1: s satisfies (A.66).

In this case, s-almost all types in $U_F(x)$ prefer $F(G_F^*(s)) = \bar{x}$ to $F(s) = x$. I claim that

$$s(U_F(x) \backslash U_F(x')) = 0 \tag{A.68}$$

for all $x' \in R_F \cap (x, \bar{x})$. To prove (A.68), it suffices to note that (A.66) and the monotonicity of $s(U_F(x'))$ and $\bar{s}_F(x')$ in x' imply

$$s(U_F(x'')) \geq s(U_F(x')) = \bar{s}_F(x') \geq \bar{s}_F(x'') = s(U_F(x''))$$

for all $x' \in R_F \cap (x, \bar{x})$ and all $x'' \in R_F \cap (x, x')$. Thus, for all such x' and x'', $s(U_F(x'') \backslash U_F(x')) = 0$. (A.68) follows because the function $x'' \rightarrow s(U_F(x'') \backslash U_F(x'))$ is right-continuous.

Consider the distribution

$$s^* := s_{T \backslash U_F(x)} + s(U_F(x)) \cdot \delta_{\bar{t}}, \tag{A.69}$$

where $\bar{t}$ is an arbitrary element of $P^*(\bar{x})$. What can one say about $F(s^*)$ and $F(G_F^*(s^*))$? By Lemma A.1, obviously,

$$F(s^*) \leq \bar{x} \text{ and } F(G_F^*(s^*)) \leq \bar{x}. \tag{A.70}$$

I claim that

$$F(s^*) = x. \tag{A.71}$$

If $F(s^*)$ were smaller than x, the group of agents with types in $P^*(\bar{x})$ could block F at s^* by coordinating their reports so that their reported types would be distributed as $s_{U_F(x)}$, rather than $s(U_F(x)) \cdot \delta_{\bar{t}}$. The overall distribution of reported types would then be s, rather than s^*, and the induced outcome would be $F(s) = x$, which they all prefer to anything below x. Alternatively, if $F(s^*)$ were greater than x, the group of agents with types in $U_F(x)$ could block F at s by coordinating their reports so that their reported types would be distributed as $s^*_{U_F(x)} = s(U_F(x)) \cdot \delta_{\bar{t}}$, rather than $s_{U_F(x)}$. The overall distribution of reported types would then be s^*, rather than s, and the induced outcome would be $F(s^*) \in (x, \bar{x}]$, which, by (A.68), s-almost all of them prefer to $F(s) = x$. Given these arguments, (A.71) follows.

I next show that

$$F(G^*_F(s^*)) \leq x. \tag{A.72}$$

If $F(G^*_F(s^*))$ were greater than x, the group of agents with types in $T \backslash U_F(x)$ could block F at $G^*_F(s^*)$ by coordinating their reports so that their reported types would be distributed as $s^*_{T\backslash U_F(x)}$, rather than $G^*_F(s^*)_{T\backslash U_F(x)}$. The overall distribution of reported types would then be s^*, rather than $G^*_F(s^*)$, and the induced outcome would be $F(s^*) = x$, which they all prefer to anything above x. To see this, it suffices to note that, because $F(s^*) = x$, one has $t \in T\backslash U_F(x)$ if and only if $g^*_F(t, s^*) \in P^*_F(x) \cup D_F(x))$ and hence

$$s^*(T\backslash U_F(x)) = G^*_F(T\backslash U_F(x))|s^*) = G^*_F(P^*_F(x) \cup D_F(x))|s^*).$$

To conclude the proof, I show that F violates part (ii) of the MBV property. We first note that, because $F(s^*) = F(s) = x$, (A.64) implies that, for any $t \in T$,

$$\pi_F(g^*_F(t, s^*)) = \pi_F(g^*_F(t, s)). \tag{A.73}$$

Given that $x^* \leq x$,

$$\begin{aligned} G^*_F(U_F(x^*)|s^*) &= G^*_F(U_F(x^*)\backslash U_F(x)|s^*) + G^*_F(U_F(x)|s^*) \\ &= s^*(\{t \in T|\pi_F(g^*_F(t, s^*)) \in (x^*, x]\}) \\ &\quad + s^*(\{t \in T|\pi_F(g^*_F(t, s^*)) > x\}). \end{aligned}$$

By (A.73) and (A.69), it follows that

$$\begin{aligned} G^*_F(U_F(x^*)|s^*) &= s(\{t \in T|\pi_F(g^*_F(t, s)) \in (x^*, x]\}) \\ &\quad + s(\{t \in T|\pi_F(g^*_F(t, s)) > x\}) \\ &= G^*_F(U_F(x^*)\backslash U_F(x)|s) + G^*_F(U_F(x)|s) \\ &= G^*_F(U_F(x^*)|s). \end{aligned} \tag{A.74}$$

By (A.69), one also has

$$G^*_F(D_F(\bar{x})|s^*) = 1 - s(U_F(x)). \tag{A.75}$$

Moreover, since $G^*(s) \in \mathcal{M}^*_F(T)$,

$$G^*_F(D_F(\bar{x})|s) = 1 - G^*_F(U_F(\bar{x}) \cup P^*_F(\bar{x})|s) \geq 1 - G^*_F(U_F(x)),$$

so, by (A.65),

$$G^*(D_F(\bar{x})|s) \geq 1 - s(U_F(x)). \tag{A.76}$$

Upon combining (A.76) and (A.75), one obtains

$$G^*_F(D_F(\bar{x})|s^*) \leq G^*(D_F(\bar{x})|s). \tag{A.77}$$

(A.77) and (A.74) contradict part (ii) of the MBV property. The assumption that $F(s) < F(G^*_F(s))$ for s satisfying $s(U_F(x)) = \bar{s}_F(x+)$ has thus led to a contradiction and must be false.

Case 2: s satisfies (A.67).
Trivially, in this case, one also has

$$s(U_F(\hat{x})) > \bar{s}_F(\hat{x}) \;\; \text{for all} \;\; \hat{x} \in R_F \cap (x, x^*] \tag{A.78}$$

and

$$s(U_F(x)) > \bar{s}_F(x). \tag{A.79}$$

By single-peakedness,

$$U_F(x) = \bigcup_{\hat{x} \in R_F \cap (x, x^*]} P(\hat{x}, x),$$

where, as before, $P(\hat{x}, x)$ is the set of types that prefer $\hat{x}$ to x. By the monotonicity property $P(\hat{x}', x) \subset P(\hat{x}, x)$ for $\hat{x} \leq \hat{x}'$, which is implied by single-peakedness, it follows that

$$s(U_F(x)) = \inf_{\hat{x} \in R_F \cap (x, x^*]} s(P(\hat{x}, x)).$$

For some $\hat{x} \in R_F \cap (x, x^*]$, therefore, $s(P(\hat{x}, x))$ is close enough to $s(U_F(x))$ so that

$$s(P(\hat{x}, x)) > \bar{s}_F(x). \tag{A.80}$$

For this $\hat{x}$, consider the type distributions

$$\hat{s} := s_{T \backslash P(\hat{x}, x)} + s(P(\hat{x}, x))) \cdot \delta_{\hat{t}}, \tag{A.81}$$

and

$$s^* := s(T \backslash P(\hat{x}, x)) \cdot \delta_t + s(P(\hat{x}, x))) \cdot \delta_{\hat{t}} \tag{A.82}$$

where, as before, $P(\hat{x}, x)$ is the set of types that prefer $\hat{x}$ to x, $\hat{t}$ is an arbitrary fixed element of $P^*(\hat{x})$, and t is an arbitary fixed element of $P^*(x)$.

What can one say about $F(\hat{s})$ and $F(s^*)$? By Lemma A.1, obviously,

$$F(\hat{s}) \leq \hat{x} \;\text{ and }\; x \leq F(s^*) \leq \hat{x}. \tag{A.83}$$

I claim that in fact

$$F(\hat{s}) = x \tag{A.84}$$

and

$$F(s^*) = x. \tag{A.85}$$

If $F(\hat{s})$ were smaller than x, the group of agents with types in $P(\hat{x}, x)$ could block F at $\hat{s}$ by coordinating their reports so that their reported types would be distributed as $s_{P(\hat{x},x)}$, rather than $s(P(\hat{x}, x)) \cdot \delta_{\hat{t}}$. The overall distribution of reported types would then be s, rather than $\hat{s}$, and the induced outcome would be $F(s) = x$, which they all prefer to anything below x. Alternatively, if $F(\hat{s})$ were greater than x, the group of agents with types in $P(\hat{x}, x)$ could block F at s by coordinating their reports so that their reported types would be distributed as $\hat{s}_{P(\hat{x},x)} = s(P(\hat{x}, x)) \cdot \delta_{\hat{t}}$, rather than $s_{P(\hat{x},x)}$. The overall distribution of reported types would then be $\hat{s}$, rather than s, and the induced outcome would be $F(\hat{s}) \in (x, \hat{x}]$, which they all prefer to $F(s) = x$. Given these arguments, (A.84) follows.

To prove (A.85), by contradiction, suppose that $F(s^*) \neq x$, hence, by (A.83), $F(s^*) > x$. Then the group of agents with type $t \in P^*(x)$ can block F at s^* by coordinating their reports so that their reported types would be distributed as $s_{T\backslash P(\hat{x},x)}$, rather than $s(T\backslash P(\hat{x}, x)) \cdot \delta_t$. The overall type distribution then would be $\hat{s}$, rather than s^*, and the induced outcome would be $F(\hat{s}) = x$, which they all prefer to anything above x. F satisfying $F(s^*) \neq x$ would thus fail to be group strategy-proof. (A.85) follows.

To conclude the proof, I show that (A.85) is incompatible with part (i) of the MBV property. By construction, the type distribution s^* belongs to $\mathcal{M}^*_F(T)$ and

$$s^*(U_F(x)) = s(P(\hat{x}, x)). \tag{A.86}$$

By (A.80), it follows that

$$s^*(U_F(x)) > \bar{s}_F(x). \tag{A.87}$$

By part (i) of the MBV property, however, $F(s^*) = x$ implies $s^*(U_F(x)) \leq \bar{s}_F(x)$. The assumption that $F(s) = x < \bar{x} = F(G^*_F(s))$ for s satisfying (A.67) for some $\hat{x} \in (x, \bar{x}]$ has thus led to a contradiction and must be false. This completes the proof of Lemma A.19. ■

A completely symmetric argument, which we leave to the reader, also establishes that $F(s) \leq F(G^*_F(s))$. Proposition 4.2 follows immediately.

B Social Choice with an Atomless Measure Space of Agents

In this appendix, I show how the type distribution formalism that is used in the body of the paper can be derived from a formalism with an atomless measure space of agents, $(A, \mathcal{A}, \alpha)$. We assume that each agent $a \in A$ has a preference ordering $\succeq_{\theta(a)}$ on the set $\mathbb{R}$ of alternatives for social choice and that this ordering can be represented by a utility function of the form $u_a(\cdot) = u(\cdot, \theta(a))$, where $\theta(\cdot)$ is a measurable mapping from A to a complete separable metric space T and the function $u(\cdot, \cdot)$ is upper semi-continuous.

A *social choice function* determines an outcome $x \in \mathbb{R}$ as a function of the mapping $\theta(\cdot)$. I say that the social choice function is *anonymous* if the chosen outcome depends only on the cross-section distribution

$$s(\theta) = \alpha \circ \theta^{-1} \tag{B.1}$$

of preference parameters.[25] Because the mapping $\theta(\cdot)$ from A to T is measurable, this cross-section distribution is well defined and is an element of the set $\mathcal{M}(T)$ of probability measures on T. Thus an anonymous social choice function is given by a mapping $F : \mathcal{M}(T) \to \mathbb{R}$ such that, for any $s \in \mathcal{M}(T)$, $F(s)$ is the outcome chosen if the cross-section distribution of preference parameters is s.

Individual Strategy Proofness. In the measure space formulation, we say that an anonymous social choice function $F : \mathcal{M}(T) \to \mathbb{R}$ is *individually strategy-proof* if, for every $\hat{a} \in A$, every measurable function $\theta : A \to T$, and every $t' \in T$,

$$u(F(s(\theta)), \theta(\hat{a})) \geq u(F(s(\hat{\theta}(\theta, \hat{a}, t'))), \theta(\hat{a})), \tag{B.2}$$

where $\hat{\theta}(\theta, \hat{a}, t')) : A \to T$ is the mapping satisfying

$$\hat{\theta}(\hat{a}|\theta, \hat{a}, t')) = t' \quad \text{and} \quad \hat{\theta}(a'|\theta, \hat{a}, t')) = \theta(a') \tag{B.3}$$

for $a' \in A \backslash \{\hat{a}\}$. By (B.1), this formulation of individual strategy proofness is obviously equivalent to the one given in the text. Moreover, because the

[25] If the measure space $(A, \mathcal{A}, \alpha)$ is homogeneous, this definition of anonymity is equivalent to the requirement that social choice is unchanged under any permutation of agents' names. If $(A, \mathcal{A}, \alpha)$ is not homogeneous, the definition in the text is stronger than the requirement of invariance under permutations of names. As discussed in Section 4 of Khan and Sun (1999), homogeneity holds if $(A, \mathcal{A}, \alpha)$ is a hyperfinite Loeb space and fails to hold if $(A, \mathcal{A}, \alpha)$ is the Lebesgue unit interval.

measure α is atomless, we have

$$s(\hat{\theta}(\theta, \hat{a}, t')) = s(\theta) \tag{B.4}$$

for every $\hat{a} \in A$, every measurable function $\theta : A \to T$, and every $t' \in T$. This observation yields the conclusion of Proposition 2.1.

Group Strategy Proofness. Turning to group strategy proofnes, I say that, given an anonymous social choice function F and a measurable function $\theta : A \to T$, a set $\hat{A} \in \mathcal{A}$ *blocks* F *at* θ if there exists a measurable function $\theta' : \hat{A} \to T$, such that

$$u(F(s(\theta)), \theta(\hat{a})) < u(F(s(\hat{\theta}(\theta, \hat{A}, \theta'))), \theta(\hat{a})), \tag{B.5}$$

for all $\hat{a} \in \hat{A}$ where $\hat{\theta}(\theta, \hat{A}, \theta')$ is the mapping from A to T such that

$$\hat{\theta}(\hat{a}|\theta, \hat{A}, \theta') = \theta'(\hat{a}) \tag{B.6}$$

for $\hat{a} \in \hat{A}$ and

$$\hat{\theta}(a'|\theta, \hat{A}, \theta') = \theta(a') \tag{B.7}$$

for $a' \in A \backslash \hat{A}$. F is said to be *group strategy-proof* if there are no $\hat{A} \in \mathcal{A}$ and $\theta : A \to T$ such that $\hat{A}$ blocks F at θ.

In terms of the space $\mathcal{M}(T)$, the domain of the social choice function, recall that in the formalism of the main text a Borel set $W \subset T$ *blocks* F *at* s if there exists $s' \in \mathcal{M}(T)$ such that

$$u(F(s), t) < u(F(s_{T \backslash W} + s(W) \cdot s'), t) \tag{B.8}$$

for all $t \in W$, where $s_{T \backslash W}$ is the restriction of s to the set $T \backslash W$. The following remark shows that the measure space formulation of group strategy proofness given here is in fact equivalent to the distribution formulation of group strategy proofness in the main text.

Remark B.1 *An anonymous social choice function F is group strategy-proof if and only if there are no $W \subset T$ and $s \in \mathcal{M}(T)$ such that W blocks F at s.*

Proof. To prove the "if" part of the remark, suppose that F is not group strategy-proof and let $\hat{A} \in \mathcal{A}$ and $\theta : A \to T$ be such that $\hat{A}$ blocks F at θ. Let $\theta' : \hat{A} \to T$ be such that (B.5) holds for all $\hat{a} \in \hat{A}$. Since (B.5) implies $F(s(\hat{\theta}(\theta, \hat{A}, \theta'))) \neq F(s(\theta))$, one must have $\alpha(\hat{A}) > 0$ so one may define

$$s' = \frac{1}{\alpha(\hat{A})} \cdot \alpha_{\hat{A}} \circ (\theta')^{-1}, \tag{B.9}$$

where $\alpha_{\hat{A}}$ is the restriction of α to $\hat{A}$. Given this specification of s', one easily verifies that (B.5) holds for $W := \theta(\hat{A}), s = \alpha \circ \theta^{-1}$,and all $t \in W$. Thus $W = \theta(\hat{A})$ blocks F at $s = \alpha \circ \theta^{-1}$. Conversely, if there are no $W \subset T$ and $s \in \mathcal{M}(T)$ such that W blocks F at s, F must be group strategy-proof.

To prove the "only if" part of the remark, suppose that there exist $W \subset T$ and $s \in \mathcal{M}(T)$ such that W blocks F at s. Using the fact that α is an atomless measure, let $\theta : A \to T$ be such that $\alpha \circ \theta^{-1} = s$ and let $\hat{A} = \theta^{-1}(W)$. Since (B.5) implies $F(s) \neq F(s_{T\setminus W} + s(W) \cdot s')$ and therefore $s(W) > 0$, one has $\alpha(\hat{A}) > 0$. Again using the fact that α is an atomless measure, one may define $\theta' : \hat{A} \to T$ so that (B.9) holds for the distribution s' with which W blocks F at s. Then $\hat{A} \in \mathcal{A}$ *blocks* F at θ. Conversely, if F is group strategy-proof, there must not exist any $W \subset T$ and $s \in \mathcal{M}(T)$ such that W blocks F at s. ■

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