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ABSTRACT

Directed Search with Phantom Vacancies

When vacancies are filled, the ads that were posted are often not withdrawn, creating "phantom" vacancies. The existence of phantoms implies that older job listings are less likely to represent true vacancies than are younger ones. We assume that job seekers direct their search based on the listing age and so equalize the expected benefit of a job application across listing age. Forming a match with a vacancy of age α creates a phantom of age α with probability β and this leads to a negative informational externality that affects all vacancies of age α and older. Thus, the magnitude of this externality decreases with the age of the listing when the match is formed. Relative to the constrained efficient search behavior, the directed search of job seekers leads them to over-apply to younger listings. We illustrate the model using US labor market data. The contribution of phantoms to overall frictions is large, but, conditional on the existence of phantoms, the social planner cannot improve much on the directed search allocation.

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1 Introduction

This paper is based on two premises. First, many listings for job openings that are advertised on job boards or newspapers or are heard about from friends and acquaintances are out of date. We use the concept of *phantom vacancies* to model this out-of-date information, where by a phantom vacancy we mean a job listing that continues to be advertised even though the vacancy has already been filled. Second, employers and job seekers are aware of this stale information and adjust their search behavior accordingly. On job boards, searchers can observe the posting date for job listings. They understand that older listings are more likely to be phantoms. They also understand that other searchers also understand this so there is likely to be more competition at younger listings. Workers take these countervailing forces into account when directing their search based on listing age. In the directed search equilibrium that we analyze, workers follow a mixed strategy with respect to listing age that trades off the probability the listing is a phantom against the extent of competition for the position. On the other side of the market, firms can renew their vacancies by paying a relisting cost. Doing so informs workers that the vacancy is still unfilled.

We argue that phantom vacancies are an important source of labor market frictions and hence unemployment. Why is there unemployment? A job seeker may fail to find an advertised position that matches his or her skills. Alternatively, appropriate positions may be advertised, but the job seeker's application may be met with the response, "Sorry, but the job has already been filled." Job search theory has focused on the former while ignoring the latter friction, i.e., phantom vacancies, which we emphasize here. And, as online job search becomes more common, we argue that the frictions caused by phantoms may become relatively more important as a source of unemployment. From the individual job seeker's perspective, with online search, it should be easier to identify appropriate advertised positions, but it may become more difficult to be sure that they haven't already been filled.

The concept of phantom vacancies was introduced in Chéron and Decreuse (2017), and we use the matching function developed in that paper. Chéron and Decreuse (2017) is a model of *random search* in the sense that a job seeker is just as likely to apply to any one listing as to any other; i.e., the unemployed are assumed to be unable to adapt to the existence of phantoms. In contrast, ours is a model of *directed search* – job seekers can direct their search based on listing age and can thus take the existence of phantoms into account.

We have two objectives in our paper. The first is to characterize the directed search equilibrium. In equilibrium, searchers allocate themselves across "submarkets" that are defined by listing age. Worker directed search satisfies a no-arbitrage condition, namely, that the expected payoffs associated with searching in the various submarkets must be equalized. On the other side of the market, firms decide how many new vacancies to list. We also allow firms to "renew" their listings – for a fee, a firm can relist its vacancy, thereby effectively resetting its age to zero.

Our second objective is to characterize the constrained efficient allocation and to understand how and why it differs from the equilibrium allocation. The nature of the constrained efficient allocation depends on the tools we allow the social planner to employ. We first suppose that the social planner can determine firms' vacancy posting and listing renewal behavior. Second, we suppose that the social planner can also choose the allocation of searchers across submarkets; i.e., we allow the social planner to direct job seekers' search. Of course, the social planner is constrained by the same informational friction that workers face, namely, the planner cannot see whether a particular listing is a real vacancy or a phantom.

Before we address these questions, we offer some evidence for our two premises. Are phantoms important in real-world labor markets? To the extent that workers apply for jobs that are already filled,[1] and casual empiricism suggests this is often the case, phantoms matter. More formally, Chéron and Decreuse (2017) present evidence of phantoms from Craigslist, an online job site, showing that the distribution of job listings by age over one month (the time at which Craigslist destroys ads) is uniform by week. This implies that ads are not withdrawn as soon as the corresponding jobs are filled; instead, job listings persist for some time as phantoms. We are not arguing that all job boards fail to remove every obsolete listing immediately, but the Craigslist evidence clearly suggests the existence of stale information in the labor market. Similarly, Davis and Samaniego de la Parra (2019) note that "...stale postings occur frequently on many prominent online platforms for posting job vacancies, and they create distinctive matching frictions and information externalities." (p.7) See also footnote 17 (pp.13-14) in their paper, which discusses the existence of phantoms on CareerBuilder.com and Monster.com.[2] Using French data, Skandalis (2019) presents evidence on phantoms by looking at the effects of media announcements of plants' hiring needs. Even though vacancy postings do not change in windows of several weeks around the dates of the announcements, the media coverage substantially increases worker applications. As she notes, this result is consistent with the idea that "job seekers are uncertain about the payoff from applying to a vacancy and direct their applications

[1] In Acharya and Wee (2018), firms that have filled their vacancies continue to advertise those positions in an effort to find more productive "replacement" workers. The longer a job has been advertised, the less likely it is that an application for that position will be successful. In this sense, these listings are similar to phantoms. The important difference between our model and that of Acharya and Wee (2018) is that we allow job seekers to respond to the existence of phantoms by directing their search.

[2] There is also evidence of phantoms in other markets with search frictions. Fradkin (2019) documents that on Airbnb, about 15% of first attempts to make a booking by prospective renters in his sample fail due to "stale vacancies," i.e., because the hosts failed to block out specific dates on their calendars promptly even though those dates were in fact not available.

to newer vacancies in order to minimize the risk of applying to a phantom." (p.5). Finally, in a recent memo the AEA ad hoc Committee on the Job Market notes that it is rare for the listings on Job Openings for Economists for searches that have been canceled to be removed.[3] That is, there are phantoms in the market for academic economists.

There is also evidence that job seekers direct their search towards recently posted job listings. First, as the following query to AskaManager.com suggests, this is what some job seekers say they do:

> I am currently on the job hunt and I had a question about applying to jobs online. You know how most websites will tell you the job has been posted 1 day ago, 28 days ago, etc. For some reason, I have concluded that I need to apply to a job the first week they post the position to have the best chances of being hired. Although I heard that it can take up to a month for the company to hire anyone for the position, I feel that applying to a job that was posted 3 weeks ago isn't that promising. What is your take on this situation?

Second, data that link applications to vacancy postings indicate that job applicants direct their search towards younger listings. Using data from a stratified 5% random sample of Dutch establishments, Van Ours and Ridder (1992) report that "Almost all applications arrive in a short period just after the vacancy has been announced." (p. 142) More recently, using data from Dice.com, the primary source for the DHI Vacancy and Flow Applications Database, Davis and Samaniego (2019) find (p.4) that "job seekers exhibit a striking propensity to target new and recently posted vacancies: 47 percent of applications flow to vacancies posted in the last 48 hours, and 63 percent go to those posted in the last 96 hours. Applications per vacancy per unit time drop sharply as postings age." The results of Belot, Kircher and Muller (2018, pp.16-17) show the same pattern. In their audit study in which paired vacancies differed only in the posted wage and the listing date, they find that job seekers were substantially more likely to save the younger listing, even if the difference in the listing date was only one or two days. Finally, another piece of evidence that workers react to listing age is that firms choose to repost their vacancies even though there is a cost to doing so. Listing renewal appears to be quite common. Employers relist their vacancies in order to let workers know that the jobs are still unfilled.

There are other explanations for why job seekers target younger listings. In stock-flow models such as Coles and Smith (1998), new job seekers flowing into the market search through

[3] The memo can be found at: https://www.aeaweb.org/joe/communications/memo-aug-11-2020

the entire stock of listed vacancies.[4] A job seeker who fails to find a match in this first search step, having already examined the extant stock, is then limited to searching through the inflow of new listings. As with phantoms, stock-flow implies that most applications go to younger listings. Two pieces of evidence suggest that stock-flow matching may not be the sole explanation for the way that job seekers direct their search. First, as we noted above, listing renewal appears to be quite common. In a stock-flow model, employers have no incentive to relist. Second, the stock-flow model can be tested by looking at how worker application behavior changes with elapsed duration of unemployment. All else equal, recently unemployed workers should be equally likely to apply to a young listing as to an old one, while workers who have been unemployed for a longer time should only apply to young listings. Data on how worker application behavior varies over an unemployment spell are scarce, but there is one study that addresses this question. Using data from SnagAJob, Faberman and Kudlyak (2014) find (p.4) that "The fraction of applicants to a newly-posted vacancy rises with duration, consistent with a stock-flow model, but it does so only slightly ..."

Another possible explanation for why worker search is concentrated on younger listings is job heterogeneity. Some jobs are "lemons" – a worker contacting such a job is unlikely to find it acceptable or they may find the employer is overly picky – while other jobs are "plums." Lemons would then be over-represented in the stock of old listings, and workers would respond by directing their applications towards young listings.[5] However, it is difficult to imagine that the existence of lemons explains why 47% of the applications observed in the Dice.com data go to listings that are at most 48 hours old. While stock-flow matching and lemons may be part of the reason that young listings receive more applications than old listings do, the evidence suggests that phantom vacancies are also an important part of the explanation.

In the model we present below, we focus on phantoms and abstract from considerations of stock-flow matching and lemons. We use a continuous-time model of sequential search in which unemployed workers apply for one job at a time. An alternative would be to allow for multiple applications in a discrete-time model, but as both Albrecht, Gautier, and Vroman (2006) and Galenianos and Kircher (2008) show, this would introduce other inefficiencies, and in this paper, we are interested in concentrating on the externality caused by phantom vacancies. Another alternative would be to assume nonsequential search in which firms actively recruit and screen workers, e.g., Wolthoff (2018), but, in order to focus clearly on the role that phantoms play in

[4] This assumption seems most applicable when an intermediary is present to help job seekers identify all appropriate job openings in the stock of vacancies. In Coles and Smith (1998), that intermediary role is played by the UK Jobs Centre.

[5] The idea that time on the market may be used as a signal of quality when search frictions are present has been explored in several papers, e.g., Kim (2017).

labor market equilibrium, we have chosen to follow the main thread of the literature and to assume that search is sequential.[6]

Our basic results are as follows. In the decentralized equilibrium allocation, workers follow a mixed strategy with respect to the choice of submarket. Specifically, workers allocate their applications across submarkets so that the values of applying to listings of different ages are equalized. Younger listings, i.e., new listings and recently renewed listings, receive a high weight and a disproportionate number of applications while older listings receive relatively few applications. If the social planner is limited to choosing how many new vacancies are posted and the age at which job listings are renewed, we show there is a generalized Hosios condition that implements the constrained efficient allocation. In this case, if firms can post and commit to a wage that is independent of the age at which the vacancy is filled, a competitive search equilibrium decentralizes the social planner allocation.[7] However, when the social planner can also choose the allocation of job seekers across submarkets, then the equilibrium allocation is generically inefficient. In equilibrium, job seekers direct their search more toward younger listings than the social planner would have them do. The equilibrium allocation of worker applications across listing ages generates a dynamic congestion externality. When matches are formed, phantoms congest the market, and a phantom that is created when a younger listing leads to a match is more costly than one that is created when an older listing does so. That is, the magnitude of the externality decreases with the listing age of the vacancy, and workers have no incentive to incorporate this dynamic effect into their decision calculus.[8] The allocation that results when the social planner can choose the flow of new vacancies, the listing renewal age and the allocation

[6] Van Ours and Ridder (1992) argue that employers search is nonsequential. Similarly, based on their analysis of the Dice.com data, Davis and Samaniego (2019) suggest that models of nonsequential search are the better tool for analyzing labor market outcomes. Phantoms are likely to be important even if firms search nonsequentially. With nonsequential search, firms advertise a position and then collect a list of applicants. At some point – when the applicant list is long enough – the firm makes an offer. The longer a position has been advertised, the more likely it is that an offer has been made and accepted, i.e., the more likely it is that the listing is a phantom.

[7] Our baseline model is one of directed search rather than competitive search. Workers direct their search based solely on listing age. Listing age is a variable that firms cannot control (except by renewal), i.e., they cannot compete by choosing listing age. In competitive search equilibrium, however, firms can compete for worker applications through their posted wages.

[8] A related effect is present in the frictionless dynamic matching model of Board, Meyer-ter-Vehn and Sadzik (2020). In their model, each firm hires a worker from a pool of applicants. Firms with more talented incumbent workers are better at recruiting (distinguishing good from bad hires) and offer higher wages in order recruit before other firms. This leads to dynamic adverse selection – the further down a firm is in the recruiting queue, the lower is the average quality of the applicant pool it faces. The analogy with our result is that the adverse selection that a firm creates through its hiring decision is greater the higher the firm's position in the recruiting queue. The externality created thus has a cascading quality as does our phantom externality.

of job seekers across submarkets can also be decentralized in competitive search equilibrium. However, this competitive search equilibrium requires firms to post wage schedules, i.e., wages that vary with the age of the vacancy at the time the match is formed, a phenomenon that we do not observe in reality.

We supplement our theoretical results with numerical illustrations. We do this in three steps. First, we select parameter values so that the decentralized equilibrium of our model matches some key US labor market outcomes over the period 2000-2008. The nature of the decentralized equilibrium depends on the value we choose for the worker bargaining power parameter, and we set this parameter to its (endogenously determined) generalized Hosios value. In our calibration, we make a baseline parameter assumption, namely, that the fraction of ads that are not removed when the corresponding vacancy is filled, β, equals 1/2. In the second step of our numerical illustrations, we examine the sensitivity of our results to changes in β. Varying this parameter has strong effects on how workers direct their search and on the overall unemployment rate. Finally, we numerically solve for the full constrained efficient allocation; i.e., we allow the social planner to allocate job seekers across listing ages in addition to choosing the level of vacancy creation and the listing renewal age. We then compare the decentralized allocation with the social planner solution. Allowing the planner to direct worker search increases aggregate steady-state consumption by a bit less than 1%, but this gain comes almost entirely from reduced vacancy management costs rather than reduced unemployment. The reason that allowing the planner to direct worker search has little effect on unemployment is that optimizing the way that workers direct their search leads to more matches, but these additional matches in turn generate more phantoms.

In the next section, we present our model of directed search with phantom vacancies. We then discuss the social planner's problem in Section 3. Section 4 contains our numerical simulations and conclusions are given in Section 5.

2 Decentralized Equilibrium

In this section, we develop an equilibrium search and matching model of unemployment with directed search by listing age, phantom vacancies, endogenous job creation and endogenous listing renewal. We set up the model and characterize the decentralized allocation.

2.1 The model setup

We focus on the stationary state of a continuous time model. The unit of time is a month. There is a continuum of workers of unit mass, and each worker can be either employed or unemployed. The endogenous mass of unemployed is u, and there is also a continuum of vacancies of endogenous mass v. There is a listing for each vacancy, and these differ by age, $a \geq 0$.

Creating a new vacancy comes at a one-time cost c. Once the vacancy is created, the firm has a listing that gradually ages with calendar time. The listing can be renewed at any time at cost k, $0 < k < c$. In exchange for the renewal cost, the firm has a new listing of age 0. We will endogenize the age, A, at which a firm that has failed to fill its vacancy renews its listing. Filled jobs produce y. All jobs, filled or vacant, are destroyed at Poisson rate λ, and newly separated workers join the pool of unemployed.

The labor market is segmented by listing age. In submarket a, $u(a)$ unemployed try to match with $v(a)$ vacancies. Each time there is a match or when a vacancy is destroyed, with probability β, the corresponding ad is not withdrawn and a phantom vacancy is created. Phantoms, once created, persist in the market. There are $p(a)$ phantoms in submarket a.

The matching process is frictional. Along with the usual search frictions, information persistence about vacancies that have already been filled or destroyed but are still advertised creates an added friction. The flow of new matches in submarket a is

$$M(a) = \pi(a)m(u(a), v(a) + p(a)),$$

where $\pi(a) = \frac{v(a)}{v(a)+p(a)}$ is the nonphantom proportion. The function, $m(\cdot)$, is strictly increasing in both arguments, strictly concave and has constant returns to scale.[9]

Workers cannot distinguish between phantoms and true vacancies. Therefore $m(\cdot)$ depends on the number of listings $v(a)+p(a)$. As no one can match with a phantom, $m(\cdot)$ is multiplied by the fraction of contacts that are with unfilled vacancies. If workers could match with phantoms as well as vacancies, then $m(\cdot)$ would be the standard DMP matching function.

We denote market tightness for listings of age a by $\theta(a) = (v(a)+p(a))/u(a)$. The job-finding rate for submarket a is $\mu(a) = m(1, \theta(a))\pi(a)$ and the job-filling rate is $\eta(a) = m(1, \theta(a))/\theta(a)$. We write $m(\theta)$ for $m(1, \theta(a))$ below when there is no risk of confusion.

Each time a vacancy is filled or destroyed, a phantom is created with probability $\beta \in [0, 1]$. For all $a \in [0, A]$, phantoms and vacancies evolve as follows:

$$\begin{aligned} \dot{v}(a) &= -(\eta(a) + \lambda)v(a), \\ \dot{p}(a) &= \beta(\eta(a) + \lambda)v(a), \end{aligned}$$

where a dot over a variable denotes its derivative with respect to the listing age. These laws of motion imply $p(a) = \beta(v(0) - v(a))$.

[9] The form of $M(a)$ is taken from Chéron and Decreuse (2017). They derive their matching function from first principles in discrete time and then extend it to continuous time. As noted in the introduction, their analysis is based on random search, whereas ours is a model of directed search based on listing age.

The nonphantom proportion therefore has the following law of motion:

$$\dot{\pi}(a) = -\left[\eta(a) + \lambda\right]\pi(a)(1 - (1-\beta)\pi(a)).$$

The nonphantom proportion decreases with age. Phantoms accumulate as employers gradually fill their jobs and as vacancies are destroyed at exogenous rate λ.

2.2 Equilibrium

Workers direct their search by listing age. The values of being unemployed, U, and employed, W, are defined as follows:

$$rU = \max_{a\in[0,A]}\{b + \mu(a)[W - U]\}, \tag{1}$$

$$rW = w + \lambda[U - W]. \tag{2}$$

In equilibrium, since jobs are homogenous, job seekers allocate themselves across the different listing ages so that the job-finding rate $\mu(a) = \pi(a)m(\theta(a))$ stays constant; i.e., since $\pi(0) = 1$,

$$\pi(a)m(\theta(a)) = m(\theta(0)) \text{ for all } a.$$

Since the nonphantom proportion decreases with listing age, this no-arbitrage condition implies that market tightness increases with listing age.

Let $V(a)$ be the value of a vacancy of age a and let J be the value of a filled job. We have

$$rV(a) = \eta(a)[J - V(a)] - \lambda V(a) + \dot{V}(a), \tag{3}$$

$$rJ = y - w - \lambda J. \tag{4}$$

The value of a vacancy changes with the listing age, reflecting the rate of applications by vacancy age and the length of time until renewal. Immediately after renewal, the listing age is reset to 0. By continuity of the value function, we have $V(A) = V(0) - k$. Let $\tau(a) = \exp(-\int_0^a(\eta(s)+\lambda)ds)$ be the survival probability for a vacancy, i.e., the probability that the vacancy is neither filled nor destroyed by age a. That is, $\tau(a) = v(a)/v(0)$. Integrating equation (3) forward with this boundary condition gives

$$V(0) = \frac{J\int_0^A \eta(s)e^{-rs}\tau(s)ds - ke^{-rA}\tau(A)}{1 - e^{-rA}\tau(A)}. \tag{5}$$

To derive the optimal renewal age, we need to discuss the firms' coordination problem. Consider a firm with the belief that all other firms set renewal age $\tilde{A} > 0$ and suppose that rational workers also hold this belief. In this case, workers would not apply to job listings older than $\tilde{A}$. They would suppose that these must be phantoms. It follows that the firm must either set a renewal age less than or equal to $\tilde{A}$. Keeping the listing after $\tilde{A}$ would be pointless because the job-filling rate would be 0.

Suppose first that $\tilde{A}$ is very large. The optimal renewal age $\hat{A}$, the firm's best-response to $\tilde{A}$, results from the condition $\dot{V}(\hat{A}) = 0$. Using equation (3), we obtain

$$V(\hat{A}) = V(0) - k = \frac{\eta(\hat{A})}{r + \lambda + \eta(\hat{A})} J. \tag{6}$$

Firms renew their listings when the rate of applications to their vacancies becomes sufficiently small.

Now suppose that $\tilde{A}$ is small, i.e., smaller than the $\hat{A}$ given by equation (6). Then the firm sets $A = \tilde{A}$. It follows that any $\tilde{A}$ belonging to $[0, \hat{A}]$ can be an equilibrium of the renewal game. Hereafter, we only focus on equilibria such that $\tilde{A} = \hat{A}$.[10] As we show later, an equilibrium of this type produces the choice of renewal age, A, that the social planner chooses when the planner can select the renewal age and the number of vacancies to create but cannot direct the job seekers' choice of submarket.

The wage is determined by Nash bargaining over the match surplus. We assume this wage can be renegotiated at any time, which explains why it is not conditional on the listing age. We also assume that the job is destroyed if the firm and the worker do not reach an agreement. This means that the firm's outside option is 0. If $\gamma \in (0, 1)$ denotes worker bargaining power, then the wage that maximizes the Nash product solves $(1 - \gamma)(W - U) = \gamma J$. Using equations (1) to (4) and solving for the wage gives

$$\begin{aligned} w &= \gamma y \frac{r + \lambda + m(\theta(0))}{r + \lambda + \gamma m(\theta(0))} + (1 - \gamma) b \frac{r + \lambda}{r + \lambda + \gamma m(\theta(0))} \text{ and} \\ J &= \frac{(1 - \gamma)(y - b)}{r + \lambda + \gamma m(\theta(0))}. \end{aligned}$$

Finally, free entry implies that firms create vacancies until $V(0) = c$.

The equilibrium allocation is characterized by the following system of equations:

$$\begin{aligned} m(\theta(0)) &= \pi(a) m(\theta(a)), & (7) \\ \dot{\pi}(a) &= - [\eta(a) + \lambda] \pi(a) [1 - (1 - \beta) \pi(a)], & (8) \\ c(1 - e^{-rA} \tau(A)) + k e^{-rA} \tau(A) &= \frac{(1 - \gamma)(y - b)}{r + \lambda + \gamma m(\theta(0))} \int_0^A \eta(a) e^{-ra} \tau(a) da, & (9) \\ c - k &= \frac{\eta(A)}{r + \lambda + \eta(A)} \frac{(1 - \gamma)(y - b)}{r + \lambda + \gamma m(\theta(0))}, & (10) \\ \mathsf{u} &= \lambda / (\lambda + m(\theta(0))), & (11) \end{aligned}$$

[10] Equilibria in the listing renewal age game must be symmetric. If there were an asymmetric equilibrium, there would be at least two distinct $\hat{A}$ solving the first-order condition (6), say A_1 and A_2, with $A_1 < A_2$ without loss of generality. This would imply that $\theta(A_1) = \theta(A_2)$, with $\theta(a) > 0$ for $a \in [A_1, A_2]$. This is impossible because the no-arbitrage condition states $\pi(a) m(\theta(a)) = m(\theta(0))$, whereas $\pi(A_1) > \pi(A_2)$.

with $\pi(0) = 1$ and $\eta(a) = m(\theta(a))/\theta(a)$.

In the free-entry condition (9), the left-hand side is the expected stock cost of a new listing. The cost of renewal is weighted by the discounted probability $e^{-rA}\tau(A)$ that the job is still available at the renewal age. The cost of a new vacancy is weighted by the complementary term, $1 - e^{-rA}\tau(A)$. The right-hand side is the value of a filled job, $\frac{(1-\gamma)(y-b)}{r+\lambda+\gamma m(\theta(0))}$, multiplied by the discounted probability that the job is filled before the renewal age is reached.

From the no-arbitrage condition (7), we can express tightness in submarket a as a function of initial tightness and the nonphantom proportion, i.e., $\theta(a) = m^{-1}(m(\theta(0))/\pi(a))$. We then insert this into the law of motion of the nonphantom proportion (8) to obtain the following Cauchy problem:

$$\dot{\pi}(a) = -\left[\frac{m(\theta(0))/\pi(a)}{m^{-1}(m(\theta(0))/\pi(a))} + \lambda\right]\pi(a)[1 - (1-\beta)\pi(a)], \pi(0) = 1.$$

We write the solution to this problem as $\pi(a, \theta(0))$ to highlight the dependence on $\theta(0)$. Solving for equilibrium reduces to finding the initial tightness $\theta(0)$ and the optimal renewal age such that the free-entry condition (9) and the optimal renewal condition (10) hold. Lastly, unemployment is determined by the Beveridge curve (11).

There exists an equilibrium provided $(1-\gamma)(y-b)/(r+\lambda) > c$. The left-hand side is the maximum value of a filled job. This maximum value must exceed the cost of job creation. Initial tightness qualitatively responds to changes in the economic environment in a standard way. It decreases with worker bargaining power, γ, unemployment income, b, the discount rate, r, the job destruction rate, λ, and the vacancy creation cost, c. It increases with output per worker, y. Unemployment varies accordingly.

The density function of listings by listing age is

$$\phi_{v+p}(a) = (v(a) + p(a))/\int_0^A (v(s) + p(s))ds$$

and the associated cumulative distribution function is

$$\Phi_{v+p}(a) = \int_0^a (v(s) + p(s))ds/\int_0^A (v(s) + p(s))ds.$$

The growth rate of the density is $\dot{\phi}_{v+p}/\phi_{v+p} = -(1-\beta)(\eta+\lambda)\pi \leq 0$. The growth rate is generally declining, reflecting the listing stock depletion as workers gradually find jobs. However, phantom formation reduces the rate of depletion. The listing density does not vary with age when $\beta = 1$ because the job-filling rate is exactly offset by the rate at which phantoms are created.

The density function of applications by listing age is $\phi_u(a) = u(a)/\mathsf{u} = (v(a)+p(a))/(\theta(a)\mathsf{u})$ and the associated cumulative distribution function is $\Phi_u(a) = \mathsf{u}^{-1}\int_0^a (v(s)+p(s))\theta(s)^{-1}ds$. The growth rate of the density is $\dot{\phi}_u/\phi_u = \dot{\phi}_{v+p}/\phi_{v+p} - \dot{\theta}/\theta < \dot{\phi}_{v+p}/\phi_{v+p}$ when $\beta > 0$, i.e., when there are phantoms in the market. Since the growth rate of the density of applications with age is less than the corresponding growth rate of listings, the model predicts a concentration of job seekers' efforts at younger listing ages than would occur in the absence of phantoms.

To conclude this section, we briefly discuss the case without phantoms. This corresponds to $\beta = 0$. The nonphantom proportion is one at all ages and the no-arbitrage condition becomes $m(\theta(a)) = m(\theta(0))$. Tightness does not change across submarkets. As the job-filling rate does not decrease with age, firms have no incentive to renew their listings. Formally, the value $V(A) = V(0)$ for all possible renewal ages so it is not worth paying the cost k, as small as it may be, to get a new listing. Thus, $A \to \infty$ and the free-entry condition becomes

$$c = \frac{\eta(\theta(0))}{r + \eta(\theta(0)) + \lambda} \frac{(1-\gamma)(y-b)}{r + \lambda + \gamma m(\theta(0))}. \tag{12}$$

This equation is analogous to the standard free-entry condition with a flow cost of posting jobs. The left-hand side is a stock cost and therefore the right-hand side is a fraction of the value of a filled job. The economic mechanism behind entry, though, is fundamentally the same as in the standard model. An increase in tightness has two effects that reduce the value of a vacancy. First, it increases the bargained wage, thereby lowering the value of a filled job. Second it decreases the job-filling rate.

3 Constrained efficient allocation

The nature of the constrained efficient allocation depends on the instruments available to the social planner. We consider two cases. First, we assume that the planner can choose the flow of new vacancies and the vacancy renewal age but cannot direct worker search by listing age. In this case, the constrained efficient allocation can be decentralized if the worker bargaining share satisfies a modified Hosios condition. Equivalently, competitive search equilibrium in which firms post and commit to a listing-age-invariant wage generates the constrained efficient allocation. In the numerical illustrations that we present in Section 4, we use this competitive search equilibrium as our decentralized allocation.

In the second case, we expand the set of instruments available to the planner by also allowing the planner to direct worker search. Again, the constrained efficient allocation can be decentralized by a competitive search equilibrium. However, in this case, the competitive search equilibrium requires firms to post wage schedules in which they commit to pay a wage that varies with the listing age of the vacancy at the time it is filled. Absent listing-age-dependent wages, which we do not in fact observe, a novel externality arises: in the decentralized equilibrium, phantoms lead job seekers to over-apply to young listings relative to the social planner optimum.

In both cases, to focus on steady states, we study the constrained efficient allocation when the discount rate, r, tends to 0, and, in both cases, the planner maximizes net flow output less vacancy creation costs and listing renewal costs, i.e.,

$$\Omega = b + (1-\mathbf{u})(y-b) - c\underbrace{[v(0) - v(A)]}_{\text{new vacancies}} - k\underbrace{v(A)}_{\text{renewals}} . \tag{13}$$

3.1 Social Planner: Case 1

In this subsection, we assume that the social planner can choose the flow of new vacancies, $v(0) - v(A)$, and the renewal age, A, but is unable to allocate job seekers across listing ages. As a result, worker directed search implies that the job-finding rate must be the same in all submarkets.

It is useful to express $v(0)$ and $v(A)$ as functions of θ_0 and A, where θ_0 is the planner's choice of initial market tightness. This allows us to write the planner's objective in a more convenient form. To do this, we use the steady-state condition that the flow of filled vacancies equals the flow of workers who find jobs, i.e.,

$$\begin{aligned} v(0)\int_0^A \eta(a)\tau(a)da &= m(\theta_0)\mathbf{u} \text{ or} \\ v(0) &= \frac{m(\theta_0)\mathbf{u}}{\int_0^A \eta(a)\tau(a)da}. \end{aligned}$$

Then, using $v(A) = v(0)\tau(A)$, we rewrite the social planner problem as

$$\max_{\theta_0, A}\left\{b + \frac{m(\theta_0)}{\lambda + m(\theta_0)}(y-b) - \frac{\lambda}{\lambda + m(\theta_0)}m(\theta_0)Z(\theta_0, A)\right\},$$

where $Z(\theta_0, A) = [c(1-\tau(\theta_0, A)) + k\tau(\theta_0, A)]/\int_0^A \eta(\theta_0, a)\tau(\theta_0, a)da$. This notation highlights the dependence of the survivor function and the job-filling rate across submarkets on θ_0.

This social planner objective differs from the standard one in three ways. First, in the standard model with no phantoms, there is never an incentive to renew a vacancy, i.e., $A \to \infty$. Second, since with no phantoms $A \to \infty$ and $\theta(a)$ is a constant, the denominator of $Z(\theta_0, A)$, i.e., $\int_0^A \eta(\theta_0, a)\tau(\theta_0, a)da$, is simply $m(\theta_0)/\theta_0$, so $v(0) = \theta_0\mathbf{u}$. In our problem, the social planner needs to take into account the fact that the choice of θ_0 not only reflects the overall level of vacancy creation but also determines the profile of the job-filling rate across submarkets. Finally, in the usual formulation, the cost associated with posting and maintaining vacancies is a constant independent of market tightness. Here, in contrast, the vacancy cost depends directly on θ_0 since an increase in initial market tightness makes it more likely that the vacancy renewal cost will be incurred.

To characterize the social planner optimum, let $\alpha_0 = \theta_0 m_\theta(\theta_0)/m(\theta_0)$ be the elasticity of $m(\cdot)$ with respect to initial market tightness, let $\sigma_0 = \theta_0 Z_\theta(\theta_0, A)/Z(\theta_0, A)$ be the elasticity of $Z(\theta_0, A)$ with respect to θ_0, and define $\varepsilon_0 = \alpha_0/(\alpha_0+\sigma_0)$. The first-order condition of the social planner problem with respect to θ_0 gives

$$\begin{aligned} Z(\theta_0, A) &= \frac{\varepsilon_0(y-b)}{\lambda+(1-\varepsilon_0)m(\theta_0)}; \text{ i.e.,} \\ c(1-\tau(\theta_0, A)) + k\tau(\theta_0, A) &= \frac{\varepsilon_0(y-b)}{\lambda+(1-\varepsilon_0)m(\theta_0)} \int_0^A \eta(\theta_0, a)\tau(\theta_0, a)da, \end{aligned} \tag{14}$$

whereas the first-order condition with respect to A gives

$$(c-k)[\eta(\theta_0, A)+\lambda] \int_0^A \eta(\theta_0, a)\tau(\theta_0, a)da = \eta(\theta_0, A)[c(1-\tau(\theta_0, A)) + k\tau(\theta_0, A)]. \tag{15}$$

Combining equations (14) and (15) gives

$$c-k = \frac{\eta(\theta_0, A)}{\eta(\theta_0, A)+\lambda} \frac{\varepsilon_0(y-b)}{\lambda+(1-\varepsilon_0)m(\theta_0)}. \tag{16}$$

A listing is renewed when the capital gain induced by renewal, $c-k$, is equal to the opportunity cost of keeping the listing alive, i.e., a term measuring the probability of finding a worker, $\frac{\eta(\theta_0,A)}{\eta(\theta_0,A)+\lambda}$, multiplied by the social value of a filled job, $\frac{\varepsilon_0(y-b)}{\lambda+(1-\varepsilon_0)m(\theta_0)}$.

There is a modified Hosios condition under which the decentralized equilibrium gives the constrained efficient allocation. Comparing equations (14) and (15) to equations (9) and (10) when $r \to 0$ implies that the two allocations coincide, i.e., initial labor market tightness and the renewal date in the decentralized equilibrium equal those chosen by the social planner, when the bargaining power $\gamma = 1-\varepsilon_0$. That is, the elasticity of the matching function in the standard Hosios condition must be replaced by $\varepsilon_0 = \alpha_0/(\alpha_0+\sigma_0)$.

This condition differs from the standard Hosios condition for several reasons. First, as noted above, the cost per vacancy varies with θ_0. This generates a modified Hosios condition for a reason similar to the one discussed in Julien and Mangin (2020).[11] Second, also as discussed above, θ_0 determines the shape of market tightness across submarkets, and hence the job-filling rate across submarkets, a feature that is absent in the model without phantoms. Finally, and less essentially, our model differs from the standard one insofar as (i) there is a fixed cost of vacancy creation rather than a flow cost and (ii) vacancies are destroyed at a constant rate. Neither (i) nor (ii) is due to the presence of phantoms.

The constrained efficient allocation – equivalently, the decentralized equilibrium with $\gamma = 1-\varepsilon_0$ – can be implemented by competitive search. Suppose that each firm, when listing its

[11] Julien and Mangin (2020) derive a modified Hosios condition that gives the constrained efficient outcome. Their modified Hosios condition accounts for the possibility that net flow output per filled vacancy may vary with market tightness. The similarity here is that the cost per vacancy varies with (initial) market tightness.

vacancy, posts and commits to a wage that does not change as the vacancy ages. The value of applying to a firm depends on the offered wage and the probability the worker's application will be successful. By the no-arbitrage condition, the job-finding rate is constant across listing ages among all firms posting the same wage, so the value of applying to a firm posting w can be written as $U(w, \theta_0)$. Similarly, let $V(0; w, \theta_0) = \max_A V(0; w, \theta_0, A)$ be the value of a new listing conditional on the firm choosing the optimal renewal age A. The competitive search equilibrium can be described as the solution to

$$\max_{w,\theta_0} U(w, \theta_0) \text{ s.to } V(0; w, \theta_0) = c.$$

When $r \to 0$, we show in Appendix A that the solution to this constrained optimization problem is given by equation (14); i.e., the competitive search allocation and the constrained efficient allocation coincide.

Finally, we again briefly discuss the case without phantoms. If the phantom birth rate is set to zero, firms do not renew their listings and $A \to \infty$. In this case, the job-filling rate is a constant over listing age and depends only on initial labor market tightness, θ_0, i.e., the job-filling rate is $\eta(\theta_0)$ and $Z(\theta_0) = c(\eta(\theta_0) + \lambda)/\eta(\theta_0)$. This in turn implies that the elasticity of $Z(\theta_0)$ with respect to θ_0 is $\sigma_0 = (1 - \alpha_0)\lambda/(\eta(\theta_0) + \lambda)$. Again, the decentralized equilibrium allocation with $\gamma = 1 - \varepsilon_0$ (equivalently, the competitive search allocation) is constrained efficient.

3.2 Social Planner: Case 2

We now turn to the case in which the planner can direct worker search by choosing the function $\theta(a)$ that allocates job seekers across submarkets as well as choosing the flow of new vacancies, $v(0) - v(A)$, and the vacancy renewal age, A. Again, it is convenient to rewrite the social planner objective, this time by expressing $v(0)$ and $1 - \mathbf{u}$ as functionals involving the functions $\theta(a)$ and $\tau(a)$.

Note first that $v(a) + p(a) = v(0)[\tau(a) + \beta(1 - \tau(a))]$, so

$$\mathbf{u} = \int_0^A u(a)da = v(0) \int_0^A [\tau(a) + \beta(1 - \tau(a))]/\theta(a)da.$$

Second, in steady state, we have

$$\lambda(1 - \mathbf{u}) = \int_0^A \eta(a)v(a)da = v(0) \int_0^A \eta(a)\tau(a)da.$$

These two conditions imply

$$\begin{aligned} v(0) &= \frac{\lambda}{\int_0^A \eta(a)\tau(a)da + \lambda \int_0^A [\tau(a) + \beta(1 - \tau(a))]/\theta(a)da} \\ 1 - \mathbf{u} &= \frac{\int_0^A \eta(a)\tau(a)da}{\int_0^A \eta(a)\tau(a)da + \lambda \int_0^A [\tau(a) + \beta(1 - \tau(a))]/\theta(a)da}. \end{aligned}$$

We can use these expressions to rewrite the social planner's objective. Let $N(a) = \int_0^a \eta(s)\tau(s)ds$ and $D(a) = \int_0^a \eta(s)\tau(s)ds + \lambda \int_0^a [\tau(s) + \beta(1-\tau(s))]/\theta(s)ds$. Then the planner's problem is

$$\max_{\theta(.),A} \left\{ b + \frac{N(A)(y-b) - c\lambda + (c-k)\lambda\tau(A)}{D(A)} \right\}$$

subject to

$$\begin{aligned} \dot{\tau} &= -(\eta(\theta(a)) + \lambda)\tau(a), \ \tau(0) = 1, & (17) \\ \dot{N} &= \eta(\theta(a))\tau(a), \ N(0) = 0, & (18) \\ \dot{D} &= \eta(\theta(a))\tau(a) + \lambda[\tau(a) + \beta(1-\tau(a))]/\theta(a), \ D(0) = 0. & (19) \end{aligned}$$

This problem can be solved by standard optimal control techniques. To keep the notation simple, we use $m(\theta), \eta(\theta), \alpha(\theta), \tau, \pi$ and θ in place of $m(\theta(a)), \eta(\theta(a)), \alpha(\theta(a)), \tau(a), \pi(a)$ and $\theta(a)$ whenever possible. Similarly, we use m_0, η_0, α_0 in place of $m(\theta_0), \eta(\theta_0)$, and $\alpha(\theta_0)$.

In Appendix B, we show that the solution to this constrained efficiency problem has the following form:

First, for all $a \in [0, A]$, the optimal profile of labor market tightness, $\theta(a)$, is defined by

$$\frac{(1-\alpha(\theta))\pi m(\theta)}{(1-\alpha(\theta))\pi m(\theta) + \alpha(\theta)\eta(\theta) + \lambda} = \frac{(1-\alpha_0)m_0}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}. \tag{20}$$

Second, the social planner choices of θ_0 and A are determined by

$$\begin{aligned} c(1-\tau(A)) + k\tau(A) &= \frac{y-b}{\lambda}\left(N(A) - \frac{(1-\alpha_0)m_0 D(A)}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}\right) & (21) \\ c - k &= \frac{\alpha(\theta(A))\eta(\theta(A))}{(1-\alpha(\theta(A))\pi(A)m(\theta(A)) + \alpha(\theta(A))\eta(\theta(A)) + \lambda}\left(\frac{y-b}{\lambda}\right). & (22) \end{aligned}$$

As was the case in the decentralized allocation, equation (20) implies that tightness, θ, is an implicit function of π and θ_0. Since $(1-\alpha(\theta))\pi m(\theta)$ strictly increases with θ and π strictly decreases with a, this equation implies that θ strictly increases with listing age.[12] Moreover, since $\alpha(\theta)\eta(\theta) = m'(\theta)$ strictly decreases with a, we also have that $(1-\alpha(\theta))\pi m(\theta)$ strictly decreases with age.

Equation (20) replaces the no-arbitrage condition, $\pi m(\theta) = m_0$, that characterizes the directed search allocation. For given θ_0, there are two differences between the two equations. The first has to do with the term $1-\alpha(\theta)$. Allocating an additional job seeker to a listing of age a increases expected employment in that submarket by $(1-\alpha(\theta))\pi m(\theta)$. The planner takes into account that adding a worker affects the elasticity of $m(\cdot)$ in that submarket (unless α is

[12] To see that $(1-\alpha(\theta))\pi m(\theta)$ increases with θ, even if $\alpha'(\theta) > 0$, note that since $\alpha(\theta)m(\theta) = \theta m'(\theta)$, we have $\alpha'(\theta)m(\theta) + \alpha(\theta)m''(\theta) = m'(\theta) + \theta m''(\theta)$; thus $\alpha'(\theta)m(\theta) < (1-\alpha(\theta))m'(\theta)$.

constant); the worker has no incentive to take this effect into account. The second, and more essential, difference has to do with the slope of market tightness with respect to a.

Differentiating equation (20) with respect to a, we can show

$$\left(\alpha(\theta) - \frac{\theta\alpha'(\theta)}{1-\alpha(\theta)}\right)\frac{\dot{\theta}}{\theta} = -\frac{\theta\pi}{\theta\pi + \frac{(1-\alpha_0)m_0}{\alpha_0\eta_0+\lambda}}\frac{\dot{\pi}}{\pi} \tag{23}$$

and that $\alpha(\theta) - \frac{\theta\alpha'(\theta)}{1-\alpha(\theta)} > 0$, which confirms that θ increases with age as in the decentralized allocation.[13] The externality is not so strong that the planner prefers older listings to younger ones. However, $\frac{\theta\pi}{\theta\pi + \frac{(1-\alpha_0)m_0}{\alpha_0\eta_0+\lambda}} > \frac{\dot{\pi}}{\pi}$. When $\alpha(\theta)$ is constant, the growth rate of θ is lower than in the decentralized allocation. The planner is less inclined to direct job seekers towards younger listings than the workers themselves are in the decentralized allocation. That is, market tightness increases more rapidly with listing age in the directed search equilibrium than it does in the constrained efficient allocation.

The reason that the planner directs job seekers towards older listings relative to the pattern observed in the decentralized allocation is that a dynamic externality arises when a vacancy is filled, and the magnitude of this externality decreases with listing age. A phantom that is created when a worker matches with a young vacancy lasts a relatively long time and impacts relatively many other job seekers. By contrast, a phantom that is created from an older listing affects relatively few workers. In the limit, as a vacancy's listing age tends to A, a phantom created by a match with the vacancy dies immediately and does not affect social welfare.

The constrained efficient allocation described above can be decentralized by competitive search with listing-age-dependent wages. Suppose firms, when they list their vacancies, post and commit to a wage schedule, $w(a)$, that promises the worker a wage that depends on the age of the listing at the time the worker is hired. The firm's problem is to choose a wage function, $w(a)$, and a renewal age, A, to maximize the value of its newly posted vacancy. This optimization is carried out subject to a market utility constraint, namely, that a worker can do no better than applying to a listing of age a with posted wage $w(a)$, and the level of new vacancy postings is determined by free entry. In Appendix C, we show that the competitive search equilibrium is characterized by equations (20), (21) and (22); i.e., the competitive search allocation and the constrained efficient allocation coincide.

The wage schedule that decentralizes the constrained efficient allocation is

$$\begin{aligned} w(a) &= b + \frac{(1-\alpha_0)m_0}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}\left(\frac{y-b}{1-u(a)}\right) \\ &= \left(\frac{(\alpha_0\eta_0+\lambda)b + (1-\alpha_0)m_0 y}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}\right)\left(\frac{\pi m(\theta)/(\pi m(\theta)+\lambda)}{m_0/(m_0+\lambda)}\right) \end{aligned} \tag{24}$$

[13] These statements are verified in Appendix B.

where $u(a) = \lambda / (\pi m(\theta) + \lambda)$ is the unemployment rate among workers who direct their search towards listings of age a. The job finding rate, $\pi m(\theta)$, decreases with listing age, so the wage increases with listing age. To decentralize the constrained efficient allocation, the wage schedule must reward workers who search for older vacancies in a particular way. Specifically, the competitive search equilibrium wage schedule satisfies $w(a)/w(0) = (1 - u(0))/(1 - u(a))$. Of course, competitive search equilibrium with age-dependent wages is a purely theoretical concept that elucidates how a social planner would ideally like to price applications to listings of different ages. In practice, firms don't post wage schedules of this sort, not least because the commitment requirement is impractical.

4 Numerical illustrations

In this section, we illustrate our model numerically. We do this in three steps. First, we solve our model so that the decentralized allocation reproduces various US labor market outcomes over the period 2000-2008. Second, using this baseline parameterization, we examine the effects of varying β. Specifically, we look at how unemployment, vacancy creation, vacancy renewal, and the listing age profile of worker search vary as this parameter changes. Finally, we solve for the full constrained efficient allocation, i.e., the allocation that arises when the social planner can direct worker search in addition to choosing the flow of new vacancies and the vacancy renewal age, and we compare the decentralized allocation to the constrained efficient allocation.

The decentralized allocation depends, of course, on γ, the worker bargaining share. We set γ to its (endogenously determined) generalized Hosios value. Our decentralized allocation is thus the allocation that would be chosen by a social planner in the presence of phantoms if the planner had the power to choose the flow of new vacancies and the vacancy renewal age but not the pattern of worker search. This means that when we compare this decentralized allocation with the full constrained efficient allocation, the usual congestion and thick-market externalities are exactly offsetting.

4.1 Baseline parametrization

The key to simulating the decentralized allocation is that knowledge of θ_0, β and A makes it possible to find the functions $\pi(a)$ and $\theta(a)$. For our baseline parameterization, we set $\beta = 1/2$, and $A = 1$, the latter corresponding to the maximum age of an ad on Craigslist. We now explain how we set θ_0. We start with a trial value for θ_0 and then iterate using an equation for vacancy duration.

The average duration of a vacancy can be written as

$$d = \int_0^A \tau^{\mathrm{ds}}(a, \theta_0)(1 - \tau^{\mathrm{ds}}(A, \theta_0))^{-1} da. \tag{25}$$

Davis et al. (2013) estimate a mean vacancy duration between 14 and 25 days. However they also report that the work typically only starts a couple of weeks later. We therefore set d equal to one month. To solve this equation for θ_0, we need an expression for $\tau(a, \theta_0)$. We derive this as follows.

We assume that $m(\cdot)$ is Cobb-Douglas, i.e., $m(\theta) = B\theta^\alpha$, $0 < \alpha < 1$. Since (i) $\dot{\tau} = -(\eta(\theta) + \lambda)\tau$, (ii) $\theta = \theta_0 \pi^{-1/\alpha}$ (using the assumption that $m(\cdot)$ is Cobb-Douglas), and (iii) $\pi = v/(v + p) = \dfrac{\tau}{\beta + (1-\beta)\tau}$, we have

$$\dot{\tau}^{\mathrm{ds}} = -\left[B\theta_0^{\alpha-1} \left(\frac{\tau^{\mathrm{ds}}}{\beta + (1-\beta)\tau^{\mathrm{ds}}} \right)^{\frac{1-\alpha}{\alpha}} + \lambda \right] \tau^{\mathrm{ds}}, \tag{26}$$

$$\tau^{\mathrm{ds}}(0) = 1. \tag{27}$$

Given a trial value for θ_0, once we set values for λ, α and B, we can solve this differential equation numerically.

To set the parameters λ, α and B, we proceed as follows. First, using Shimer's (2005) methodology, we compute the mean job-finding rate and the mean job separation rate for the period 2000-2008. This gives $\mu = 0.5$ and $\lambda = 0.03$. This implies a corresponding unemployment rate of $\mathsf{u} = \lambda/(\lambda + \mu) = 0.0563$. Second, we set α to match the aggregate elasticity of hires with respect to the aggregate ratio v/u. The typical estimate when regressing the log job-finding probability on the log vacancy-to-unemployment ratio is about 0.3 (see also Shimer 2005). This value for the elasticity overestimates α when there are phantoms. The reason is that a marginal increase in the vacancy-to-unemployment ratio not only affects $m(\cdot)$, but also reduces the phantom proportion and modifies the allocation of job-seekers across listing ages. To see this, note that

$$\frac{d \ln \mu}{d \ln \mathsf{v}/\mathsf{u}} = \frac{d \ln \left(\int_0^A \phi_u(a)\pi(a)m(\theta(a))da \right)}{d \ln \mathsf{v}/\mathsf{u}} \tag{28}$$

and using the equilibrium condition $\mu(a) = \pi(a)m(\theta(a)) = m(\theta_0)$ for all $a \geq 0$ that

$$\frac{d \ln \mu}{d \ln \mathsf{v}/\mathsf{u}} = \alpha \frac{d \ln \theta_0}{d \ln \mathsf{v}/\mathsf{u}}. \tag{29}$$

In the standard model without phantoms, the source of the change in v/u does not matter. In this case, we have seen that $A \to \infty$, $\theta(a) = \theta_0$ and $\pi(a) = 1$ for all $a \geq 0$. Therefore $\mathsf{v}/\mathsf{u} = \theta_0$ and $d \ln \mu / d \ln \mathsf{v}/\mathsf{u} = \alpha$. However, with phantoms, this equality does not hold because θ_0 differs from v/u, so the source of shock is likely to impact the elasticity. Another issue in our numerical exercise is that phantoms may impact the number of reported vacancies. Econometricians may use $(\mathsf{v}+\mathsf{p})/\mathsf{u}$ instead of v/u. Therefore we also compute the elasticity $\frac{d\ln\mu}{d\ln(\mathsf{v}+\mathsf{p})/\mathsf{u}}$. Finally, once α is fixed, we can set B, the scale parameter of the matching function using $B\theta_0^\alpha = \mu$; i.e., $B = \mu\theta_0^{-\alpha}$.

To solve for θ_0, we start with a trial value. Given this trial value, we compute α by setting the right-hand side of equation (29) equal to 0.3. Given values for θ_0, α, λ and B, we then solve equation (26) numerically to obtain $\tau(a, \theta_0)$. Finally, we use this numerical solution in equation (25) and update θ_0 to iterate to a solution.

Once we have solved for θ_0, we have our numerical solution for the differential equation that determines $\tau^{\mathrm{ds}}(a, \theta_0)$, and we can solve for π^{ds} and θ^{ds}. We know that $\pi = \tau/(\beta + (1-\beta)\tau)$ and that the equilibrium condition $\pi m(\theta) = m(\theta_0)$ implies $\theta = \theta_0\pi^{-1/\alpha}$. This gives π^{ds} and θ^{ds} for all $a \in [0, A]$, namely,

$$
\begin{aligned}
\pi^{\mathrm{ds}}(a, \theta_0) &= \frac{\tau^{\mathrm{ds}}(a, \theta_0)}{\beta + (1-\beta)\tau^{\mathrm{ds}}(a, \theta_0)}, && (30)\\
\theta^{\mathrm{ds}}(a, \theta_0) &= \theta_0\pi^{\mathrm{ds}}(a, \theta_0)^{-1/\alpha}. && (31)
\end{aligned}
$$

We now set the remaining model parameters: r, y, b, c and k, and we solve for γ. First, we normalize $y = 1$ and set $r = 0$ to be able to compute the constrained efficient allocation. We set $b = 0.7$, a standard value in the literature. Then, we set γ, c and k so that the free-entry condition holds, $A = 1$ is optimally chosen and the bargaining power decentralizes the constrained efficient allocation when the planner does not direct worker search. That is γ, c and k are chosen to solve

$$
\begin{aligned}
\gamma &= 1 - \varepsilon_0,\\
c &= J(\theta_0)\left\{\int_0^1 \eta(a)e^{-ra}\tau^{\mathrm{ds}}(a, \theta_0)da + \frac{\eta(1)}{r+\lambda+\eta(1)}e^{-rA}\tau^{\mathrm{ds}}(1, \theta_0)\right\},\\
k &= J(\theta_0)\left\{\int_0^1 \eta(a)e^{-ra}\tau^{\mathrm{ds}}(a, \theta_0)da - \frac{\eta(1)}{r+\lambda+\eta(1)}\left(1 - e^{-rA}\tau^{\mathrm{ds}}(1, \theta_0)\right)\right\}.
\end{aligned}
$$

Table 1 gives the parameters along with values for key endogenous variables in our baseline calibration.

Calibrated parameters								
λ	B	α	β	y	b	r	c	k
0.03	0.70	0.15	0.5	1	0.7	0	2.29	0.11
Key endogenous variables								
		θ_0	A	γ	u			
		0.12	1	0.15	0.0563			

Table 1: Baseline Calibration

The elasticity of $m(\cdot)$ with respect to the total number of advertised jobs $v+p$ is $\alpha = 0.15$. The corresponding elasticity of the job-finding rate with respect to the vacancy-to-unemployed ratio is $d\ln\mu/d\ln(\mathsf{v}/\mathsf{u}) = 0.29$. When v/u is replaced by $(\mathsf{v}+\mathsf{p})/\mathsf{u}$, this elasticity is 0.37. Worker bargaining power is $\gamma = 0.15$. This value is low because the elasticity $\sigma_0 = 0.026$ is small relative to the calibrated value of α, which in turn implies that the optimal $\gamma = \sigma_0/(\alpha+\sigma_0)$ is also low. To understand why σ_0 is small, consider the case without phantoms where $\sigma_0 = (1-\alpha)\lambda/(\eta(\theta_0)+\lambda)$. The job-filling rate $\eta(\theta_0) >> \lambda$, so that σ_0 is low. Other key parameters are c and k. The cost of creating a new job is a bit more than twice one month's value added. The cost of renewing ads is 11% of one-month of value added. This is larger than the direct cost of ads, but seems in the correct order of magnitude once the wage and time costs of editing and managing these ads are taken into account.

We now show some features of the baseline allocation. Figure 1 compares the distributions of listings, vacancies and job seekers by listing age. It highlights the bias of job seekers towards younger listings. The distribution of vacancies reflects the pattern of hirings, which progressively deplete the stock of vacancies. The distribution of listings is closer to uniform because half the listings survive job filling and exogenous destruction and become phantoms. The distribution of job seekers is heavily distorted towards young listings because workers fear the phantoms contaminating older listings. In the absence of phantoms, the three distributions would coincide.

A way to quantify the bias towards young listings is to compute job queues at various listing ages. The mean job queue between two dates, say a_1 and a_2, is $(a_2-a_1)^{-1}\int_{a_1}^{a_2}\theta(a)^{-1}da$. In the baseline allocation, Table 2 shows that the mean job queue is 6.58 job seekers per listing for listings aged less than 24 hours. It falls to 4.51 for listings aged between 24 and 48 hours. The first week concentrates 68% of the job seekers' efforts. These figures are broadly in line with evidence reported in Davis and Samaniego (2019).

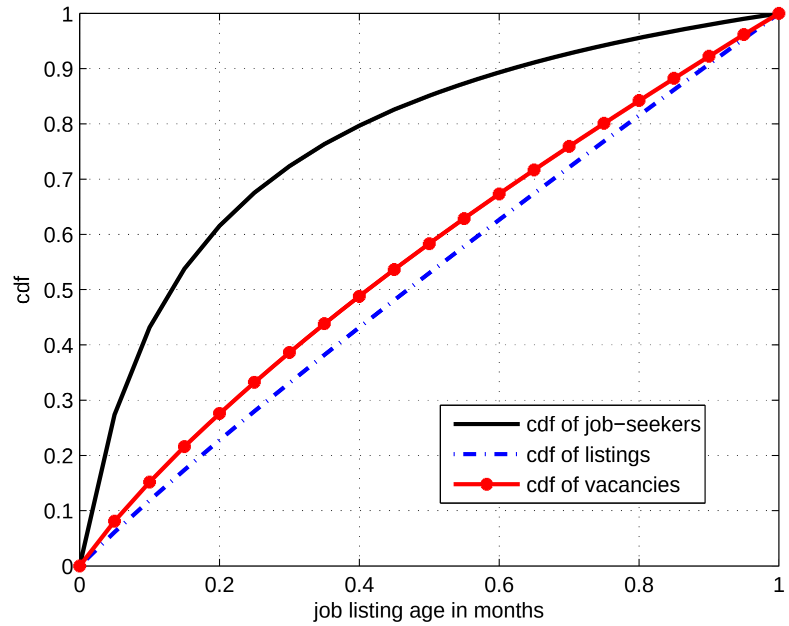

Figure 1: Distributions of listings and job-seekers by listing age

	Density of job seekers	Mean job queue
Day 1	0.21	6.58
Day 2	0.14	4.51
Day 3	0.10	3.35
Day 4	0.07	2.62
Week 1	0.68	0.96
Week 2	0.18	0.52
Week 3	0.09	0.57

Table 2: Mass of job seekers and job queues at various intervals of listing age

4.2 Phantom birth probability

We now examine some of the effects of changes in the phantom birth probability, β. We consider three scenarios: the baseline one with $\beta = 0.5$, the pessimistic one with $\beta = 0.75$ and the optimistic one with $\beta = 0.25$. Otherwise, we use the parameter values set in our baseline calibration. Changing β has a large effect on the decentralized allocation. A different value of β translates into a different worker bargaining power γ. Formally, the elasticity σ_0 is impacted by the new pattern of phantom birth and this in turn affects the optimal $\gamma = \sigma_0/(\alpha + \sigma_0)$.

Varying β while holding A fixed would, of course, imply a different value for θ_0, but $A = 1$ would no longer be the optimal relisting age. To take this into account, we adapt the solution method used for our baseline calibration. We start with a trial value for γ. Then, given a initial value for θ_0, we numerically find the functions $\pi(a)$ and $\theta(a)$ and compute the associated optimal renewal age A using equation (10). Next, we update θ_0 (and the associated A) so that both equation (10) and the free-entry condition (9) hold. Lastly we take advantage of the fact that the decentralized allocation coincides with the constrained efficient allocation when the planner can choose θ_0 and A but cannot direct worker search. We do this by varying γ (together with the associated values for θ_0 and A) to find the constrained efficient allocation. The worker bargaining power that decentralizes the constrained efficient allocation satisfies $\gamma = 1 - \varepsilon_0$.

	$\beta = 0.25$	$\beta = 0.50$	$\beta = 0.75$
γ	0.08	0.15	0.26
θ_0	2.77	0.12	0.01
A	1.49	1.000	2.35
u	0.036	0.056	0.112

Table 3: The three scenarios. The model is calibrated with $\beta = 0.5$. Then β is decreased to 0.25 or increased to 0.75, all other parameters being the same.

Table 3 shows the differences between the three scenarios in terms of worker bargaining power, initial tightness, renewal age and unemployment rate. The results are driven by changes in worker bargaining power, which is almost halved in the optimistic scenario and close to doubled in the pessimistic one. The differences in γ translate into similar differences in the unemployment rate, which more than triples between the pessimistic and the optimistic scenarios.

Initial tightness strongly decreases with β, reflecting the increase in worker bargaining power and the increased prevalence of phantoms at higher listing ages. The effect on the renewal age is ambiguous. On the one hand, job seekers' preference for younger listings increases with β, so the job-filling rate declines more rapidly with a. This gives firms an incentive to renew their vacancies more quickly. On the other hand, worker bargaining power is higher, so the value of a filled job is lower, and this tends to lengthen the renewal age. In our calibration, the former effect dominates when the phantom birth probability is decreased to $\beta = 0.25$, while the latter effect dominates when β is increased to 0.75.

Figure 2 shows tightness as a function of listing age in the three scenarios. Initial tightness decreases with β. Workers increasingly concentrate their search on younger listings when β increases. Figure 3 shows the nonphantom proportion as a function of listing age in the three

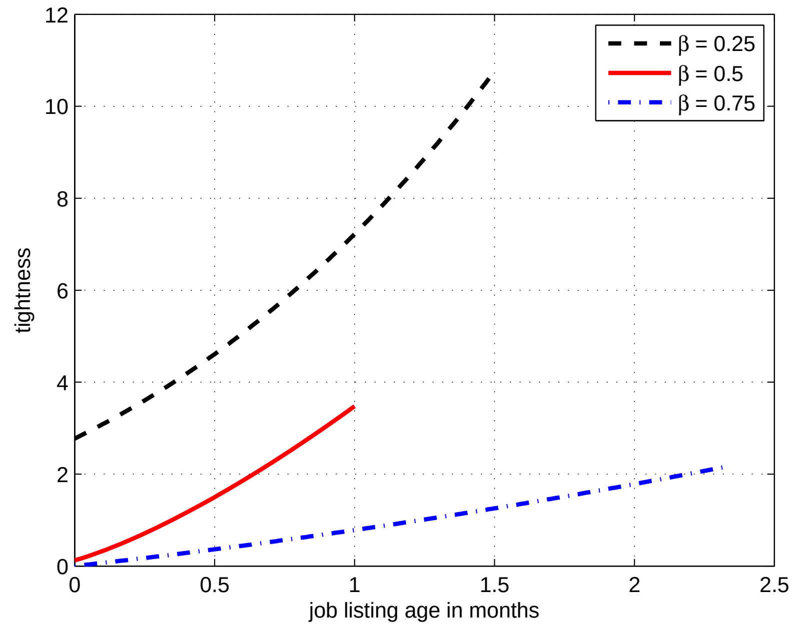

Figure 2: Tightness as a function of listing age and workers' bargaining power

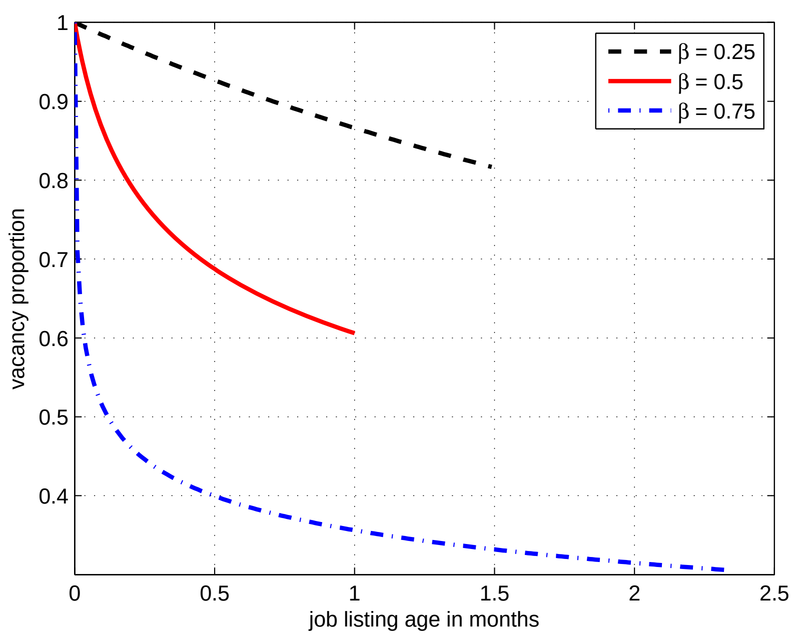

Figure 3: Vacancy proportion as a function of listing age and workers' bargaining power

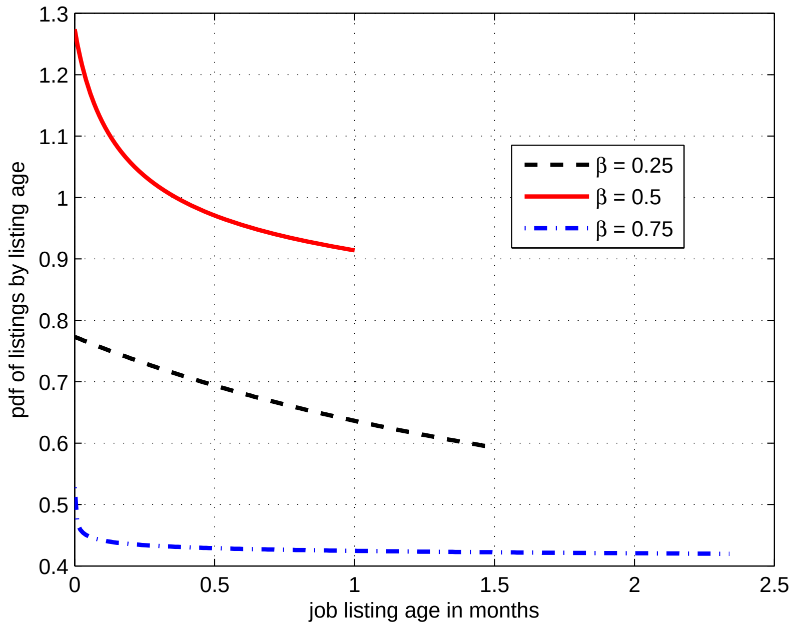

Figure 4: Density of listings by listing age

scenarios. As expected, this proportion decreases with β. The decline with age is very strong when $\beta = 0.75$. This effect is driven by a large decrease in initial labor market tightness. As β increases, the increase in worker bargaining power makes vacancy creation less attractive, while at the same time the strong decrease in the nonphantom proportion with age pushes worker search towards younger listings. The latter effect implies that the job-filling rate is very large in the beginning of a listing's existence. In turn, many phantoms are created. Figure 4 shows how the density of listings by age varies with β. As β increases, job seekers are naturally more interested in recent listings. Finally, Figure 5 shows the ratio of the pdf of job seekers to the pdf of listings by listing age. Figure 5 shows that job seekers focus heavily on young listings when $\beta = 0.75$. When $\beta = 0.25$, the distribution of job seekers gets closer to the distribution of listings by age. Under random search, e.g., if workers were unable to observe listing age, the the ratio of the two pdf's would always equal one.

4.3 Constrained efficient allocation

We now turn to the case in which the planner can optimally allocate job seekers across listings by age as well as choosing the flow of new vacancies and the renewal age.

We again use the auxiliary variable τ. We start with a guess for θ_0 and then numerically

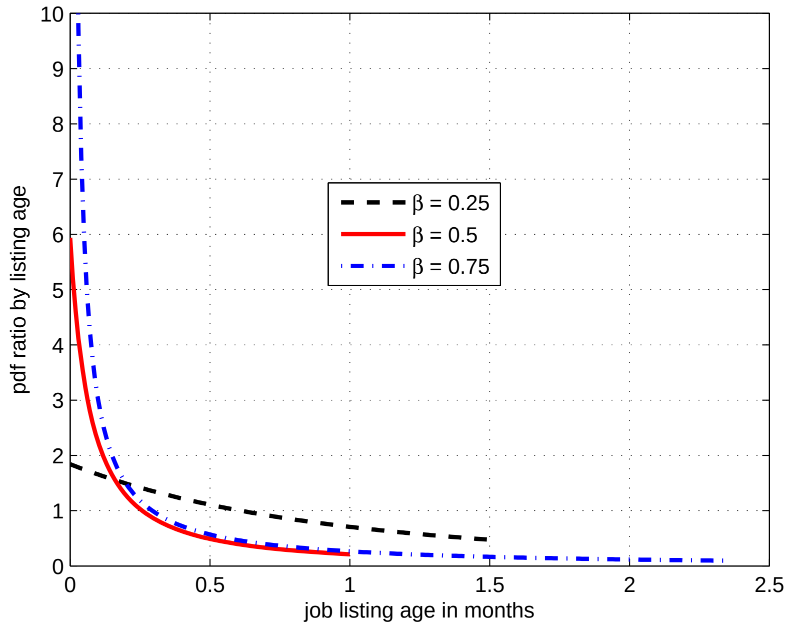

Figure 5: Job-seeker to listing pdf ratio by listing age

integrate the following system of differential equations:

$$\dot{\tau} = -(B\theta^{\alpha-1} + \lambda)\tau; \quad \tau(0) = 1, \tag{32}$$

$$\alpha\frac{\dot{\theta}}{\theta} = \frac{\theta\pi(B\theta^{\alpha-1} + \lambda)(1 - (1-\beta)\pi)}{\theta\pi + (1-\alpha)B\theta_0^{\alpha-1}/(\alpha B\theta_0^{\alpha} + \lambda)}; \quad \theta(0) = \theta_0. \tag{33}$$

We then find the associated renewal age using equation (22) and iteratively update θ_0 so as to maximize stationary consumption (using gradient descent).

	Ω	A	θ_0	u
decentralized	0.905	1.0	0.12	0.056
centralized	0.913	1.06	0.50	0.056

Table 4: Centralized vs decentralized allocations

Table 4 displays the key differences between the allocations. Stationary aggregate consumption, Ω, is increased by 0.8% in the social planner allocation. This gain comes from lower costs of vacancy management. The unemployment rate is essentially the same in the two allocations (slightly lower in the centralized one). The table features two additional findings: the

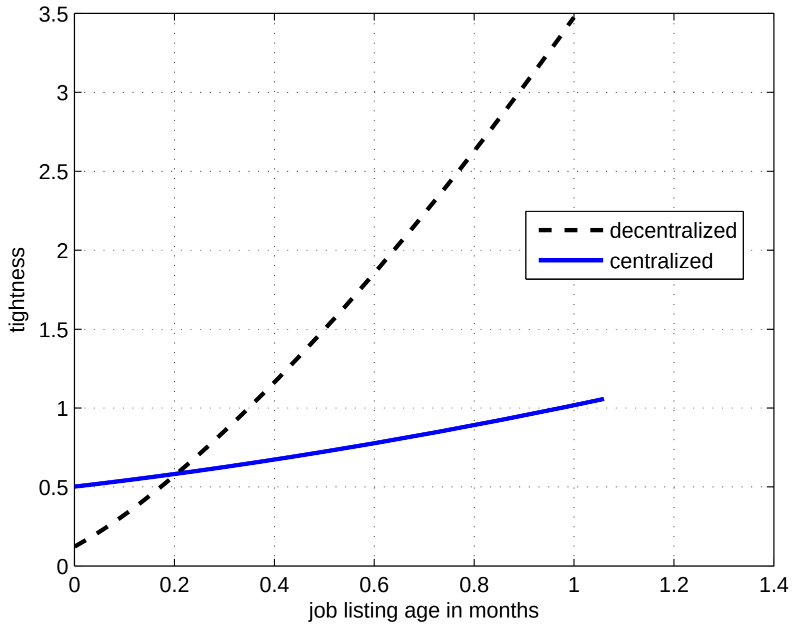

Figure 6: Tightness in the centralized vs decentralized allocations

renewal age is larger by 6% in the centralized case and initial tightness is four times larger in the centralized allocation.

Figure 6 confirms that the planner has a lower preference for young listings than the job seekers have in the decentralized allocation. Tightness is initially higher in the centralized case but becomes lower after a week. This pattern implies that workers search for jobs at older listings. In turn this justifies the planner in setting a later renewal age.

Figure 7 shows the job-finding rate by listing age. The equilibrium condition $\mu(a) = \mu(0)$ for all $a \leq A$ is not satisfied in the centralized allocation. Instead, workers who search for jobs at young listing ages have a better chance of finding one.

Figure 8 shows the pattern of the optimal wage by listing age. Here again, we note that this pattern is not realistic. While we don't observe listing-age-dependent wages in reality, it is useful to see the wage function that could decentralize the constrained efficient allocation. The wage increases by 0.7% in one month. This is needed to make the job seekers accept lower job-finding rates when they search for older listings.

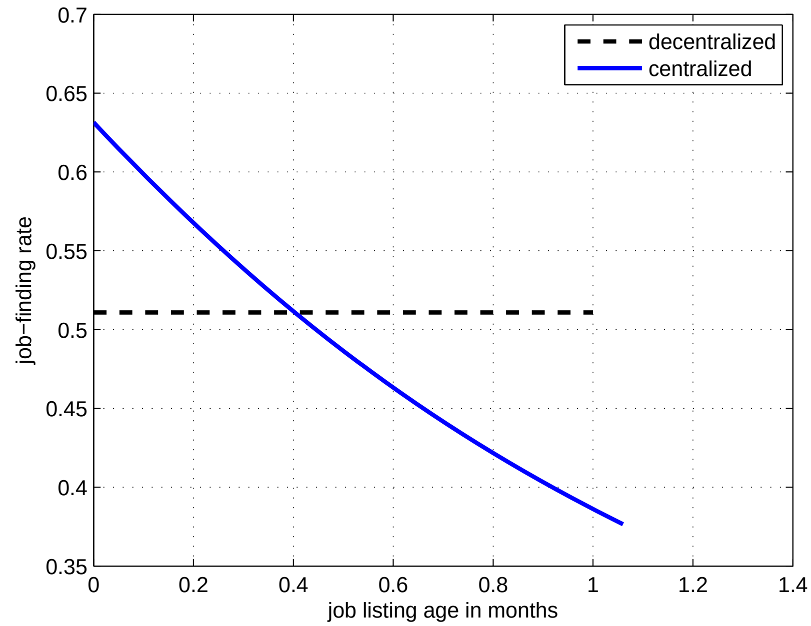

Figure 7: Job-finding rate by listing age in the centralized vs decentralized allocations

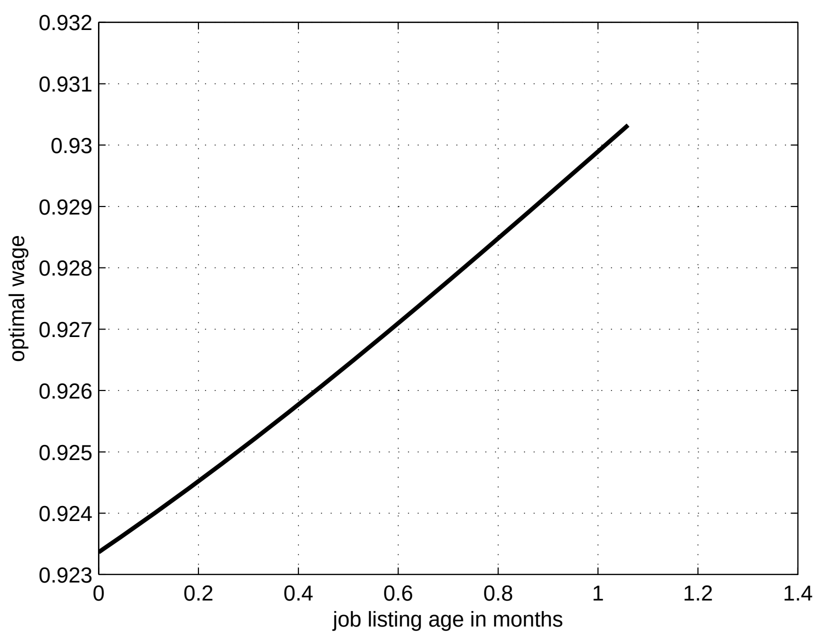

Figure 8: Optimal Wage Schedule

5 Conclusions

In this paper, we have characterized the directed search equilibrium of a model with phantom vacancies, i.e., listings that remain after the actual vacancy has been filled or destroyed. In equilibrium, searchers allocate themselves across submarkets that are defined by listing age, and worker directed search satisfies a no-arbitrage condition, namely, that the expected payoffs associated with searching in the various submarkets must be equalized. On the other side of the market, firms decide how many new vacancies to list and how long to wait before relisting their vacancies. We contrast the decentralized equilibrium allocation with its constrained efficient counterpart, but the nature of the constrained efficient allocation depends on the tools we allow the social planner to employ. We first suppose that the social planner can determine firms' vacancy posting and listing renewal behavior. Second, we suppose that the social planner can also direct job seekers' search. In both cases, the social planner is constrained by the same information friction that workers face, namely, the planner cannot see whether a particular listing is a real vacancy or a phantom.

In the decentralized equilibrium allocation, workers follow a mixed strategy with respect to the choice of submarket. Since older listings are more likely to be phantoms, relatively many applications are sent to younger listings – more than the social planner would like. If the social planner is limited to choosing how many new vacancies are posted and the age at which job listings are renewed, we show there is a generalized Hosios condition that implements the constrained efficient allocation. In this case, if firms can post and commit to a wage that is independent of the age at which the vacancy is filled, a competitive search equilibrium decentralizes the social planner allocation. However, when the social planner can also choose the allocation of job seekers across submarkets, the equilibrium allocation is generically inefficient. The equilibrium allocation of worker applications across listing ages generates a dynamic congestion externality. When matches are formed, phantoms congest the market, and a phantom that is created when a younger listing leads to a match is more costly than one that is created when an older listing does so. That is, the magnitude of the externality decreases with the listing age of the vacancy, and workers have no incentive to incorporate this dynamic effect into their decision calculus. The allocation that results when the social planner can choose the flow of new vacancies, the listing renewal age and the allocation of job seekers across submarkets can also be decentralized in competitive search equilibrium. However, this competitive search equilibrium requires firms to post wage schedules, i.e., wages that vary with the age of the vacancy at the time the match is formed, a phenomenon that we do not observe in reality.

We supplement our theoretical results with numerical illustrations. We do this in three steps. First, we select parameter values so that the decentralized equilibrium of our model matches some key US labor market outcomes over the period 2000-2008. The nature of the decentralized equilibrium depends on the value we choose for the worker bargaining power parameter, and we set this parameter to its (endogenously determined) generalized Hosios value. In our calibration, we make a baseline parameter assumption, namely, that the fraction of ads that are not removed when the corresponding vacancy is filled, β, equals 1/2. In the second step of our numerical illustrations, we examine the sensitivity of our results to changes in β. Varying this parameter has strong effects on how workers direct their search and on the overall unemployment rate. That is, the existence of phantoms has a strong effect on the labor market. Finally, we numerically solve for the full constrained efficient allocation; i.e., we allow the social planner to allocate job seekers across listing ages in addition to choosing the level of vacancy creation and the listing renewal age. We then compare the decentralized allocation with the social planner solution. Allowing the planner to direct worker search increases aggregate steady-state consumption by a bit less than 1%, but this gain comes almost entirely from reduced vacancy management costs rather than reduced unemployment. The reason that allowing the planner to direct worker search has little effect on unemployment is that optimizing the way that workers direct their search leads to more matches, but these additional matches in turn generate more phantoms.

If phantoms are potentially so important, why do they persist? The answer is simply that while each firm would like other firms to delist their vacancies immediately once they are filled or destroyed, there is little incentive for individual employers to do this. Indeed, each firm has an incentive to keep their listings online until the job is "definitively" filled, i.e., until its selected applicant has shown up for his or her first days of work. Job boards would like to weed out phantoms, but, although some specialized websites are able to do this, most face the same information friction as job seekers – it is difficult to determine whether a particular listing still corresponds to a viable vacancy. In any event, it is clear that phantoms are present in the labor market just as it is clear that job seekers react to their existence. Our paper is the first to incorporate both of these realistic features into a search-theoretic model of the labor market.

APPENDIX

A Competitive search equilibrium

The competitive search equilibrium is the solution to

$$\max_{w,\theta_0} U(w,\theta_0) \text{ s.to } V(0;w,\theta_0) = c,$$

From equations (1) and (2), the unemployment value is

$$U(w,\theta_0) = \frac{1}{r}\left(\frac{(r+\lambda)b + m(\theta_0)w}{r+\lambda+m(\theta_0)}\right),$$

and, from equation (5), the value of a new listing is

$$V(0;w,\theta_0)=\frac{\left(\frac{y-w}{r+\lambda}\right)\int_0^A \eta(a)e^{-ra}\tau(a)da-ke^{-rA}\tau(A)}{1-e^{-rA}\tau(A)}.$$

Setting this equal to c gives

$$\begin{aligned} w &= y-(r+\lambda)\frac{c(1-e^{-rA}\tau(A))+ke^{-rA}\tau(A)}{\int_0^A \eta(a)e^{-ra}\tau(a)da}\equiv y-(r+\lambda)\Gamma(\theta_0,A), \text{ i.e.,} \\ U(\theta_0) &= \frac{1}{r}\left(\frac{(r+\lambda)b+m(\theta_0)[y-(r+\lambda)\Gamma(\theta_0,A)]}{r+\lambda+m(\theta_0)}\right) \end{aligned}$$

The necessary condition for the competitive search problem can thus be written as

$$m'(\theta_0)[y-b-(r+\lambda)\Gamma(\theta_0,A)]-m(\theta_0)\Gamma_{\theta_0}(\theta_0,A)(r+\lambda+m(\theta_0))=0 \tag{34}$$

To show that this is equivalent to the necessary condition for constrained efficiency when the social planner chooses θ_0 and A, we let $r\to 0$. In this case,

$$\Gamma(\theta_0,A)\to\frac{c(1-\tau(A))+k\tau(A)}{\int_0^A \eta(a)\tau(a)da}=Z(\theta_0,A),$$

and equation (34) becomes

$$m'(\theta_0)[y-b-\lambda Z(\theta_0,A)]-m(\theta_0)Z_{\theta_0}(\theta_0,A)(\lambda+m(\theta_0))=0.$$

Multiplying through by $\dfrac{\theta_0}{m(\theta_0)}$ gives

$$\alpha_0(y-b)-(\alpha_0+\sigma_0)\lambda Z(\theta_0,A)]-\sigma_0 Z(\theta_0,A)m(\theta_0)=0.$$

Finally, dividing by $\alpha_0+\sigma_0$ (and recalling that $\varepsilon_0=\alpha_0/(\alpha_0+\sigma_0)$) gives

$$\varepsilon_0(y-b)-\lambda Z(\theta_0,A)+(1-\varepsilon_0)Z(\theta_0,A)m(\theta_0)=0.$$

This is equivalent to equation (14), i.e.,

$$Z(\theta_0,A)=\frac{\varepsilon_0(y-b)}{\lambda+(1-\varepsilon_0)m(\theta_0)}.$$

B Social Planner Case 2: Derivations

In this appendix, we work through the optimal control problem discussed in Section 3.2. Consider the problem

$$\max_{\theta(.),A}\left\{b+\frac{N(A)(y-b)-c\lambda+(c-k)\lambda\tau(A)}{D(A)}\right\}$$

subject to

$$\begin{aligned}
\dot{\tau} &= -(\eta(\theta)+\lambda)\tau,\ \tau(0)=1, &(35)\\
\dot{N} &= \eta(\theta)\tau,\ N(0)=0, &(36)\\
\dot{D} &= \eta(\theta)\tau+\lambda[\tau+\beta(1-\tau)]/\theta,\ D(0)=0 &(37)
\end{aligned}$$

where $N(a)=\int_0^a \eta(s)\tau(s)ds$ and $D(a)=\int_0^a \eta(s)\tau(s)ds+\lambda\int_0^a[\tau(s)+\beta(1-\tau(s))]/\theta(s)ds$.

The Hamiltonian is

$$H=-\sigma_1(\eta(\theta)+\lambda)\tau+\sigma_2\eta(\theta)\tau+\sigma_3(\eta(\theta)\tau+\lambda[\tau+\beta(1-\tau)]/\theta), \tag{38}$$

where σ_1, σ_2 and σ_3 are the cofactors associated to the differential equations (35) to (37). The optimality conditions are as follows:

- Adjoint equations

$$\begin{aligned}
\frac{\partial H}{\partial \tau} &= -\sigma_1(\lambda+\eta(\theta))+(\sigma_2+\sigma_3)\eta(\theta)+\sigma_3\lambda(1-\beta)/\theta=-\dot{\sigma}_1, &(39)\\
\frac{\partial H}{\partial N} &= 0=-\dot{\sigma}_2, &(40)\\
\frac{\partial H}{\partial D} &= 0=-\dot{\sigma}_3. &(41)
\end{aligned}$$

- Maximum principle

$$\frac{\partial H}{\partial \theta}=(-\sigma_1+\sigma_2+\sigma_3)\eta'(\theta)\tau-\sigma_3\lambda[\tau+\beta(1-\tau)]/\theta^2=0. \tag{42}$$

- Transversality conditions

$$\begin{aligned}
\sigma_1(0) &= \rho_1, &(43)\\
\sigma_2(0) &= \rho_2, &(44)\\
\sigma_3(0) &= \rho_3, &(45)\\
\sigma_1(A) &= \frac{(c-k)\lambda}{D(A)}=(c-k)v(0), &(46)\\
\sigma_2(A) &= \frac{y-b}{D(A)}=\frac{y-b}{\lambda}v(0), &(47)\\
\sigma_3(A) &= -\frac{N(A)(y-b)-c\lambda+(c-k)\lambda\tau(A)}{D(A)^2} &(48)\\
&= -\left((1-\mathbf{u})(y-b)-cv(0)+(c-k)v(A)\right)\frac{v(0)}{\lambda}.
\end{aligned}$$

- Optimal choice of A

$$H(A) = 0. \tag{49}$$

As the Hamiltonian does not explicitly depend on a, we also have $H(a) = 0$ for all $a \in [0, A]$.

From the adjoint equations and the transversality conditions, we obtain that for all $a \in [0, A]$,

$$\sigma_2(a) = \frac{y-b}{D(A)}, \tag{50}$$

$$\sigma_3(a) = -\frac{N(A)(y-b) - c\lambda + (c-k)\lambda\tau(A)}{D(A)^2}. \tag{51}$$

Equation (42) implies

$$(\sigma_1 - \sigma_2 - \sigma_3)(1-\alpha(\theta))\pi m(1,\theta) = \sigma_3\lambda, \tag{52}$$

where $\pi = \tau/(\tau + (1-\beta)\tau)$ is the nonphantom proportion.

Using $H(a) = 0$ for all $a \in [0, A]$, we obtain

$$-(\sigma_1 - \sigma_2 - \sigma_3)(\eta(\theta) + \lambda) - \sigma_2\lambda - \sigma_3\lambda + \sigma_3\lambda\eta(\theta)/(\pi m(1,\theta) = 0. \tag{53}$$

Combining this equation with the maximum principle (52), we obtain

$$\frac{(1-\alpha(\theta))\pi m(1,\theta)}{(1-\alpha(\theta))\pi m(1,\theta) + \alpha(\theta)\eta(\theta) + \lambda} = -\frac{\sigma_3}{\sigma_2} = \frac{(1-\alpha_0)m_0}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}, \tag{54}$$

i.e., equation (20) in the text. Note that equation (54) implies

$$\left(\alpha - \frac{\theta\alpha'}{1-\alpha}\right)\frac{\dot{\theta}}{\theta} = -\frac{\theta\pi}{\theta\pi + \frac{(1-\alpha_0)m_0}{\alpha_0\eta_0+\lambda}}\frac{\dot{\pi}}{\pi} > 0. \tag{55}$$

To derive equations (21) and (22) in the text, we use equations (47) and (48). These two transversality conditions imply

$$-\frac{\sigma_3}{\sigma_2} = \frac{N(A)(y-b) - c\lambda + (c-k)\lambda\tau(A)}{(y-b)D(A)} = \frac{(1-\alpha_0)m_0}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}, \tag{56}$$

which is equivalent to

$$c(1-\tau(A)) + k\tau(A) = \frac{y-b}{\lambda}\left(N(A) - \frac{(1-\alpha_0)m_0 D(A)}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}\right). \tag{57}$$

Next, using equations (46) and (48), we have

$$-\frac{\sigma_3}{\sigma_{1A}} = \frac{N(A)(y-b) - c\lambda + (c-k)\lambda\tau(A)}{(c-k)\lambda D(A)} = \frac{(1-\alpha_A)\theta_A\pi_A}{\alpha_A}, \tag{58}$$

where the subscript A indicates a function evaluated at $a = A$. Combining (56) and (57) gives

$$c - k = \frac{\alpha_A\eta_A}{(1-\alpha_A)\mu_A + \alpha_A\eta_A + \lambda}\frac{y-b}{\lambda}. \tag{59}$$

Equations (57) and (59) coincide with equations (21) and (22) in the text.

To verify equation (23) in the text, let $K = \dfrac{(1-\alpha_0)m_0}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}$ and note that equation (54) can be rewritten as

$$(1-\alpha(\theta))\pi m(\theta)(1-K) = \alpha(\theta)\eta(\theta)K + \lambda K. \tag{60}$$

Differentiating with respect to age gives

$$\begin{aligned}(1-\alpha(\theta))\pi m(\theta)(1-K)\left(-\frac{a'\dot\theta}{1-\alpha} + \alpha\frac{\dot\theta}{\theta} + \frac{\dot\pi}{\pi}\right) &= \alpha(\theta)\eta(\theta)K\left(\frac{a'\dot\theta}{1-\alpha}\frac{1-\alpha}{\alpha} + (\alpha-1)\frac{\dot\theta}{\theta}\right) \quad (61)\\ &= -\frac{1-\alpha}{\alpha}\alpha(\theta)\eta(\theta)K\left(-\frac{a'\dot\theta}{1-\alpha} + \alpha\frac{\dot\theta}{\theta}\right). \quad (62)\end{aligned}$$

It follows that

$$[\pi m(\theta)(1-K) + \eta(\theta)K]\left(-\frac{a'\theta}{1-\alpha} + \alpha\right)\frac{\dot\theta}{\theta} = -\pi m(1,\theta)(1-K)\frac{\dot\pi}{\pi}, \tag{63}$$

which can be rearranged to give equation (23).

To verify that $\dot\theta/\theta > 0$, it suffices to show that $\alpha > \alpha'\theta/(1-\alpha)$. To see this, note that since $\alpha(\theta)m(\theta) = \theta m_\theta(\theta)$, we have $\alpha'(\theta)m(\theta) > (1-\alpha(\theta))m_\theta(\theta)$. Multiplying both sides by $\theta/(1-\alpha(\theta))m(\theta)$ gives the result.

C Competitive Search Equilibrium with Age-Dependent Wages

As discussed in the text, the firm's problem is to choose the wage function, $w(a)$, and the renewal age, A, to maximize the value of its newly posted vacancy. This optimization is done subject to the market utility constraint, namely, that a worker who pursues a job in submarket a can do no better directing his or her search to a different submarket.

On the worker side, the value of searching for a job of age a that is committed to pay $w(a)$ is such that

$$\begin{aligned} rU(a,w(a)) &= b + \pi(a)m(\theta(a,w(a)))[W(w(a)) - U(a,w(a))], \quad (64)\\ rW(w) &= w + \lambda[U - W(w)], \quad (65)\end{aligned}$$

where $U = \max_a U(a,w(a))$. Replacing (65) in (64) gives

$$U(a,w(a)) = \frac{b}{r+\pi(a)m(\theta(a,w(a)))} + \frac{\pi(a)m(\theta(a,w(a)))}{r+\pi(a)m(\theta(a,w(a)))}\frac{w(a)+\lambda U}{r+\lambda}. \tag{66}$$

On the firm side, $V(a)$ denotes the value of a vacancy of age a. Let $r \to 0$. We have

$$V(0) = \max_{w(.),A} \frac{-k\tau(A) + \int_0^A \eta(\theta(a))\frac{y-w(a)}{\lambda}\tau(a)da}{1-\tau(A)}. \tag{67}$$

Firms maximize the value of their newly posted vacancies subject to the market utility constraint, namely,

$$rU(a,w(a)) = rU \Leftrightarrow b + \pi(a)m(\theta(a,w(a)))\frac{w(a)-rU}{\lambda} = rU,$$

and they post new vacancies up to the point that $V(0)=c$.

To put the problem in standard form, we define $z(a)=\int_0^a \eta(\theta(s))\frac{y-w(s)}{\lambda}\tau(s)ds$. The market utility constraint implies that

$$w(a) = rU + \lambda\frac{rU-b}{m(\theta(a))}\frac{\tau+\beta(1-\tau)}{\tau}. \tag{68}$$

Therefore the problem is

$$\max_{\theta(.),A}\frac{-k\tau(A)+z(A)}{1-\tau(A)} = c, \tag{69}$$

subject to

$$\dot{\tau} = -(\eta(\theta)+\lambda)\tau;\ \tau(0)=1, \tag{70}$$

$$\dot{z} = \eta(\theta)\frac{y-rU}{\lambda}\tau - (rU-b)\frac{\tau+\beta(1-\tau)}{\theta};\ z(0)=0. \tag{71}$$

The Hamiltonian for this optimization problem is

$$H = -\sigma_1(\eta(\theta)+\lambda)\tau + \sigma_2\left(\eta(\theta)\frac{y-rU}{\lambda}\tau - (rU-b)\frac{\tau+\beta(1-\tau)}{\theta}\right). \tag{72}$$

The maximum principle states that

$$H_\theta = 0 \Leftrightarrow \left(\sigma_1 - \sigma_2\frac{y-rU}{\lambda}\right)(1-\alpha(\theta))\pi m(1,\theta) = -\sigma_2(rU-b). \tag{73}$$

The optimal choice of A obtains when $H(A)=0$. Again, as the Hamiltonian does not explicitly depend on age, we also have that $H(a)=0$ for all $a\in[0,A]$. Using $H(a)=0$ and $H_\theta=0$, we can eliminate the co-states σ_1 and σ_2. Doing so and re-arranging gives

$$\frac{(1-\alpha(\theta))\pi m(1,\theta)}{(1-\alpha(\theta))\pi m(1,\theta)+\alpha(\theta)\eta(\theta)+\lambda} = \frac{rU-b}{y-b} = \frac{(1-\alpha_0)m_0}{(1-\alpha_0)m_0+\alpha_0\eta_0+\lambda}. \tag{74}$$

Equations (20) and (74) are identical; i.e., the age-profile of worker search that characterizes the social planner optimum is implemented in the competitive search allocation, and the wage schedule that decentralizes the social planner allocation (equation (24) in the text) can be inferred from equations (68) and (74).

Given that workers allocate themselves efficiently across submarkets, the competitive search equilibrium level of vacancy creation and the choice of renewal age will also be optimal. We can verify this formally from the social planner's optimal control problem as follows. Free entry, i.e., $V(0)=c$, implies

$$c(1-\tau(A)+k\tau(A) = z(A) = \int_0^A\left(\eta(\theta(a))\frac{y-rU}{\lambda} - (rU-b)\frac{1}{\pi(a)\theta(a)}\right)\tau(a)da. \tag{75}$$

Using

$$rU = b + \frac{(1-\alpha_0)m_0}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}(y-b), \tag{76}$$

we have

$$c(1-\tau(A) + k\tau(A) = \frac{y-b}{\lambda}\left(N(A) - \frac{(1-\alpha_0)m_0 D(A)}{(1-\alpha_0)m_0 + \alpha_0\eta_0 + \lambda}\right), \tag{77}$$

which matches equation (21).

Lastly, we need to verify that the social planner and competitive search equilibrium values of A coincide. The equilibrium value of A is determined by the condition that the capital gain induced by renewal, $c-k$, equals the opportunity cost of keeping the listing alive. Since the equilibrium profile of labor market tightness matches that of the social planner, this opportunity cost can be inferred from the transversality conditions in the social planner's problem. These transversality conditions are as follows:

$$\sigma_1(A) = \frac{-k + z(A)}{(1-\tau(A))^2}, \tag{78}$$

$$\sigma_2(A) = \frac{1}{1-\tau(A)}. \tag{79}$$

Therefore

$$\frac{\sigma_{1A}}{\sigma_{2A}} = \frac{\partial z(A)}{\partial \tau(A)} = \frac{-k + z(A)}{1-\tau(A)} = c - k. \tag{80}$$

Combining this equation with $H_\theta(A) = 0$, i.e., the maximum principle, gives

$$c - k = \frac{\alpha_A \eta_A}{(1-\alpha_A)\mu_A + \alpha_A\eta_A + \lambda}\frac{y-b}{\lambda}. \tag{81}$$

Equations (74), (77) and (81) describe the competitive search allocation with listing-age-dependent wages. These are the same as equations (20), (21) and (22) describing the constrained efficient allocation. Therefore the two allocations coincide. The constrained efficient allocation is decentralized.

References

[1] Acharya, Sushant and Shu Lin Wee (2018), "Replacement Hiring and the Productivity-Wage Gap," Federal Reserve Bank of New York Staff Reports 860, June 2018.

[2] Albrecht, James, Pieter Gautier and Susan Vroman (2006), "Equilibrium Directed Search with Multiple Applications," *Review of Economic Studies,* 73(4), 869-91.

[3] Belot, Michèle, Philipp Kircher and Paul Muller (2018), "How Wage Announcement Affect Search Behavior – A Field Experiment," working paper

[4] Board, Simon, Moritz Meyer-ter-Vehn and Tomasz Sadzik (2020), "Recruiting Talent," working paper.

[5] Chéron, Arnaud and Bruno Decreuse (2017), "Matching with Phantoms," *Review of Economic Studies,* 84, 1041–1070

[6] Coles, Melyvn and Eric Smith (1998), "Marketplaces and Matching," *International Economic Review* 39(1), 239-254.

[7] Davis, Steven, Jason Faberman and John Haltiwanger (2013), "The Establishment-Level Behavior of Vacancies and Hiring," *Quarterly Journal of Economics* 128(2), 581-622.

[8] Davis, Steven and Brenda Samaniego de la Parra (2018), "Application Flows," working paper

[9] Faberman, Jason and Mariana Kudlyak (2014), "The Intensity of Job Search and Search Duration," Federal Reserve Bank of Richmond Working Paper #14-12.

[10] Fradkin, Andrey (2019) "Search, Matching, and the Role of Digital Marketplace Design in Enabling Trade: Evidence from Airbnb," working paper

[11] Galenianos, Manolis and Philipp Kircher (2009), "Directed Search with Multiple Job Applications," *Journal of Economic Theory*, 114(2), 445-71

[12] Kim, Kyungmin (2017) "Information about Sellers' Past Behavior in the Market for Lemons," *Journal of Economic Theory*, 169, 365-399.

[13] Mangin, Sephora and Benoit Julien (2020) "Efficiency in Search and Matching Models: A Generalized Hosios Condition," working paper.

[14] Shimer, Robert (2005), "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review* 95(1), 25-49.

[15] Skandalis, Daphné (2019), "Breaking News: The Role of Information in Job Search and Matching," working paper

[16] Van Ours, Jan and Geert Ridder (1992), "Vacancies and the Recruitment of New Employees," *Journal of Labor Economics 10(2), 138-55.*

[17] Wolthoff, Ronald (2018) "Applications and Interviews: Firms' Recruiting Decisions in a Frictional Labor Market," *Review of Economic Studies*, 85(2), 1314–1351.