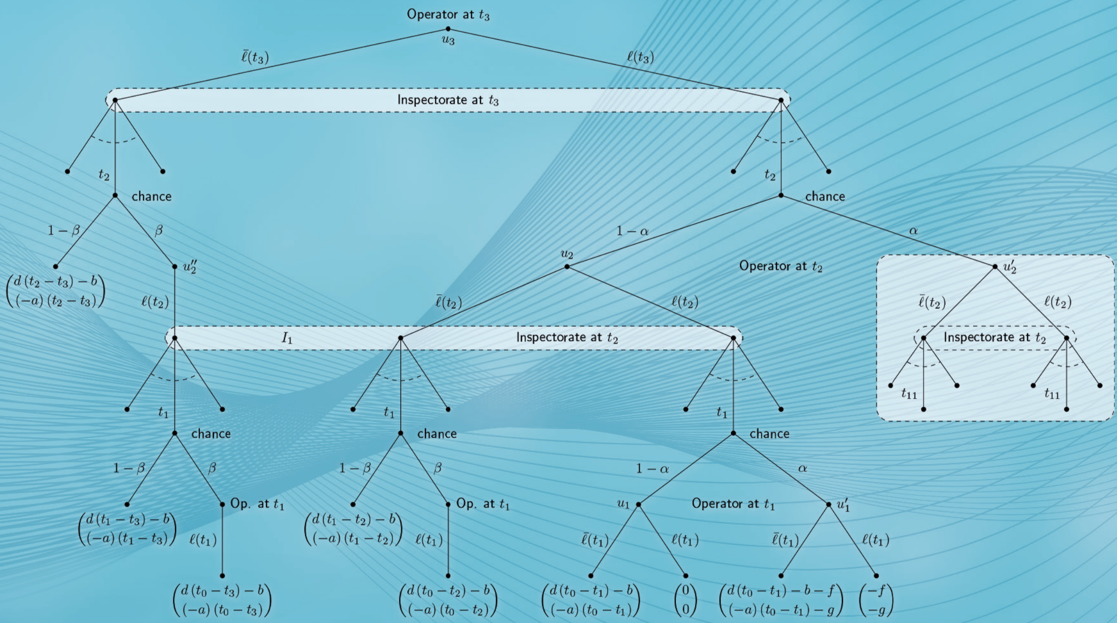

Inspection Games over Time
Fundamental Models and Approaches

Rudolf Avenhaus, Thomas Krieger

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Authors:	**Institutions:**
Prof. Dr. Rudolf Avenhaus	Universität der Bundeswehr München Fakultät für Informatik Werner-Heisenberg-Weg 39 85579 Neubiberg, Germany Email: rudolf.avenhaus@unibw.de
Dr. Thomas Krieger	Forschungszentrum Jülich GmbH Institut für Energie- und Klimaforschung Nukleare Entsorgung und Reaktorsicherheit (IEK-6) Abteilung Kernmaterialüberwachung und nukleare Sicherung Wilhelm-Johnen-Straße 52425 Jülich, Germany Email: t.krieger@fz-juelich.de

This monograph was prepared as an account of work sponsored by the Federal Ministry for Economic Affairs and Energy of the Federal Republic of Germany within the Joint Program on the Technical Development and Further Improvement of IAEA Safeguards between the Federal Republic of Germany and the IAEA, and in the context of the German Support Programme task "Optimization Approaches to Inspection/Verification Design".

Foreword

Game theory is now recognized as an indispensable tool to describe and analyse real-world problems of interdependent decision making in many fields, not limited to social sciences but also biology, engineering, computer sciences, and so on. One of the most important fields is that of inspection games. Game theory has been greatly successful in modelling problems and finding practical solutions related to this field. This monograph is written by two experts on the subject, including Rudolf Avenhaus, who has been an active leading researcher in this field for decades. It explains the basic concepts and analytical methodologies of inspection games, and extends them to more advanced problems.

Specifically, this monograph presents an overview of the fundamental models and approaches related to inspection games over time. It also provides useful guidance for practitioners to decide on the models that they could apply to their inspection problems. To accomplish these two purposes, the authors develop a hierarchy of assumptions regarding inspection philosophy, time, planning, and sampling, for all game theoretic models in review, and ascertain equivalences, relationships, and differences between these models and their equilibria. More importantly, beyond the review of existing models, the monograph presents new inspection models that fill the gaps in the existing ones.

The basic structure of an inspection game is as follows. There are two players called an Inspectorate and an Operator. The purpose of the Inspectorate is to prevent the Operator from violating certain rules such as those related to law, treaty, agreement, contract, or promise. On the other hand, the Operator has an economic, political, and/or military incentive to violate these rules. The Inspectorate cannot observe and has only imperfect information on the Operator's behaviour while the Operator strategically tries to conceal his violation. Given the Inspectorate's limited resources, only partial verification is possible.

There are many real-world problems that can be described as inspection games. A serious case is the Nuclear Non-Proliferation Treaty. The International Atomic Energy Agency (IAEA) performs inspections to prevent member countries from diverting nuclear material for military use. Arms control and disarmament, accounting and auditing, and environmental control are major applications of inspection games. In everyday life, inspection games arise in traffic control on roads, ticket inspection on public trains and buses, custom control in airports, and so on. Among many applied fields of game theory, the uniqueness of inspection games is that its theory has developed in response to problems raised by practitioners. Indeed, theorists and practitioners have interacted fruitfully to describe and solve real inspection problems.

In general, an Inspectorate has to decide whether or not an Operator has behaved legally based on imperfect information that is obtained by random sampling procedures. In the terminology of a classical testing hypothesis problem, the Inspectorate has to decide between the null hypothesis (H_0: legal behaviour) and alternative hypothesis (H_1: illegal behaviour) about the distribution of a random variable Z, based on the observation z of Z. The Inspectorate may sound an alarm (rejecting H_0) or no alarm (accepting H_0). In such an uncertain situation, an Inspectorate may make two kinds of errors. An error of the first kind is a false alarm, i.e. rejecting H_0 when the Operator has behaved legally. An error of the second kind is one of no-detection, i.e. accepting H_0 when the Operator has behaved illegally.

This monograph consists of three parts: discrete, continuous, and critical time inspection games. Each part begins by clarifying general model assumptions and proceeds to analyse the four classes of the models, depending on whether an Inspectorate and an Operator behave either sequentially or non-sequentially. In the case of zero-sum games and under the presumption that the Operator behaves illegally, the Inspectorate and the Operator want to minimize and maximize the detection time, i.e. the time between start and detection of the illegal activity, respectively. The optimal strategies for an Inspectorate and an Operator are characterized in the most complete way for each class of zero-sum inspection games through knowledge updates. Readers would find it mathematically challenging to derive an explicit formula of the optimal strategies in many inspection games over time. The optimal strategies depend only on technical parameters such as detection probabilities. This is very important for practical applications. If the question of illegal behaviour deterrence is raised, political parameters, in the form of utility functions, should be introduced, and inspection problems should be formulated as non-zero-sum games. After all, it is often the case that a false alarm inflicts costs on both the Inspectorate and the Operator.

Providing well-organized materials from the fundamentals to the state of the art, this monograph serves as an invaluable resource for experts, practitioners, and theoreticians who are working on real-world inspection problems for inspection authorities. The collected materials benefit 'modellers' who wish to learn the art of modelling for application to their own problems. Game theorists who are keen to apply the theory to determine convincing solutions to real-world problems would also greatly benefit and be inspired by the authors' work. Moreover, the monograph can form a basis for academic lectures geared towards advanced students who are interested in inspection problems and/or in successful applications of game theory to a special but important class of real-world problems. Indeed, the authors are to be warmly congratulated for this excellent monograph.

I first met Rudolf Avenhaus in 1988 at the Center for Interdisciplinary Research (Zentrum für interdisziplinäre Forschung, ZiF) at the University of Bielefeld, Germany. Both of us were invited to participate in the one-year research project titled, 'Game Theory in the Behavioral Sciences' organized by Reinhard Selten, a 1994 Nobel laureate in economics who passed away two years ago. Lively interactions took place among interdisciplinary participants comprising economists, biologists, mathematicians, political scientists, psychologists, and even a philosopher. Given Selten's profound influence on the thinking of many participants, the research project was highly productive. Specifically, he taught us many invaluable things related to and beyond game theory, and encouraged us to collaborate on inspection games; see Avenhaus et al. (1991). Our research benefited considerably from his continued advice and personal thoughtfulness. It is my great pleasure and honour to pen a foreword to this monograph, which Avenhaus also considers an homage to Reinhard Selten.

Akira Okada
Tokyo, April 2018

Preface

Having worked for so many years in the field of modelling inspections, in particular in inspections over time, we felt a need for writing a monograph on this subject for basically two reasons.

In the last thirty years a large number of game theoretical models have been developed and analysed, which describe similar or related inspection problems. As a consequence it has become increasingly difficult to maintain an overview on what exists already, how these models are related to each other and where possible gaps might exist.

More than that are in many cases the assumptions for these models not documented very well, to say the least, which means that in particular for practitioners, who wants to use the results of the analyses of these models, it is very difficult to decide whether or not they describe their inspection problems and procedures properly.

This monograph tries to solve these two kinds of problems. Most of the inspection models presented here, but not all of them, have been published already elsewhere, sometimes in conference proceedings, sometimes as PhD dissertations or just as technical reports which means, not easily accessible. Also, as indicated above, underlying assumptions were not complete or lacking at all, and references to related work was missing. But of course there are also publications in which assumptions, analyses and results are so carefully described that we simply, with due reference, used their wording.

Beyond the collection of published inspection models over time we present *nine new* inspection models which close obvious gaps and – this we consider most important – we structure the material: We develop a hierarchy of assumptions for all models, and we describe equivalences, relations and differences between game theoretical models and their solutions.

When talking about mathematical models, the question of their applicability will be raised immediately. Without discussing this issue here in detail – this will be done in the main text – it should be answered already here that indeed most of the inspection models have their origin in a practical problem, but that they range from those which lend themselves to immediate application to those which have been developed primarily for theoretical purposes. In the introductory first chapter we will say more about this central issue. Here we just express our hope that both practitioners and theoreticians will become interested into our work which we performed with so much enthusiasm for the subject.

Rudolf Avenhaus and Thomas Krieger
München and Jülich 2020

Contents

Chapter 1

Introduction

An inspection game is a mathematical model of a situation where a person or an organisation, in the following called *Inspectorate*, verifies that another party, maybe a person, an organisation or even a State, and in the following called *Operator*, adheres to certain agreed or legal rules. This agreed or legal behaviour may be defined, for example by some code of conduct, by some formal agreement or even by an international treaty. The Operator may, however, have an interest in violating these agreed or legal rules. Typically, the Inspectorate's resources are limited so that verification can only be partial. A mathematical analysis helps in designing an optimal inspection scheme, where it must be assumed that an illegal activity is executed strategically. This defines a game theoretical problem, usually with two players, Inspectorate and Operator. In some cases, several Operators are considered as individual players.

The principal objective of the Inspectorate is, let us use a wording different from that given above and which has been taken from an international treaty, to deter the Operator from behaving illegally "by the risk of early detection"; see IAEA (1972). This means that an illegal activity, once started, shall be detected with as high a probability and as early as possible. But what does the latter postulate mean? There are various ways to model the timeliness capability of routine inspection regimes. For instance one can choose objective functions which depend on the time between the start of the illegal activity and its detection, or objective functions which are simple dichotomies that are based on some imposed critical detection goal. In addition, one can assume unobservable inspections such as might be associated with instrumental or remote surveillance, or alternatively, that the inspections are observable (in the sense of inspections on site) so that the Operator can make his decisions conditional on the inspection time points of the Inspectorate. Furthermore, statistical errors may or may not be taken into account.

Before elaborating on this in Section 1.1, two remarks, the first one on related subjects: Inspection games should be distinguished from inspections for quality control, or for prevention of other kinds of random accidents, for which there is no adversary who acts strategically; and inspections that are search problems, where an adversary attempts to escape a searcher with well-defined and legitimate strategies, like a submarine escaping a destroyer in war. Neither situation is described by an inspection game in the sense of this monograph; here, let us repeat this, the salient feature is that the Inspectorate tries to prevent the Operator from behaving illegally in terms of the rules of conduct, or agreement, or treaty. Nevertheless, it will be shown that in some cases, e.g., in quality control, models are used which are related to inspection games considered in this monograph; see Diamond's inspection game in Chapter 9.

The second remark deals with this monograph's specialization to *inspection games over time*: There are books, e.g., Avenhaus and Canty (1996), and review articles, e.g., Avenhaus et al. (1996) and Avenhaus et al. (2002), on inspection games in general. During the last 15 years, however, so many inspection games over time have been published, all of them based on similar, partly similar or different assumptions such that it became increasingly difficult to maintain an overview. Therefore we felt the need to organize this wealth of models and to present it in a way as coherent as possible.

1.1 Classification of assumptions for inspection models over time

Since there exist so many game theoretical models for inspections over time, it is one of the objectives of this monograph to classify these inspection models, in other words, to list all assumptions which define these models. Figure 1.1 represents the classification of assumptions for the inspection games over time treated in this monograph.

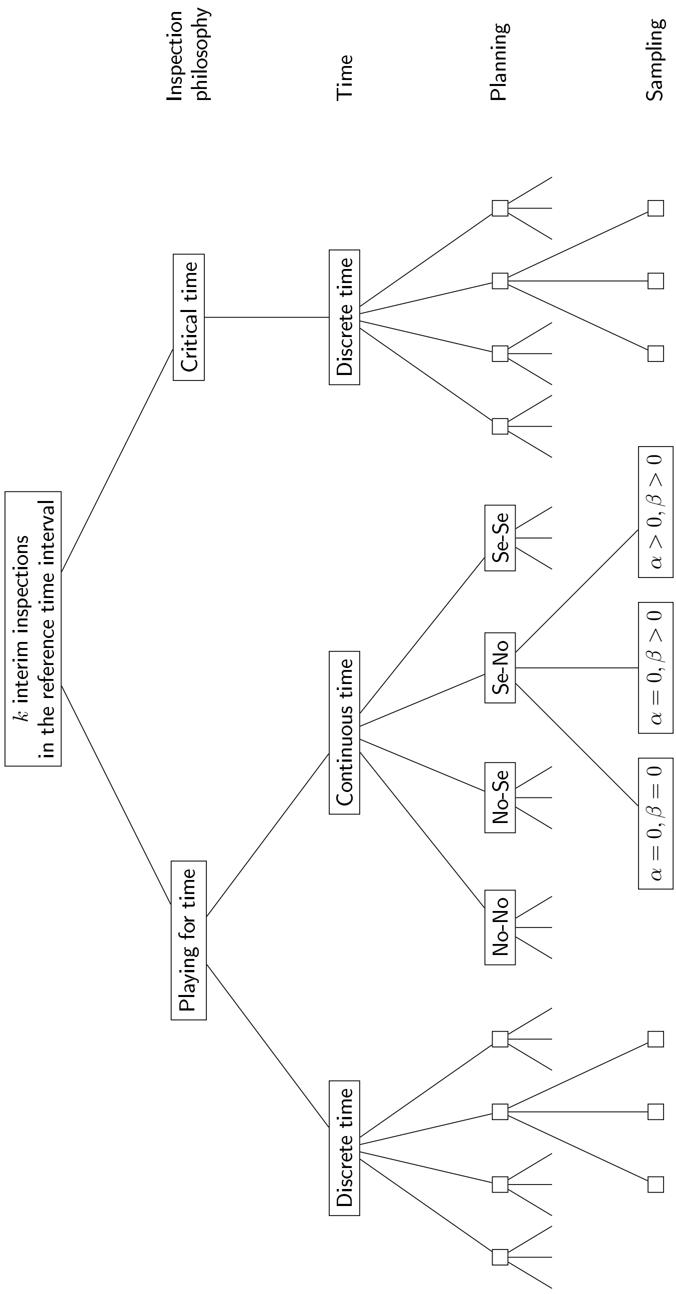
Quite generally a reference time interval, e.g., a calendar year, is considered within which the Inspectorate performs k interim inspections. Let us mention that at first sight this description of an interim inspection over time may look very special, but that in fact it covers, with appropriate interpretations, all inspection problems over time considered in this monograph.

The four dimensions in Figure 1.1 have the following meaning:

- *Inspection philosophy:* The objective of the Inspectorate is to detect any illegal activity of the Operator, if he intends to do so, as early as possible – *playing for time*, or to detect it within a given time interval – *critical time*. In Chapter 17 inspection problems are considered which are formally equivalent to critical time games even though their context is totally different.
- *Time:* The Inspectorate can perform its interim inspections at any point of time during the (open) reference time interval, and the Operator can start the illegal activity at any point of time during the (open) reference time interval – *continuous time*, or the interim inspections are performed only at discrete and equidistant points of time during the reference time interval, and the illegal activity can start only at discrete and equidistant points of time during the reference time interval – *discrete time*.
- *Planning:* The Operator may plan the illegal activity, if he intends to do so, non-sequentially (No) at the beginning of the reference time interval, or sequentially (Se) during the reference time interval. The same possibilities exist for the Inspectorate. Thus, four variants are distinguished (in the order: Operator-Inspectorate): No-No, No-Se, Se-No and Se-Se.
- *Sampling:* Interim inspections may be performed without committing any statistical error ($\alpha = \beta = 0$), or with committing only errors of the second kind (no detection of illegal behaviour, $\alpha = 0, \beta > 0$), or with committing both errors of the first (false alarms) and second kind ($\alpha > 0, \beta > 0$).

Four remarks on this classification of assumptions, see also Krieger and Avenhaus (2018a) and Krieger and Avenhaus (2018b):

Figure 1.1 Classification of assumptions for inspection models over time.



First, the inspection philosophies are derived from the International Atomic Energy Agency's objective of safeguards: "The Agreement should provide that the objective of safeguards is the timely detection of diversion of significant quantities of nuclear material from peaceful nuclear activities to the manufacture of nuclear weapons or of other nuclear explosive devices or for purposes unknown, and deterrence of such diversion by the risk of early detection"; see IAEA (1972) and p. 1. While the *playing for time* concept is related to the "early detection" objective, the *critical time* concept is related to the "timely detection" objective and has its origin in the so-called conversion time introduced by the IAEA; see IAEA (2002). The conversion time is the time required to convert different forms of nuclear material to the metallic components of a nuclear explosive device. Using this definition it appears to be natural to assume that the Inspectorate has "won" the game if any illegal activity is detected within some "critical" time, otherwise it has "lost" it. In practical applications, which are quite different from IAEA safeguards and which of course are considered in this monograph as well, the critical time may have quite different meanings: Whereas for example in the context of the Non-Proliferation Treaty it means the conversion time for nuclear material, in the case of the control of waterways by Customs it may mean a night during which Smugglers have a chance to cross the water without being detected; see Chapter 17. Note that inspection games based on other objective functions, such as the overall non-detection probability as payoff to the Operator if nuclear material can be diverted from several strata, have been analysed, e.g., in Avenhaus (1986), Avenhaus and Canty (1996) or Avenhaus and Krieger (2011a), but do not fit in this monograph because of its different objective function.

Second, the advantage of discrete time inspection games is that any kind of time resolution can be modelled, e.g., years, month, weeks, days, hours, and seconds. Thus, these games seem to be appropriate for applications as boundary conditions and organizational aspects such as working days could be taken into account. Because in continuous time inspection games the interim inspections can be performed at *any* time point during the reference time interval, the disadvantage of these games is that the implementation of the game theoretical results in practice becomes difficult because access might not be granted or even impossible at *any* time point, e.g., Sunday at 2:17 a.m. The advantage of continuous time inspection games is that they are easier to be analysed even in case first and second kind errors are taken into account.

Third, the sequential inspection planning requires quite a flexibility on the Inspectorate's part.

Fourth, considering the sampling dimension, the second case is relevant, if items are counted on a random sampling basis, i.e., if *Attribute Sampling Procedures* are used. The third case is given when quantitative measurements are performed, i.e., when *Variable Sampling Procedures* are used, where correct data or behaviour may be declared as wrong ones. We do not consider the first and second case as specializations of the third one since inspections games with errors of the first kind and their Nash equilibria are much more complicated than those without first kind errors, in other words, one would never start with the complicated form and then specialize to the simpler one. A finer argument shows that one cannot simply change the error first kind probability α without changing that of the error second kind β: In general β tends towards one if α tends towards zero; see Chapter 20.

According to Figure 1.1 there exist $3 \times 4 \times 3 = 36$ inspection models over time. Not all of them have been, or will be analysed, but on the other hand, even more assumptions will have to be made in some cases; see, e.g., the "General Assumptions" Chapters 2, 8 and 14 or the assumption if the Operator decides to behave legally or illegally during the course of the game or at the beginning of the game. The structure of Parts I – III will be deduced from Figure 1.1.

1.2 Applications

Since, according to Heraklit (520 - 460 bC), war is the father of all things, it is not surprising that the first game theoretical models of inspections after the path breaking work by von Neumann and Morgenstern (1947) were developed during the time of the Cold War and in the context of arms control and disarmament studies. Probably the first genuine inspection game published in the open literature was the recursive game developed by Dresher (1962) which will be discussed in detail in the Chapter 16. Dresher himself suggested several possible arms control problems as applications for his model, in particular verification of a test ban treaty.

In a survey on inspection games in Arms Control by Avenhaus et al. (1996), it was pointed out that three phases of development in the application of inspection models to arms control and disarmament may be identified: In the first of these, roughly from 1961 to 1968, studies that focused on inspecting a nuclear test ban treaty emphasized game theory, with less consideration given to statistical aspects associated with data acquisition and measurement uncertainty. The second phase, from 1968 to about 1985, involves work stimulated by the Treaty on the Non-Proliferation of Nuclear Weapons (NPT). Here, the verification principle of material accountancy came to the fore, along with the need to include the formalism of statistical decision theory within the inspection models. The third phase, 1985 to the present, has been dominated by challenges posed by such far-reaching verification agreements as the Intermediate Range Nuclear Forces Agreement (INF), the Treaty on Conventional Forces in Europe (CFE) and the Chemical Weapons Convention (CWC), as well as perceived failures of the NPT system in Iraq and North Korea. The Comprehensive Test Ban Treaty (CTBT) has been agreed upon by a UN resolution in 1996, but it is not yet in force because not enough States have ratified it.

In particular in the context of the NPT and the verification system of the International Atomic Energy Agency (IAEA) a wealth of inspection games has been developed and studied. Since the mid-nineteens, when the Additional Protocol, see IAEA (1997), has been agreed upon, the importance of interim inspections in nuclear facilities has been emphasized and consequently, game theoretical models have been developed with the aim to determine the optimal timing and intensity of these inspections. These models represent best the idea of inspection games over time therefore, the concept of interim inspections will be used as a general metaphor, with appropriate interpretations whenever necessary.

Because of the importance of this application, let us add two remarks. First, both inspection philosophies are relevant here: On one hand, the playing for time concept meets, as mentioned on p. 1, one of the IAEA safeguards criteria "...by the risk of early detection"; see IAEA (1972). On the other, and since the purification of nuclear material and subsequent manufacture of a nuclear weapon takes time, the IAEA has explicitly defined critical times for all forms of fissionable material that can arise in commercial nuclear activities. These range from seven days for metallic plutonium to one year for low enriched uranium; see IAEA (2002).

Second, in order to avoid the wrong impression that interim inspections cover the whole area of nuclear safeguards, two other areas are mentioned here. The one is the verification of the measurement of data reported by the Operator/States with the help of independent measurements performed by an Inspectorate. In Statistics this wide area is known under the technical term stratified variable sampling. The other area is nuclear material accountancy which represents the core of nuclear material safeguards: With the help of the Operator's declared data, book and physical inventories are compared at the end of some material balance period for a so-called material balance area in order to decide whether the difference between these two inventories can be justified by the statistical measurement errors or else, can only be explained by loss or hidden inventories or diversion of nuclear material.

Until today the verification system for the NPT is the most elaborate international one, in fact, it has served as a model for the verification of the CWC and others; see Avenhaus et al. (2006). In particular the concepts of on-site and interim inspections have been adapted and elaborated for other Arms Control and Disarmament Treaties; see Melamud et al. (2014). It is, however, just fair to mention that for the description and analysis of verification systems other than that for the NPT, inspection games over time have not yet been used to that degree they would deserve.

The control of straits or waterways is another widely studied area of application: It is assumed that Smugglers will try to cross a strait with boats, and that customs officers try to catch them. Most of the analyses are based on, or are variations of Dresher's model, and we will describe them in detail in Part III. It has to be admitted, however, that to our best knowledge so far we have found no real applications in the open literature even though we assume that there are some which may not have been published for reasons of confidentiality.

Quite a different application is given by reliability and production control problems: Even though, as mentioned on p. 1, there is no adversary who may behave illegally, one assumes in the sense of a pessimistic approach that the technical equipment will produce failures which are most disastrous, and that one has to optimize inspection procedures under this assumption. We will discuss the seminal paper by Diamond (1982) in major detail in Chapter 9.

There are other applications, e.g., in the environmental control area, but a word of caution is just fair: What is an application? Theorists claim that an application is the next lower level of a theory. In that sense, e.g., in Probability Theory the De Moivre-Laplace Theorem is just an application of the Central Limit Theorem. An engineer, on the other hand, may consider an application of a theory something which helps him to construct a better device or to reduce production costs. Inspection games over time are somewhere between these two extremes: In some cases they give concrete advice, as we shall see, but in other cases problems are analysed in an idealized way, and real applications are not known to us.

In fact the same may be said about the application of game theory in general: Even though there is a continuing enthusiasm that this theory is uniquely suited for the description and analysis of rational behaviour in human interaction, especially in conflict situations, serious applications like those of inspection games are still rare, probably more time and patience is needed. Even Leibniz's and Newton's calculus needed more than a century for becoming an indispensable tool of natural sciences, technology and economy.

1.3 The art of modelling

When analysts who have a reasonable background in Statistics, Game Theory and Operations Research in general, and who have some experience with applied work, are asked for answers by practitioners who know their problems but not the difficulties of quantitative modelling, the following frequently happens: Either analysts arrive at formal models and their solutions which are satisfying from a mathematical, not to say aesthetic point of view, but not so much from a practical one, or the problem and its so solutions is so special, that it is interesting only for that single case, or the analysts have to work with approximations and present second best solutions which may answer the practitioners' questions but are not satisfying analysts' standards of rigorousness and non-triviality.

Only at rare occasions – lucky moments in their lives – analysts come across problems and their solutions which are satisfying both from the practical and theoretical points of view and beyond, they are of more general interest, in other words, these analysts have found some interesting general structure.

In this monograph, in some sense we have come across all these situations: As an example, let us take the continuous time inspection game by Diamond (1982) which is presented in Chapter 9. There, in Theorem 9.1 optimal strategies for any number of interim inspections during the reference time interval are derived, and its proof is by no means trivial. More than that, the optimal strategies of the Inspectorate can be used as approximations for the discrete time inspection game discussed in Chapter 3, where for large numbers of possible time points and for more than one interim inspections during the reference time optimal strategies have not yet been found. But there are also other cases where optimal strategies so far have only be found for limited parameter values, for example in Chapter 6, even though optimal strategies for more general cases would be interesting from a practical point of view.

If the practitioner, or more generally the person or organization which proposes to analysts a study of a problem, formulates the problem in form of very precise assumptions, then it is just good luck or bad luck whether or not the analysis will become interesting from the analysts' point of view. The real situation, however, is different in general. The practitioner may not know so precisely which assumptions are required for the quantitative analysis, or he may not be so certain about all assumptions required, or he may change his opinions in the course of the analysts' work. This, in turn gives room to the latter one to formulate assumptions himself and ask for their confirmation. This way both the practitioner and the analyst can guide the work in directions which renders the analysis feasible and convincing for both sides: Sometimes the model itself may be justified by an interesting solution which, by the way, requires that the analyst is able to explain this solution in a way which convinces the practitioner.

All this shows that modelling is a kind of art which cannot be taught and learned with the help of textbooks and courses alone. It has to be practised for long times under favourable circumstances; a key word for those who have or want to encourage studies of the kind an example for which is presented here, is capacity building for both sides, the practitioner and the analyst.

In this sense, and also having in mind that the application of game theoretical methods is one of the main objectives of this monograph, let us conclude these considerations with a statement by the eminent game theorist and first Director of the International Institute for Applied Systems Analysis (IIASA) in Laxenburg near Vienna, Howard Raiffa, which he made in the context of a justification of and an advertisement for formal analyses of international negotiations, see Raiffa (2002):

> ... Not enough research on the processes of international negotiations is being done. What is being done is not adequately coordinated and disseminated. Present research efforts are not cross-fertilized: across disciplines, between practitioners and researchers, and across national boundaries. Regrettably, a lot of profound theorizing by economists, mathematicians, philosophers, and game theorists on topics related to negotiation analysis has had little or no impact on practice.
>
> ... An important reason is clearly the lack of effective communication and dissem-

> ination of theoretical research results. Such communication could be improved if there were more intermediaries who are comfortable in both worlds and who could act as inventive go-betweens to facilitate the transfer of information that shows how theory can influence practice and how practice can influence the research agendas of theorists. The information must flow in both directions: many practitioners have developed valid, extremely useful, and often profound insights and analyses, which should help to guide the agenda of researchers in this field.

It goes without saying that these memorable thoughts of someone who lived indeed in both worlds hold in the area of inspection games as well.

1.4 Structure of the monograph

Looking again at the classification in Figure 1.1 one realizes that there is no royal way for structuring the inspection models over time. Using the *inspection philosophy* and the *time* aspect in the classification, we arrive at three main classes of inspection games which are the subjects of the three parts: First, discrete time playing for time games. Second, continuous time playing for time games. And third, discrete time critical time games. Within these three classes, four variants are considered which describe the planning both of the Operator and the Inspectorate: Both plan their activities at the beginning of the reference time interval (No-No), or both proceeds sequentially (Se-Se), or only the Inspectorate proceeds sequentially (No-Se), or only the Operator proceed sequentially (Se-No). It should be mentioned already here, that only in a few cases and under rather limiting assumptions in regards to the possible interim inspection time points and the number of interim inspections all four variants have been studied. The same holds for the consideration of statistical errors of the first and second kind: Since in the most general case the analyses get very complicated, only in a few cases satisfying solutions have been obtained.

There is an issue the analysis of which represents another important objective of this monograph: In Parts I and II, where the payoff to the Operator is the detection time, i.e., the time between start and detection of the illegal activity – the payoff to the Inspectorate being its negative value – it is assumed that the Operator behaves illegally. This means that only *technical parameters* like detection probabilities, inspection costs and others enter the models, and that optimal inspection strategies can be determined which depend only on these parameters. Obviously this is very important for practical applications. If, however, the question of deterring the Operator from illegal behaviour is raised which is, as stated on p. 1, so important for the whole area of inspections, then this procedure is no longer sufficient: In addition, so-called *political parameters* in form of utility functions have to be introduced which describe the gains and losses of the Operator in case of undetected and detected illegal behaviour; the analysis will lead to conditions, e.g., concerning inspection costs, for legal behaviour of the Operator.

There is another aspect of this issue. In case of variable sampling procedures false alarms may happen, and they cause problems, at least costs, to both the Operator and the Inspectorate. Thus, the zero-sum assumption does not hold any more. This means that also in these cases, independently of the question of legal or illegal behaviour of the Operator, payoff parameters have to be introduced.

In Part III idealized payoffs, in technical terms *utilities*, – in contrast to the detection time in Parts I and II which has a physical meaning – have to be taken into account from the very beginning of the analysis. They describe the gains or losses of both players in case of detected or undetected illegal behaviour, or legal behaviour, of the Operator. There is another difference between Part III and Parts I and II: So far, we used the concept of interim inspections within some reference time interval as metaphor for all kinds of inspections over time. If we consider, however, for example the control of waterways by customs officers, then the terms critical time and interim inspections do not make any sense even though the mathematical models are the same. Therefore, Chapter 17 in Part III is devoted to these and related inspection problems.

There are a number of additional structural aspects which deserve attention at this place: First, in contrast to abstract game theoretical analyses the pure strategies of the Operator and Inspectorate in Parts I and II are the time points for the start of the illegal activity and for the interim inspections, respectively. Thus, they have a physical meaning. Therefore, beside of the game theoretical results it is also interesting to determine and to analyse *system quantities* because they are intuitively appealing and therefore used by practitioners; see Table 1.1.

Table 1.1 Overview of system parameters, results of the game theoretical analysis and system quantities used in Parts I and II.

system parameters	results of the game theoretical analysis	system quantities
time points, number of interim inspections, number of facilities, non-detection probabilities, false alarm probabilities	optimal strategies and optimal expected detection time, equilibrium strategies and respective payoffs	cut-off value/time, optimal expected detection time, optimal expected start of the illegal activity, optimal expected interim inspection time point(s)

Second, in Parts I and II frequently inspection problems are considered which are modelled as zero-sum games which means in general that we are looking only for one optimal strategy of each player. In those games in which we are able to present more than one optimal strategy, selection procedures are discussed which take into account practical needs and differ fundamentally from those proposed, e.g., by van Damme (1987) or Harsanyi and Selten (1988). Also, it is admitted that in case of non-zero-sum games we did not always pay attention to the uniqueness of equilibria: Even though we have reasons to assume that the equilibria presented are indeed unique, e.g., on the basis of special cases, there is room for further work.

Third, the contents of this monograph described so far looks like a compilation of existing inspection games over time. There is, however, more and this should be stressed already here: *Nine new* inspection models are developed which are published in this monograph for the first time:

- discrete time playing for time No-Se inspection game; see Lemma 4.1,
- discrete time playing for time Se-No inspection game; see Theorem 4.1 and Lemma 4.4,
- discrete time playing for time Se-Se inspection game; see all Lemmata and Theorems of Chapter 5,

- discrete time playing for time No-No inspection game with errors of the second kind; see Theorem 6.1,
- continuous time playing for time Se-No inspection game with any number of interim inspections and with facility-independent errors of the second kind; see Theorem 11.2 and Lemma 11.3,
- critical time No-No inspection game with errors of the first and second kind; see Theorem 15.2,
- critical time Se-Se inspection game with errors of the second kind; see Theorem 16.2,
- generalized Thomas-Nisgav inspection game; see Theorem 17.1 with Lemmata 17.1 and 17.2,
- critical time Se-No inspection game with an *expected* number of inspections; see Lemma 24.1.

Furthermore, the Conjectures 5.1, 9.1, 17.1 and 24.1 are novel. Also surprising relations between different game theoretical models and their Nash equilibria are discussed extensively, and considerations on the choice of the optimal false alarm probability in Sections 9.5, 12.4, 15.5 and 16.4 are presented.

Fourth, there are inspection problems and issues which have to be described by models which do not fit into the classification in Figure 1.1. Let us only mention the *inspector leadership* idea, according to which the Inspectorate announces its inspection strategy to the Operator in a credible way and thus, may induce him to legal behaviour. This kind of model is discussed in Sections 9.5, 12.4, 15.5 and 16.4.

Fifth, a word on notation: Throughout this monograph we have spent an effort to develop a consistent and convincing set of symbols for the more important quantities and concepts which are used. This was necessary since in contrast to areas as Probability Theory, Mathematical Statistics and others this consistency of notation has not yet been achieved properly in Game Theory; see Myerson (1991), Fudenberg and Tirole (1991), Peters and Vrieze (1992), Aumann and Hart (1992) or Osborne and Rubinstein (1994).

Finally, a remark on the references collected at the end of this monograph: Game theoretical inspection problems are widely distributed in the published literature, and there are approaches which differ considerably from the ones presented in this monograph, let us mention as examples Pradiptyo (2007), Friehe (2008), Rauhut (2009), Andreozzi (2010), Deutsch et al. (2011), Jiang et al. (2013), Kolokoltsov et al. (2013), Delle Fave et al. (2014), Zhang and Luo (2014), Katsikas et al. (2016), Rossiter and Hester (2017) and Kolokoltsov and Malafeyev (2019). Thus, even though we have spent a major effort to collect most of the important publications, we do not guarantee their completeness.

1.5 Who is the client?

We think that there may be at least four categories of clients to this monograph.

First, there are experts dealing with problems where inspections serve the purpose to deter people, organisations or States from illegal behaviour in the sense of some agreement or treaty, or to say it positively, to induce them to legal behaviour. These experts may be practitioners who are looking for applications of the solutions of the inspection models, e.g., optimal distribution of given inspection resources on different locations, or optimal amount of resources for the indicated objectives. In other words these practitioners want help for their problem to organize inspections effectively and efficiently. But these experts may also be theoreticians, who are working for inspection authorities and who have to study these inspection models in order to adapt them to the authorities' needs, or to develop new models.

In this context the subtitle of this monograph may be explained: If one tries to describe a real inspection problem with the help of a game theoretical model, then one detects always special features which have to be taken into account and which require new assumptions. Also the mathematical models may get too complicated for a rigorous analysis. Therefore, we present in this monograph only what we deemed to be fundamental models and approaches, and considered only in a few cases extensions which go beyond analytical feasibility, see, e.g., Sections 9.4 and 16.3, in order to demonstrate their complexity.

Second, there are "modellers" who work in related or different areas, and who might want to learn what has been achieved in the admittedly special field dealt within this monograph in order to find out if they can use the models for their somewhat different inspection problems – see, e.g., Chapter 17 – or even for other problems where adversaries are involved as well which, however, act also legally, e.g., battleships searching for submarines.

Third, there might be theoreticians, e.g., game theorists, who are primarily interested in genuine mathematical problems and their solutions, but who might also be interested in applications in the sense explained in Section 1.2. Indeed, let us repeat this, game theory is until now not so rich in convincing applications.

And last but not least, we address the younger generation. In order that students are not frightened by the longer and more demanding proofs of some Theorems and Lemmata they – as well as other clients – may skip them at first reading and instead, try to familiarize with the inspection models and their game theoretical solutions. More importantly, we hope that this monograph may serve as a basis for academic lectures for those students who are interested in this special but important field, and also for those students who simply want to learn what abstract concepts like non-cooperative extensive form games and Nash equilibria mean when they are confronted with real world problems.

Part I
Playing for Time: Discrete Time

The three main parts of this monograph differ essentially by their inspection philosophy and by their treatment of time. In Part I the objective of the Inspectorate is to detect with the help of interim inspections the illegal activity as early as possible whereas that of the Operator is to start the illegal activity at a time point in such a way that it is detected as late as possible. Furthermore, it is assumed that the interim inspections and the start of the illegal activity can take place only at discrete points of time during the reference time interval.

Since one important goal of this monograph is to formulate the assumptions for the inspection games as complete and as clear as possible, in Chapter 2 a qualitative description and a list of assumptions is given. Even though these assumptions hold for all inspection games described and analysed in Part I, later on more assumptions are given which hold only for those games considered then.

The inspection games treated in Chapters 3 – 5 differ by the planning of the interim inspections and by the planning of the start of the illegal activity. Some models are treated in full generality and some are only solved for special cases. Throughout Part I, various relations between the inspection games analysed in Part I and relations to the continuous time inspection games treated in Part II, such as optimal/equilibrium strategies, optimal/equilibrium payoffs and system quantities given in Table 1.1, are discussed.

In Chapter 6 errors of the second kind are taken into account and all four variants of the inspection game (No-No, No-Se, Se-No and Se-Se) are analysed for the case of three possible time points for two interim inspections. Also the cases of five possible time points for one or two interim inspections are treated for illustration for the No-No and Se-No inspection game. The case of any number of possible time points for one interim inspection is solved completely for the No-No and the Se-No inspection game.

Chapter 7 finally deals with legal behaviour, the concepts of effectiveness and efficiency, utilities for variable sampling inspection schemes, i.e., when false alarms are taking into account, and extensions such as leadership considerations.

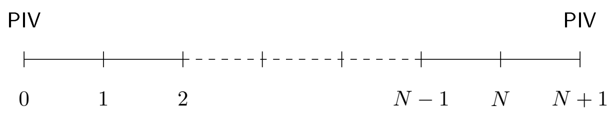
Chapter 2

General assumptions

We consider a simple inspected object, for example a production line, or a nuclear or chemical facility (for short: facility) which is subject to inspections in the framework of agreed rules, formal agreements or an international treaty, and a reference time interval of one time unit, e.g., a week, a month, or a calender year. As announced on p. 1, we call the person, or organisation or even the State which is responsible for the object or facility, the *Operator*[1], and the person or organisation which is responsible for the inspections, the *Inspectorate*.

In order to separate the timeliness aspect of routine inspections from the overall goal of detecting failures or an illegal activity, it is assumed that a thorough and unambiguous inspection takes place at the beginning and end of the reference time interval with the help of which failures or an illegal activity will be detected with certainty once they have occurred. Such an inspection is called, according to some agreed wording, Physical Inventory Verification (PIV); see IAEA (2002) and Figure 2.1.

Figure 2.1 Time line for the discrete time inspection models.



In addition, it is assumed that by agreement k less intrusive interim inspections are strategically placed during the reference time interval to reduce the time between start and detection of failures or the illegal activity below the length of the reference time interval. We assume that for natural or technical reasons interim inspections can only take place at N equally distant time points $1, 2, \ldots, N$. Thus, we have $k \leq N$. Thereafter, at the end of the reference time interval, i.e., at time point $N+1$, the above mentioned PIV takes place; see Figure 2.1. At an interim inspection a preceding failure or illegal activity will eventually be detected with some probability lower or equal than one.

For reasons to be discussed subsequently, see below, it is assumed that the Operator will start the illegal activity at one of the $N+1$ time points $0, 1, \ldots, N$ with certainty; see Figure 2.1. Of

[1] Frequently, the Operator is called inspectee, but this will not be done in this monograph in order to avoid confusion between the similar words inspectee and inspector.

course, the highest priority of the Inspectorate is to deter the Operator from illegal behaviour, or to say it positively, to induce the Operator to legal behaviour. These two aspects, that seem to contradict each other at first sight, cannot be addressed in purely technical terms like number of interim inspections, detection probabilities etc. Instead, so-called utilities have to be introduced which describe the gains and losses of both players in case of legal and illegal behaviour of the Operator. We discuss this important issue in Section 7.2.

Since the Operator is assumed to behave illegally during the reference time interval, and even may plan this strategically, and since the Inspectorate is assumed to place the interim inspections strategically, this inspection problem is described by a non-cooperative two-person game between an Operator and the Inspectorate. In this Part I, the objective of the Operator is to place the start of the illegal activity such that the *detection time* – the time between the start of the illegal activity and its detection – is as long as possible. This is the reason, see above, why we assume that an illegal activity is started immediately after an interim inspection. The objective of the Inspectorate is to place its interim inspections such that the detection time is as short as possible. This means that we consider a two-person zero-sum game with the detection time as payoff to the Operator. Note that the zero-sum assumption simplifies the analysis considerably since we need not care for the uniqueness of game theoretical solutions to be developed subsequently; see the interchangeability property on p. 399.

It should be mentioned already here that in cases, where, e.g., failures in production lines have to be detected, the term illegal activity is misleading, and that those failures cannot be assumed to be planned strategically by somebody. Nevertheless we will see that these cases will be covered by the following analyses as well.

At first sight, having read the description of the inspection game so far, one might assume that it specifies the problem completely and one may start immediately the formal analysis. It turns out, however, that more assumptions have to be made. Since it is a general major objective of this monograph to classify the many inspection games which have been developed so far and since in the literature necessary assumptions are frequently not explicitly mentioned, we will now list and number all assumptions which are necessary for the analysis of the formal models in this part. We will not repeat them for any specific model. More than that, in subsequent chapters we will refer to this list of assumptions and eventually add more or change some of the ones presented here. Thus, we consider this list of assumptions to be central for this whole part.

(i) There are two players: the Operator of the facility under consideration and the Inspectorate.

(ii) The Inspectorate performs k interim inspections at N possible time points $1, 2, \ldots, N$.

(iii) The Inspectorate performs at the beginning and at the end of the reference time interval a regular inspection (Physical Inventory Verification, PIV) at which the illegal activity of the Operator is detected with certainty if it is not detected at a previous interim inspection.

(iv) The Operator starts once at one of the $N + 1$ time points $0, 1, \ldots, N$ an illegal activity. In Chapter 7 legal behaviour of the Operator will also be considered.

(v) During an interim inspection the Inspectorate may commit an error of the second kind with probability β, i.e., the illegal activity, see assumption (iv), is not detected during the next interim inspection with probability β. Note that if there is no interim inspection left, then it is detected with certainty at the final PIV; see assumption (iii). This non-detection probability is the same for all k interim inspections. Errors of the second kind are only considered in Chapter 6. Note that game theoretical models taking errors of the first kind into account are only considered in Parts II and III.

(vi) The number k of interim inspections is known to the Operator.

(vii) The Operator decides at the beginning of the reference time interval, i.e., at time point 0, when to start the illegal activity, or he only decides whether to start the illegal activity immediately at time point 0 or to postpone the start; in the latter case he decides again after the first interim inspection; and so on.

The Inspectorate decides at the beginning of the reference time interval when to perform its interim inspections, or it decides only when to perform the first interim inspection, and after the first one when to perform the second interim inspection; and so on.

(viii) Both players decide independently of each other, i.e., no binding agreements are made.

(ix) The payoff to the Operator is the detection time, i.e., the time between start and detection of the illegal activity. The payoff to the Inspectorate is the negative one (zero-sum game). In Chapter 7 this assumption is generalized such that the payoffs to both players are linear functions of the detection time.

(x) An (interim) inspection does not consume time. In case of the coincidence of the start of the illegal activity and the interim inspection, the illegal activity may be detected at the occasion of the next interim inspection or, with certainty, at the final PIV. In this sense the wording "... right after an interim inspection ..." is equivalent to "... at an interim inspection ...".

(xi) The game ends either at the interim inspection at which the illegal activity is detected or at the final PIV; see (iii).

Let us comment some of these assumptions. First, assumption (iii) represents an idealization: In practice even at the occasion of a PIV the illegal activity may not be detected with certainty. The idea here is that the PIV is much more accurate than any interim inspection, i.e., the detection probability at the PIV is usually higher than $1 - \beta$.

Second, assumption (iv) sounds strange at first sight, since an inspected party in general, and in particular in international treaties, voluntarily submits to inspections. However, the *raison d'être* (reason for existence) of any inspection authority must be the assumption that the inspected party has a real incentive to violate its commitments. Let us quote Grümm (1983): "This diversion hypothesis should not been understood – and in general is not understood – as an expression of distrust directed against States in general or any State in particular. Any misunderstanding might be dispelled by comparing diversion hypothesis with the philosophy of airport control. In order to be effective, airport control has to assume a priori and without any suspicion against a particular passenger that each handbag might contain prohibited goods". A different view of this problem is given by the famous saying *Trust but verify* which is attributed to V. I. Lenin. Finally, the less emotional view of the scientific modeller will be discussed in detail at the end of Section 7.3.

Third, assumption (v) typically describes Attribute Sampling schemes; see Thyregod (1988). They usually occur when random sampling schemes are used, where items are counted, and where errors arise only when falsified or wrong items are not contained in the sample. As indicated, inspection models which take into account errors of the first kind, i.e., false alarms, are not considered here since so far practitioners did not ask for them. They will, however, be taken into account in Parts II and III.

Fourth, assumption (vi) deserves some justification: Is it appropriate to assume that the Operator *knows* the number k of interim inspections? We distinguish two cases. First, if the "agreed rules" or "formal agreements", see p. 15, directly refer to a single inspected object, and if they specify the number k of interim inspections, then k is known to the Operator of the inspected object. Second, if the "agreed rules", "formal agreements" or the "international treaty" refer to a State, then the State knows k. In contrast, the Operator of a single inspected object can find out the number k, e.g., via observations or asking colleagues about the number of interim inspections in the remaining facilities of the State. Although, even if he knows k, he does not know how k is distributed to the single facilities, i.e., he knows only that not more than k interim inspections are performed in his facility. That means in this second case, that the Operator is faced with a random number of interim inspections for his facility; see p. 146 and Chapter 11. In this case one has to consider a larger game where the choice of the single facility and the number of interim inspections in that facility represent just a part of a pure strategy of the Inspectorate. As a result, the latter number may be randomized.

Note that in the inspection games between Customs and Smuggler in Chapter 17 it is also assumed that the number of controls is known to the Smuggler. In these conflict situations, however, there exists no agreed rules, formal agreements or international treaties the Smuggler has to adhere. Nevertheless, since Customs has to obey rules given by its State, one may justify this assumption by the Smugglers' long term observation of Customs' activities.

Also, in assumption (vi) the number k is implicitly assumed to be a deterministically fixed integer $1, 2, \ldots$. The possibility that an *expected*, eventually non-integer number of inspections for *one* facility is fixed and known to the Operator is addressed only in Chapter 24, where a Se-No critical time inspection game with an expected number of inspections in *one* facility is analysed.

There are two reasons why in this monograph – except of Chapter 24 – only inspection games with a fixed number of interim inspections resp. controls are considered. First, these inspection games were in the focus of research interests from the very beginning, when models with only a few parameters and assumptions and unique solutions – fundamental models – were asked for and analysed. Second, practitioners, not familiar with game theory, got only interested, if at all, in this type of inspection games. Only recently expected numbers of inspections for single facilities gained the attention of representatives of responsible organizations. For this reason, Chapter 24 has been added to this monograph, last but not least in order to give an example for further research directions.

Finally, calling the behaviour of a player *non-sequential* if the player only decides at the beginning of the reference time interval, and *sequential* if the player also decides during the reference time interval, assumption (vii) implies four variants: No-No, No-Se, Se-No and Se-Se as indicated in Table 2.1.

It should be mentioned that for $k = 1$ interim inspection and any number N of possible time points, the No-No and No-Se inspection games on one hand and the Se-No and Se-Se inspection games on the other lead to the same game theoretical results due to the fact that for $k = 1$ there is no difference between sequential and non-sequential behaviour of the Inspectorate.

Table 2.1 Four variants of the general inspection game and their abbreviations.

Operator \ Inspectorate	Non-sequential	Sequential
Non-sequential	No-No	No-Se
Sequential	Se-No	Se-Se

Of course, it can not be decided without more information about the real situation to be described which of the four variants is the appropriate one. Nevertheless, it will turn out, that for those cases which have been analysed in the literature for a given behaviour of the Operator, No or Se, the behaviour of the Inspectorate does not influence the optimal expected detection time.

Let us repeat: Assumptions (i) to (xi) will be used in all chapters of this part. Only those assumptions which for some reason or other deserve special attention will be mentioned again, but of course all assumptions together represent the basis of the game theoretical models to be developed and analysed subsequently.

Chapter 3

No-No inspection game

We analyse in this chapter the first of the four variants presented in Table 2.1, where assumptions (v) and (vii) of Chapter 2 are specified as follows:

(v') During an interim inspection the Inspectorate does not commit an error of the second kind, i.e., the illegal activity, see assumption (iv), is detected with certainty during the next interim inspection or with certainty during the final PIV; see assumption (iii).

(vii') The Operator decides at the beginning of the reference time interval, i.e., at time point 0, at which of the possible time points $0, 1, \ldots, N$ he will start the illegal activity.

The Inspectorate decides at the beginning of the reference time interval at which of the possible time points $1, \ldots, N$ it will perform its k interim inspection(s).

The remaining assumptions of Chapter 2 hold throughout this chapter. Note that in Section 6.1, the No-No inspection game with uncertain detection of an illegal activity at an interim inspection, i.e., $\beta \geq 0$, is considered.

In Section 3.1 we analyse the case of any number N of possible time points for $k = 1$ interim inspection for which a game theoretical solution exists; see Krieger (2007) and Krieger (2008). Also in Section 3.1 properties of the optimal payoff and its relations to system quantities, see Table 1.1, as well as the asymptotic behaviour of the optimal strategies and the optimal payoff are examined. Section 3.2 presents game theoretical solutions for the case of $k = 2$ interim inspections and those number N of possible time points which have been treated analytically or numerically. Again, the relation between the optimal payoff and system quantities as well as the asymptotic behaviour of the optimal payoff are investigated.

3.1 Any number of inspection opportunities and one interim inspection

Since this is a first place in this monograph where game theoretical models for inspection problems are developed and analysed, we present some general concepts and solution techniques which will be used in subsequent chapters. To those readers who are not familiar with normal form games we recommend to study Section 19.1 first.

Let for the purpose of illustration N be equal to 3, i.e., we consider 3 possible time points and of course, according to assumption (iii) of Chapter 2, the two inspection time points 0 and 4 for the initial and final PIV. This means that there remain the three time points 1, 2 and 3 for scheduling the interim inspection. Therefore, the set $J_{3,1}$ of pure strategies of the Inspectorate is given by[1]

$$J_{3,1} := \{1, 2, 3\}, \tag{3.1}$$

i.e., the set of time points at which it can perform its interim inspection. Note that in order to be consistent with subsequent sections and chapters in which inspection games with more than one interim inspection are considered, we subscript the Inspectorate's set of pure strategies $J_{3,1}$, where the first index indicates the number of possible time points and the second one the number of interim inspection(s).

The Operator can start the illegal activity potentially at any time point of the reference time interval. However, he will start the illegal activity only at the time points $0, 1, 2$ and 3, since otherwise the time elapsed between the start of the illegal activity and its detection would become shorter. Therefore, the set I_3 of pure strategies of the Operator is given by

$$I_3 := \{0, 1, 2, 3\}, \tag{3.2}$$

i.e., the set of time points at which he can start the illegal activity. Note that this notation – in contrast to the Inspectorate's one in (3.1) – does only depend on the number of possible time points and not on the number of interim inspections, because the Operator's pure strategies in the No-No inspection game (and also in the No-Se inspection game; see Section 4.1) are not influenced by the number of interim inspections.

What do the players gain in case the Operator starts the illegal activity at time point i and the Inspectorate inspects at time point j?[2] According to assumption (ix) of Chapter 2, the payoff $Op_{3,1}(i,j)$ to the Operator is the time elapsed between the start and the detection of the illegal activity: It is $j-i$ for $j > i$ and $4-i$ for $j \leq i$, where it should be kept in mind that for instance in case $i = j$ the illegal activity is only detected at the next occasion; see assumption (x) of Chapter 2. If this assumption is violated, i.e., if the illegal activity is detected immediately, then $Op_{3,1}(i,i) = 0$ for all $i = 1, 2, 3$, and a different No-No inspection game is considered; see p. 25. Furthermore, a second illegal activity is not allowed to be started (for instance in case of $i = 0$ and $j = 1$ another one could be started at, say, $i = 2$) according to assumption (iv). Because of assumption (ix) the payoff to the Inspectorate is $In_{3,1}(i,j) := -Op_{3,1}(i,j)$, i.e., we are dealing with a zero-sum game.

This conflict situation is depicted in Table 3.1. In the first column the pure strategies of the Operator are given, namely starting the illegal activity at time point 0, 1, 2 or 3. In the first row the pure strategies of the Inspectorate are shown, i.e., the time at which it will perform its interim inspection. An entry in this *payoff matrix* means that if the Operator starts the illegal activity at time point i and the Inspectorate performs its interim inspection at time point j, then the entry in the matrix indicates the detection time. Note that the pure strategy "starting the illegal activity at time point 3" is a strictly dominated strategy, and can thus be eliminated; see the comment on p. 24.

[1] In order to discern definitions from equations, we use throughout this monograph defining double points, i.e., $A := B$ means that A is defined by B.

[2] Although in this part the interim inspection time points are denoted by $j_k, \ldots, j_1$, we write in this section j instead of j_1 because only the case of $k = 1$ interim inspection is treated.

Table 3.1 Payoff matrix of the No-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection.

	1	2	3
0	1	2	3
1	3	1	2
2	2	2	1
3	1	1	1

We first want to answer the question, if there is a pure strategy combination which leads to a stable situation of this game, i.e., a pair of pure strategies from which no player has an incentive to deviate. The answer is no. Formally, we are looking for a pair (i^*, j^*) of optimal pure strategies, i.e., a pair of pure strategies which fulfils the so-called saddle point property for pure strategies, see (19.10),

$$Op_{3,1}(i, j^*) \leq Op_{3,1}(i^*, j^*) \leq Op_{3,1}(i^*, j) \tag{3.3}$$

for any $i \in I_3$ and any $j \in J_{3,1}$. The left hand inequality specifies the Operator's goal of maximizing his payoff, i.e., the time elapsed between the start and the detection of the illegal activity, while the right hand inequality specifies the Inspectorate's goal of minimizing that time. Suppose there exists a pair (i^*, j^*) of optimal pure strategies. Then we would obtain, using Table 3.1,

$$Op_{3,1}(i^*, j^*) = \max_{i=0,1,2,3} Op_{3,1}(i, j^*) = \max\{j^*, 4 - j^*\} \geq 2$$

and

$$Op_{3,1}(i^*, j^*) = \min_{j=1,2,3} Op_{3,1}(i^*, j) = 1\,,$$

i.e., (3.3) cannot be fulfilled. This argumentation shows, that in this game no stable situation, i.e., optimal pure strategies, exists. Therefore, we have to introduce the concept of mixed strategies. Let p_i, $i = 0, \ldots, 3$, denote the probability that the illegal activity is started at time point i and q_j, $j = 1, 2, 3$, denote the probability to perform the interim inspection at time point j. A mixed strategy of a player is a probability distribution over his set of pure strategies. Thus, the Operator's set of mixed strategies is given by

$$P_3 := \left\{ \mathbf{p} := (p_0, p_1, p_2, p_3)^T \in [0,1]^4 : \sum_{i=0}^{3} p_i = 1 \right\} \tag{3.4}$$

and that for the Inspectorate by

$$Q_{3,1} := \left\{ \mathbf{q} := (q_1, q_2, q_3)^T \in [0,1]^3 : \sum_{j=1}^{3} q_j = 1 \right\}. \tag{3.5}$$

If the players decide to play the mixed strategy combination $(\mathbf{p}, \mathbf{q})$, then the Operator's (expected) payoff, i.e., the expected detection time, is, using Table 3.1, given by

$$\begin{aligned} Op_{3,1}(\mathbf{p}, \mathbf{q}) := p_0\,(q_1 + 2\,q_2 + 3\,q_3) + p_1\,(3\,q_1 + q_2 + 2\,q_3) \\ + p_2\,(2\,q_1 + 2\,q_2 + q_3) + p_3\,, \end{aligned} \tag{3.6}$$

see (19.3) in the Annex. According to assumption (ix) of Chapter 2, the Inspectorate's payoff is $In_{3,1}(\mathbf{p}, \mathbf{q}) = -Op_{3,1}(\mathbf{p}, \mathbf{q})$.

In analogy to the saddle point criterion (3.3) for pure strategies, the mixed strategy combination $(\mathbf{p}^*, \mathbf{q}^*)$ is a pair of optimal strategies, if and only if $(\mathbf{p}^*, \mathbf{q}^*)$ fulfils the a saddle point criterion

$$Op_{3,1}(\mathbf{p}, \mathbf{q}^*) \le Op_{3,1}(\mathbf{p}^*, \mathbf{q}^*) \le Op_{3,1}(\mathbf{p}^*, \mathbf{q}) \tag{3.7}$$

for any $\mathbf{p} \in P_3$ and any $\mathbf{q} \in Q_{3,1}$; see also (19.10). It is well-known, that any matrix game possesses a saddle point in mixed strategies, i.e., the existence of an optimal mixed strategy for each player can be ensured. A pair of optimal strategies together with the optimal payoff constitutes the game theoretical solution of the game.

There may, however, exist different optimal strategies in a matrix game. Fortunately, these optimal strategies are interchangeable; see p. 399. This property is used for arguing that presenting one pair of optimal strategies is sufficient, i.e., a selection of optimal strategies is not necessary. Therefore, we are in general only interested in finding one instead of all optimal strategies for each player. We will show in Section 4.2, however, that a selection of optimal strategies based on practical considerations may be useful.

It has been mentioned on p. 22 that the pure strategy "starting the illegal activity at time point 3", i.e., $\mathbf{p}_1 := (0,0,0,1)^T$, is a strictly dominated strategy. Indeed, using (3.6) and the mixed strategy $\mathbf{p}_2 := (0, 1/2, 1/2, 0)^T$, we get for any $\mathbf{q} \in Q_{3,1}$

$$\begin{aligned} Op_{3,1}(\mathbf{p}_2, \mathbf{q}) &= \frac{1}{2}\Big(5\,q_1 + 3\,(q_2 + q_3)\Big) = \frac{1}{2}\Big(5\,q_1 + 3\,(1 - q_1)\Big) = \frac{1}{2}\Big(2\,q_1 + 3\Big) \\ &\ge \frac{3}{2} > 1 = Op_{3,1}(\mathbf{p}_1, \mathbf{q})\,, \end{aligned}$$

i.e., $\mathbf{p}_1$ is a strictly dominated strategy. Note that strictly dominated pure strategies are never used in an optimal strategy, i.e., $p_3^* = 0$ for any optimal strategy $\mathbf{p}^*$; see Myerson (1991) or Morris (1994).

The game theoretical solution of this inspection game, see Krieger (2007) and Krieger (2008), is presented in

Lemma 3.1. *Given the No-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection. The sets of mixed strategies are given by (3.4) and (3.5), and the payoff to the Operator by (3.6).*

Then an optimal strategy of the Operator is given by

$$\mathbf{p}^* = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}, 0\right)^T\,,$$

and an optimal strategy of the Inspectorate by

$$\mathbf{q}^* = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)^T\,. \tag{3.8}$$

The optimal payoff to the Operator is

$$Op^*_{3,1} := Op_{3,1}(\mathbf{p}^*, \mathbf{q}^*) = \frac{11}{6}\,.$$

Proof. In order to prove that $(\mathbf{p}^*, \mathbf{q}^*)$ constitutes a saddle point of the game, we have to show that the inequalities

$$Op_{3,1}(i, \mathbf{q}^*) \leq Op_{3,1}(\mathbf{p}^*, \mathbf{q}^*) \leq Op_{3,1}(\mathbf{p}^*, j) \tag{3.9}$$

are fulfilled for any $i \in I_3$ and any $j \in J_{3,1}$; see (19.11). Indeed, using (3.6), we get $Op_{3,1}(i, \mathbf{q}^*) = 11/6$ for all $i = 0, 1, 2$ and $Op_{3,1}(3, \mathbf{q}^*) = 1 < 11/6$, and $Op_{3,1}(\mathbf{p}^*, j) = 11/6$ for all $j = 1, 2, 3$, i.e., (3.9) is fulfilled as equality. □

In general, and there will be found many examples in this monograph, it is much easier to prove that proposed strategies indeed fulfil the saddle point criterion, than to find them. A useful techniques for the constructive determination of optimal strategies is provided by the best-response criterion or indifference principle, see Theorem 19.1: Roughly speaking the saddle point criterion is satisfied if the optimal strategy of one player is determined in such a way that the adversary is rendered indifferent as regards to his pure strategies which he plays with positive probability.

Note that in the case of $N = 3$ possible time points for $k = 1$ interim inspection common sense would propose to place the interim inspection into the middle of the reference time interval which would lead to the payoff 2. Lemma 3.1 and in particular (3.8), however, indicates that the Inspectorate can slightly do better. Also, the appealing Inspectorate's strategy $\mathbf{q} = (1/3, 1/3, 1/3)^T$ is – because $Op_{3,1}(0, \mathbf{q}) = 2 > Op^*_{3,1}$ contradicts (3.9) – not an optimal strategy although $Op_{3,1}(\mathbf{p}^*, \mathbf{q}) = 11/6 = Op^*_{3,1}$.

We mentioned on p. 22 that a violation of assumption (x) leads to a different No-No inspection game. The game theoretical solution of that game – the proof of which goes along the same lines as that of Lemma 3.1 – is given by

$$\mathbf{p}^* = \left(\frac{2}{3}, 0, \frac{1}{3}, 0\right)^T, \qquad \mathbf{q}^* = \left(\frac{2}{3}, \frac{1}{3}, 0\right)^T \qquad \text{and} \qquad Op^*_{3,1} = \frac{4}{3}\,,$$

i.e., the optimal strategies and the optimal payoff are sensitive regarding the modelling assumption. This is the reason why we put so much emphasize on modelling aspects. The fact that a slight change of a modelling assumption leads to a considerable change in the game theoretical solutions is also demonstrated in Section 15.4 and Sections 16.1 and 17.1; see p. 327.

Let us now turn to the general case of N possible time points. Let in line with the case $N = 3$

$$I_N := \{0, 1, \ldots, N\} \qquad \text{and} \qquad J_{N,1} := \{1, \ldots, N\} \tag{3.10}$$

be the sets of pure strategies of the Operator and the Inspectorate, i.e., the sets of time points where they either start the illegal activity or perform the interim inspection. If i is the time point for the start of the illegal activity, and j the time point of the interim inspection, then we obtain – according to the model assumptions of Chapter 2 as well as assumptions (v') and (vii') on p. 21 – for the detection time

$$Op_{N,1}(i, j) := \begin{cases} j - i & \text{for} \quad 0 \leq i < j < N+1 \\ N + 1 - i & \text{for} \quad 1 \leq j \leq i < N+1 \end{cases}\,. \tag{3.11}$$

Explicitly, the payoff matrix $A := (Op_{N,1}(i,j))_{i=0,\dots,N,j=1,\dots,N}$ of this inspection game is shown in Table 3.2. As already pointed out, the pure strategies of the Operator resp. the Inspectorate are depicted the first column resp. the first row of the payoff matrix and the entries are the payoffs to the Operator as given by (3.11).

Table 3.2 Payoff matrix A of the No-No inspection game with N possible time points for $k=1$ interim inspection.

	1	2	3	$\cdots$	n	$n+1$	$\cdots$	$N-1$	N
0	1	2	3	$\cdots$	n	$n+1$	$\cdots$	$N-1$	N
1	N	1	2	$\cdots$	$n-1$	n	$\cdots$	$N-2$	$N-1$
2	$N-1$	$N-1$	1	$\cdots$	$n-2$	$n-1$	$\cdots$	$N-3$	$N-2$
$\vdots$				$\ddots$			$\vdots$		
n	$N-n+1$	$N-n+1$	$N-n+1$	$\cdots$	$N-n+1$	1	$\cdots$	$N-n-1$	$N-n$
$n+1$	$N-n$	$N-n$	$N-n$	$\cdots$	$N-n$	$N-n$	$\cdots$	$N-n-2$	$N-n-1$
$\vdots$				$\vdots$			$\ddots$		
$N-1$	2	2	2	$\cdots$	2	2	$\cdots$	2	1
N	1	1	1	$\cdots$	1	1	$\cdots$	1	1

Proceeding as on p. 23, we first investigate if there exists a pair of pure strategies from which no player has an incentive to deviate, i.e., a pure strategy combination (i^*,j^*) that fulfils the saddle point condition, see (19.10),

$$Op_{N,1}(i,j^*) \le Op_{N,1}(i^*,j^*) \le Op_{N,1}(i^*,j) \tag{3.12}$$

for any $i \in I_N$ and any $j \in J_{N,1}$. With such a strategy combination we would obtain, using Table 3.2,

$$Op_{N,1}(i^*,j^*) = \max_{i=0,\dots,N} Op_{N,1}(i,j^*) = \max\{j^*, N+1-j^*\} \ge \frac{N+1}{2}$$

and

$$Op_{N,1}(i^*,j^*) = \min_{j=1,\dots,N} Op_{N,1}(i^*,j) = 1\,,$$

i.e., (3.12) cannot be fulfilled for $N \ge 2$, i.e., it exists no pair (i^*,j^*) of optimal pure strategies. Again, let p_i, $i=0,\dots,N$, denote the probability that the illegal activity is started at time point i and q_j, $j=1,\dots,N$, denote the probability to perform the interim inspection at time point j. Then the set of mixed strategies of the Operator is given by

$$P_N := \left\{ \mathbf{p} := (p_0,p_1,\dots,p_N)^T \in [0,1]^{N+1} : \sum_{i=0}^{N} p_i = 1 \right\} \tag{3.13}$$

and for the Inspectorate by

$$Q_{N,1} := \left\{ \mathbf{q} := (q_1,\dots,q_N)^T \in [0,1]^{N} : \sum_{j=1}^{N} q_j = 1 \right\}. \tag{3.14}$$

The i-th resp. the j-th pure strategy of the Operator resp. the Inspectorate corresponds to the $(i+1)$-th resp. the j-th unit vector. In order to avoid problems with the enumeration we will write – although mathematical slightly incorrect – i instead of e_{i+1} and j instead of e_j. If the players decide to play the mixed strategy combination $(\mathbf{p},\mathbf{q})$, the Operator's (expected) payoff, i.e., the expected detection time, defined on the set $P_N \times Q_{N,1}$ is, using (19.3), given by

$$Op_{N,1}(\mathbf{p},\mathbf{q}) := \mathbf{p}^T A\, \mathbf{q} = \sum_{i=0}^{N} \sum_{j=1}^{N} p_i\, q_j\, Op_{N,1}(i,j)\,. \tag{3.15}$$

According to assumption (ix) of Chapter 2, the Inspectorate's payoff is given by $In_{N,1}(\mathbf{p},\mathbf{q}) = -Op_{N,1}(\mathbf{p},\mathbf{q})$. Because the existence of optimal strategies in matrix games can be guaranteed, we are looking in analogy to (3.7) for a pair $(\mathbf{p}^*,\mathbf{q}^*) \in P_N \times Q_{N,1}$ with

$$Op_{N,1}(\mathbf{p},\mathbf{q}^*) \le Op_{N,1}(\mathbf{p}^*,\mathbf{q}^*) \le Op_{N,1}(\mathbf{p}^*,\mathbf{q})$$

for any $\mathbf{p} \in P_N$ and any $\mathbf{q} \in Q_{N,1}$, see (19.10), where $Op_{N,1}(\mathbf{p},\mathbf{q})$ is given by (3.15).

The game theoretical solution of this inspection game, see Krieger (2007) and Krieger (2008), is presented in

Theorem 3.1. *Given the No-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection. The sets of mixed strategies are given by (3.13) and (3.14), and the payoff to the Operator by (3.15) using (3.11). Define the cut-off value n^* by*[3]

$$n^* := \min \left\{ n : n \in \{1,\ldots,N\} \quad \textit{with} \quad \sum_{j=1}^{n} \frac{1}{N+1-j} \ge 1 \right\}. \tag{3.16}$$

Then an optimal strategy of the Operator is given by

$$p_i^* = \begin{cases} \dfrac{1}{N}\,(N+1-n^*) & \textit{for} \quad i = 0 \\[2ex] \dfrac{(N+1-n^*)}{(N+1-i)\,(N-i)} & \textit{for} \quad i = 1,\ldots,n^*-1 \\[2ex] 0 & \textit{for} \quad i = n^*,\ldots,N \end{cases}, \tag{3.17}$$

and an optimal strategy of the Inspectorate by

$$q_j^* = \begin{cases} \dfrac{1}{N+1-j} & \textit{for} \quad j = 1,\ldots,n^*-1 \\[2ex] 1 - \displaystyle\sum_{j=1}^{n^*-1} \dfrac{1}{N+1-j} & \textit{for} \quad j = n^* \\[2ex] 0 & \textit{for} \quad j = n^*+1,\ldots,N \end{cases}. \tag{3.18}$$

The optimal payoff to the Operator is

$$Op_{N,1}^* := Op_{N,1}(\mathbf{p}^*,\mathbf{q}^*) = \sum_{j=1}^{n^*} \frac{N+1-n^*}{N+1-j}\,. \tag{3.19}$$

---

[3] Nagell (1923) has proven – in the notation used here – that $\sum_{j=1}^{n} (N-j+1)^{-1} \notin \mathbb{N}$ for any $N > 1$ and all $n = 1,\ldots,N$, i.e., the $\ge$ sign in (3.16) can be replaced by the $>$ sign.

Proof. The proof is presented in several steps. For the sake of brevity we write $Op(\mathbf{p},\mathbf{q})$ instead of $Op_{N,1}(\mathbf{p},\mathbf{q})$.

1. We first show that for any $\mathbf{p}=(p_0,p_1\ldots,p_N)^T\in P_N$ and any $\mathbf{q}=(q_1,\ldots,q_N)^T\in Q_{N,1}$ the following recursive relations hold: For any $j\in\{1,\ldots,N-1\}$ we have

$$Op(\mathbf{p},j+1)=Op(\mathbf{p},j)-(N+1-j)\,p_j+\sum_{i=0}^{j}p_i\,, \tag{3.20}$$

and for any $i\in\{0,\ldots,N-1\}$ we get

$$Op(i+1,\mathbf{q})=Op(i,\mathbf{q})+(N-i)\,q_{i+1}-1\,. \tag{3.21}$$

This can be seen as follows: Using (3.11), we get for any $\mathbf{p}=(p_0,p_1\ldots,p_N)^T\in P_N$ and any $j\in\{1,\ldots,N\}$

$$Op(\mathbf{p},j)=\sum_{i=0}^{j-1}(j-i)\,p_i+\sum_{i=j}^{N}(N+1-i)\,p_i\,. \tag{3.22}$$

Let us fix an index $j\in\{1,\ldots,N-1\}$ and a $\mathbf{p}\in P_N$. Then we obtain from (3.22)

$$\begin{aligned}Op(\mathbf{p},j+1)&=\sum_{i=0}^{j}(j+1-i)\,p_i+\sum_{i=j+1}^{N}(N+1-i)\,p_i\\&=\sum_{i=0}^{j}(j-i)\,p_i+\sum_{i=0}^{j}p_i+\sum_{i=j}^{N}(N+1-i)\,p_i-(N+1-j)\,p_j\\&=Op(\mathbf{p},j)-(N+1-j)\,p_j+\sum_{i=0}^{j}p_i\,,\end{aligned}$$

i.e., recursive relation (3.20). For the proof of (3.21) we first get from (3.11) for any $\mathbf{q}=(q_1,\ldots,q_N)^T\in Q_{N,1}$

$$Op(i,\mathbf{q})=\begin{cases}\sum_{j=1}^{N}j\,q_j & \text{for}\quad i=0\\(N+1-i)\sum_{j=1}^{i}q_j+\sum_{j=i+1}^{N}(j-i)\,q_j & \text{for}\quad i=1,\ldots,N-1\\1 & \text{for}\quad i=N\end{cases}\,. \tag{3.23}$$

Using (3.23), we obtain

$$Op(1,\mathbf{q})=N\,q_1+\sum_{j=2}^{N}(j-1)\,q_j=Op(0,\mathbf{q})+N\,p_1-1\,,$$

and

$$Op(N,\mathbf{q})=\sum_{j=1}^{N}q_j=2\sum_{j=1}^{N-1}q_j+2\,q_N-1=Op(N-1,\mathbf{q})+q_N-1\,,$$

i.e., (3.21) for $i = 0$ and $i = N-1$. For a fixed index $i \in \{1, \ldots, N-2\}$ we get again from (3.23)

$$\begin{aligned} Op(i+1,\mathbf{q}) &= (N-i) \sum_{j=1}^{i+1} q_j + \sum_{j=i+2}^{N} (j-(i+1))\, q_j \\ &= (N+1-i) \sum_{j=1}^{i} q_j + (N-i)\, q_{i+1} - \sum_{j=1}^{i} q_j + \sum_{j=i+1}^{N} (j-i)\, q_j - \sum_{j=i+1}^{N} q_j \\ &= Op(i,\mathbf{q}) + (N-i)\, q_{i+1} - 1\,, \end{aligned}$$

i.e., (3.21) for $i \in \{1, \ldots, N-2\}$.

2. From (3.17) and (3.18) it can be directly seen that the components of $\mathbf{p}^*$ and $\mathbf{q}^*$ are greater or equal to 0 and that both vectors are correctly normalized, i.e., $\mathbf{p}^*$ and $\mathbf{q}^*$ are probability distributions over I_N resp. $J_{N,1}$.

3. Saddle point inequalities: Using (3.17), we have for a fixed index $j \in \{1, \ldots, n^*-1\}$

$$\sum_{i=0}^{j} p_i^* = (N+1-n^*) \left(\frac{1}{N} + \sum_{i=1}^{j} \left(\frac{1}{N-i} - \frac{1}{N+1-i} \right) \right) = (N+1-n^*)\, \frac{1}{N-j}$$

and hence

$$\sum_{i=0}^{j} p_i^* = \begin{cases} (N+1-n^*)\, \dfrac{1}{N-j} = (N+1-j)\, p_j^* & \text{for} \quad j = 1, \ldots, n^*-1 \\ 1 & \text{for} \quad j = n^*, \ldots, N \end{cases} \quad . \tag{3.24}$$

This leads together with recursive relation (3.20) to

$$Op(\mathbf{p}^*, 1) = Op(\mathbf{p}^*, 2) = \ldots = Op(\mathbf{p}^*, n^*) \tag{3.25}$$

and

$$Op(\mathbf{p}^*, N) > Op(\mathbf{p}^*, N-1) > \ldots > Op(\mathbf{p}^*, n^*+1) > Op(\mathbf{p}^*, n^*)\,.$$

On the other hand we obtain, using (3.18) and (3.21),

$$Op(0, \mathbf{q}^*) = Op(1, \mathbf{q}^*) = \ldots = Op(n^*-1, \mathbf{q}^*) \tag{3.26}$$

and

$$Op(N, \mathbf{q}^*) < Op(N-1, \mathbf{q}^*) < \ldots < Op(n^*-1, \mathbf{q}^*)\,. \tag{3.27}$$

Combining (3.25) and (3.26) gives us

$$\begin{aligned} Op(\mathbf{p}^*, \mathbf{q}^*) &= (\mathbf{p}^*)^T A\, \mathbf{q}^* \\ &= Op(\mathbf{p}^*, 1) = Op(\mathbf{p}^*, 2) = \ldots = Op(\mathbf{p}^*, n^*) \\ &= Op(0, \mathbf{q}^*) = Op(1, \mathbf{q}^*) = \ldots = Op(n^*-1, \mathbf{q}^*)\,. \end{aligned} \tag{3.28}$$

Now, (3.25) – (3.27) together with (3.28) imply

$$Op(i, \mathbf{q}^*) \le Op(\mathbf{p}^*, \mathbf{q}^*) \le Op(\mathbf{p}^*, j)$$

for any $i \in I_N$ and any $j \in J_{N,1}$. Making use of (19.11) we see that $(\mathbf{p}^*, \mathbf{q}^*)$ is a saddle point of the game.

4. The optimal payoff to the Operator: (3.11) and (3.28) imply

$$\begin{aligned} Op(\mathbf{p}^*, \mathbf{q}^*) = Op(n^* - 1, \mathbf{q}^*) &= (N + 2 - n^*) \sum_{j=1}^{n^*-1} q_j^* + q_{n^*}^* \\ &= (N + 1 - n^*) \sum_{j=1}^{n^*-1} q_j^* + 1 \,, \end{aligned}$$

i.e., (3.19), which completes the proof. □

Before discussing the results of Theorem 3.1 – see p. 34 – we derive in Lemma 3.2, see Krieger (2007) and Krieger (2008), lower and upper bounds of $n^*(N)$ and $Op^*_{N,1}$, and discuss in Lemma 3.3 the behaviour of the normalized cut-off value $n^*(N)/(N + 1)$ and the normalized optimal payoff $Op^*_{N,1}/(N + 1)$. Note that in order to indicate the dependence on N, we write now $n^*(N)$.

Lemma 3.2. *Given the No-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection analysed in Theorem 3.1.*

Then the following bounds hold for the cut-off value $n^*(N)$ and for the optimal expected detection time $Op^*_{N,1}$:

$$\left(1 - \frac{1}{e}\right) N < n^*(N) < \left(1 - \frac{1}{e}\right) (N + 1) + 1 \tag{3.29}$$

and

$$N + 1 - n^* < Op^*_{N,1} < N + 2 - n^* \,. \tag{3.30}$$

Proof. For any $n \in \{1, \dots, N - 1\}$ we have

$$\int_1^{n+1} \frac{1}{N + 2 - x}\, dx \le \sum_{j=1}^{n} \frac{1}{N + 1 - j} \le \int_1^{n+1} \frac{1}{N + 1 - x}\, dx$$

which is equivalent to

$$\ln\left[\frac{N + 1}{N + 1 - n}\right] \le \sum_{j=1}^{n} \frac{1}{N + 1 - j} \le \ln\left[\frac{N}{N - n}\right] . \tag{3.31}$$

From (3.16) we obtain, using the footnote on p. 27, the inequalities

$$\sum_{j=1}^{n^*} \frac{1}{N + 1 - j} > 1 \qquad \text{and} \qquad \sum_{j=1}^{n^*-1} \frac{1}{N + 1 - j} < 1 \,. \tag{3.32}$$

Thus, (3.31) yields

$$1 < \ln\left[\frac{N}{N-n^*}\right] \qquad \text{and} \qquad \ln\left[\frac{N+1}{N+2-n^*}\right] < 1\,.$$

Combining both inequalities we get (3.29). To prove (3.30), note that (3.32) is equivalent to

$$1 < \sum_{j=1}^{n^*} \frac{1}{N+1-j} < 1 + \frac{1}{N+1-n^*}\,.$$

Multiplying these two inequalities by $N+1-n^*$ and making use of (3.19) leads immediately to (3.30). □

In order to get an idea of the behaviour of the cut-off value $n^*(N)$ and the optimal payoff $Op^*_{N,1}$ to the Operator, we present in Table 3.3 these quantities together with the corresponding normalized quantities $n^*(N)/(N+1)$ and $Op^*_{N,1}/(N+1)$.

Table 3.3 Behaviour of the cut-off value $n^*(N)$, the optimal payoff $Op^*_{N,1}$ to the Operator, and its normalized values relative to $N+1$ (rounded).

N	$n^*(N)$	$n^*(N)/(N+1)$	$Op^*_{N,1}$	$Op^*_{N,1}/(N+1)$
2	2	0.666667	1.5	0.5
3	3	0.75	1.83333	0.458333
4	3	0.6	2.16667	0.433333
5	4	0.666667	2.56667	0.427778
6	5	0.714286	2.9	0.414286
7	5	0.625	3.27857	0.409821
8	6	0.666667	3.65357	0.405952
10	7	0.636364	4.38254	0.398413
12	8	0.615385	5.09939	0.392261
13	9	0.642857	5.484	0.391714
14	10	0.666667	5.84114	0.38941
20	13	0.619048	8.03906	0.382812
30	20	0.645161	11.7262	0.378265
40	26	0.634146	15.4047	0.375725
100	64	0.633663	37.4743	0.371032

It can be seen in Table 3.3, that the normalized cut-off value $n^*(N)/(N+1)$ is neither an increasing nor a decreasing function of N. In the Lemma 3.3, see Krieger (2007) and Krieger (2008), we show, that $n^*(N)$ and $Op^*_{N,1}$ are increasing functions of N, while $Op^*_{N,1}/(N+1)$ is a decreasing function of N.

Lemma 3.3. *Given the No-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection analysed in Theorem 3.1.*

Then we obtain for $n^*(N)$, $Op^*_{N,1}$ and $Op^*_{N,1}/(N+1)$:

$$n^*(N) \le n^*(N+1) \le n^*(N) + 1 \tag{3.33}$$

as well as

$$Op^*_{N,1} < Op^*_{N+1,1} \qquad \text{and} \qquad \frac{1}{N+1}\, Op^*_{N,1} > \frac{1}{N+2}\, Op^*_{N+1,1}\,. \tag{3.34}$$

Proof. According to the four asserted inequalities we proceed in four steps.

1. The inequality $n^*(N) \le n^*(N+1)$ follows immediately from the definition (3.16) of $n^*(N)$.
2. Next we consider the inequality $n^*(N+1) \le n^*(N) + 1$. We obtain from (3.16)

$$\sum_{j=1}^{n^*(N)} \frac{1}{N+1-j} \ge 1$$

and therewith

$$\sum_{j=1}^{n^*(N)} \frac{1}{(N+1)+1-j} \ge 1 + \frac{1}{N+1} - \frac{1}{N+1-n^*(N)}\,. \tag{3.35}$$

Let us suppose that there exists a natural number N with $n^*(N+1) > n^*(N)+1$. This leads, using (3.16) applied to $n^*(N+1)$ and (3.35), to

$$1 > \sum_{j=1}^{n^*(N+1)-1} \frac{1}{(N+1)+1-j} = \sum_{j=1}^{n^*(N)} \frac{1}{(N+1)+1-j} + \sum_{j=n^*(N)+1}^{n^*(N+1)-1} \frac{1}{(N+1)+1-j}$$
$$\ge 1 + \frac{1}{N+1} - \frac{1}{N+1-n^*(N)} + \sum_{j=n^*(N)+1}^{n^*(N+1)-1} \frac{1}{(N+1)+1-j}\,,$$

which implies

$$\frac{1}{N+1-n^*(N)} - \frac{1}{N+1} > \sum_{j=n^*(N)+1}^{n^*(N+1)-1} \frac{1}{(N+1)+1-j}\,.$$

In case of $n^*(N)+1 = n^*(N+1)-1$ resp. $n^*(N)+1 < n^*(N+1)-1$ we obtain the inequalities

$$-\frac{1}{N+1} > 0 \quad \text{resp.} \quad -\frac{1}{N+1} > \sum_{j=n^*(N)+2}^{n^*(N+1)-1} \frac{1}{(N+1)+1-j}\,,$$

which both cannot be fulfilled. So we conclude that $n^*(N+1) \le n^*(N)+1$ for all $N \ge 2$.

3. In order to prove $Op^*_{N+1,1} - Op^*_{N,1} > 0$ we consider – because of (3.33) – the two cases $n^*(N+1) = n^*(N) =: n^*$ and $n^*(N+1) = n^*(N) + 1 = n^* + 1$ separately. In the first case we have from (3.19)

$$Op^*_{N+1,1} - Op^*_{N,1} = \sum_{j=1}^{n^*} \left(\frac{N+2-n^*}{N+2-j} - \frac{N+1-n^*}{N+1-j} \right) .$$

Now it can be seen with the help of simple manipulations that the terms in the sum are greater than zero for $j = 1, \dots, n^* - 1$ and zero for $j = n^*$, thus we have $Op^*_{N+1,1} - Op^*_{N,1} > 0$.

In the second case, (3.19) implies

$$\begin{aligned} Op^*_{N+1,1} - Op^*_{N,1} &= (N+1+1-(n^*+1)) \sum_{j=1}^{n^*+1} \frac{1}{N+1-(j-1)} - Op^*_{N,1} \\ &= (N+1-n^*) \sum_{j=0}^{n^*} \frac{1}{N+1-j} - Op^*_{N,1} \\ &= (N+1-n^*) \left(\sum_{j=1}^{n^*} \frac{1}{N+1-j} + \frac{1}{N+1} \right) - Op^*_{N,1} \\ &= \frac{N+1-n^*}{N+1} > 0 \,. \end{aligned}$$

4. We consider the difference $(N+2)\, Op^*_{N,1} - (N+1)\, Op^*_{N+1,1}$ but only present the results which are obtained after some lengthy calculations. Again we distinguish the two mentioned cases and apply (3.19) and (3.30). For the first case we obtain

$$\begin{aligned} &(N+2)\, Op^*_{N,1} - (N+1)\, Op^*_{N+1,1} \\ &= (N+2)\, Op^*_{N,1} - (N+1)\,(N+1+1-n^*) \sum_{j=1}^{n^*} \frac{1}{N+1-(j-1)} \\ &= \frac{n^*}{N+1-n^*} \left(-Op^*_{N,1} + N + 2 - n^* \right) > 0 \,, \end{aligned}$$

and for the second case

$$\begin{aligned} &(N+2)\, Op^*_{N,1} - (N+1)\, Op^*_{N+1,1} \\ &= (N+2)\, Op^*_{N,1} - (N+1)\,(N+1-n^*) \sum_{j=1}^{n^*+1} \frac{1}{N+1-(j-1)} \\ &= Op^*_{N,1} - (N+1-n^*) > 0 \,, \end{aligned}$$

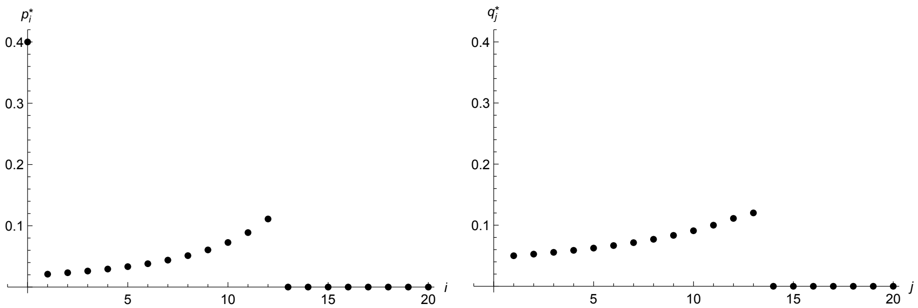
which completes the proof. □

Let us comment the results of Lemma 3.3: Whereas the left hand inequality for n^* in (3.33) is intuitive, the right hand one is not; it is, however, illustrated very well by Table 3.3. Also, the left hand inequality for $Op^*_{N,1}$ in (3.34) was to be expected, but not the right hand one, and again, it is illustrated convincingly by Table 3.3. More than that, this inequality will be used subsequently.

Let us now discuss the results of Theorem 3.1 and Lemma 3.2: First, the optimal strategies $\mathbf{p}^*$ and $\mathbf{q}^*$ have an interesting property: We see that the pure strategies $n^*, \ldots, N$ for the Operator, and $n^*+1, \ldots, N$ for the Inspectorate are cut-off and are never played. That means that the Operator will never perform an illegal activity after time point n^* and the Inspectorate will never inspect after time point n^*+1. This makes sense since detection is guaranteed to occur at the end of the reference time interval and the Operator will not wish to wait too long before starting the illegal activity. This property is unique to discrete time and continuous time No-No inspection games; see also Section 3.2 and Chapter 9, and Tables 13.1 and 13.2. (3.29) shows another interesting property of n^*: Because of $1 - 1/e \approx 0.63$, for large N only about 63% of the N possible time points are eventually used for the interim inspection. Note that the results in Theorem 3.1 can be extended to the situation in which the illegal activity is only detected with probability $1-\beta$ at the interim inspection; see Theorem 6.1.

Second, in Figure 3.1 the optimal strategies of both players are depicted for the case of $N = 20$ possible time points. Note that $n^*(20) = 13$; see also Table 3.3.

Figure 3.1 Optimal strategies $\mathbf{p}^*$ and $\mathbf{q}^*$ for $N = 20$ possible time points.



For any $N \geq 3$ we obtain from (3.17)

$$p_0^* > p_1^* \quad \text{and} \quad p_1^* < p_2^* < \ldots < p_{n^*-1}^* .$$

The relation between p_0^* and $p_{n^*-1}^*$, however, is not so obvious. A lengthy calculation, using (3.29), shows that $p_0^* > p_{n^*-1}^*$ for $N \geq 10$. An examination of the cases $N = 6, \ldots, 9$, however, shows that we get for $N = 6$ the equality $p_0^* = p_{n^*-1}^* = 1/3$, and for $N \geq 7$ the result $p_0^* > p_{n^*-1}^*$.

Furthermore, we get, using (3.18),

$$q_1^* < q_2^* < \ldots < q_{n^*-1}^* .$$

In general nothing can be said about the ratio between $q_{n^*}^*$ and q_j^*, $j = 1, \ldots, n^*-1$: In case of $N = 4$ possible time points we have $n^*(4) = 3$ and

$$\mathbf{q}^* = (q_1^*, q_2^*, q_3^*, q_4^*)^T = \left(\frac{1}{4}, \frac{1}{3}, \frac{5}{12}, 0\right)^T , \qquad \text{i.e.,} \qquad q_1^* < q_2^* < q_3^* ,$$

while in case of $N = 5$ possible time points we have $n^*(5) = 4$ and

$$\mathbf{q}^* = (q_1^*, q_2^*, q_3^*, q_4^*, q_5^*)^T = \left(\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{13}{60}, 0\right)^T , \quad \text{i.e.,} \quad q_1^* < q_4^* < q_2^* < q_3^* .$$

For large N, see (3.30), the optimal expected detection time is the time between $N+1$ and n^*, in other words, the time remaining after the Inspectorate does not inspect any more, except the final PIV. We will come back to this point on p. 41 and also in Chapter 9.

Third, it has been mentioned after Lemma 3.1 that for the case $N = 3$ possible time points the equal distribution is not an optimal strategy of the Inspectorate. This holds here even more: Were $\mathbf{q} = (1/N, \ldots, 1/N)^T$ an optimal strategy, then according to (19.11) the inequality $(N+1)/2 = Op_{N,1}(0, \mathbf{q}) \leq Op^*_{N,1}$ needed to hold. With the right inequality of (3.34) we get

$$\frac{1}{N+1}\, Op^*_{N,1} < \frac{1}{N}\, Op^*_{N-1,1} < \frac{1}{N-1}\, Op^*_{N-2,1} < \ldots < \frac{1}{3}\, Op^*_{2,1} = \frac{1}{2}\,,$$

see also Table 3.3, which is a contradiction to $(N+1)/2 = Op_{N,1}(0, \mathbf{q}) \leq Op^*_{N,1}$.

Finally, as mentioned in Section 1.4, the interpretation of the pure strategies makes it possible to determine the optimal expected time point for the start of the illegal activity $\mathbb{E}_{\mathbf{p}^*}(S)$ and the optimal expected interim inspection time point $\mathbb{E}_{\mathbf{q}^*}(T_1)$. Using (3.17) we get

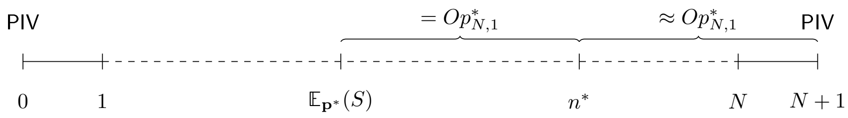
$$\begin{aligned}
\mathbb{E}_{\mathbf{p}^*}(S) &:= \sum_{i=0}^{N} i\, p_i^* = \sum_{i=1}^{n^*-1} i\, p_i^* = (N+1-n^*) \sum_{i=1}^{n^*-1} \frac{i}{(N+1-i)\,(N-i)} \\
&= (N+1-n^*) \left(-1 + \frac{n^*}{N+1-n^*} - \sum_{i=1}^{n^*-1} \frac{1}{N+1-i} \right), \qquad (3.36)
\end{aligned}$$

or, using (3.19),

$$\mathbb{E}_{\mathbf{p}^*}(S) = n^* - Op^*_{N,1}\,. \qquad (3.37)$$

Figure 3.2 illustrates the relation between (3.30) and (3.37).

Figure 3.2 Illustration of (3.30) and (3.37).



Note that a similar relation like (3.37) will be shown for the continuous time No-No inspection game; see (9.15). For $k \geq 2$ interim inspections an expression similar to (3.37) does not exist for the discrete time No-No inspection game, see the discussion in Section 3.2, and can only be conjectured for the continuous time No-No inspection game; see (9.37). Let us note that all discrete and continuous time Se-No inspection games discussed in Sections 4.2 and Chapter 10 lead to simple expressions for the expected value of S; see also Table 13.2.

Also, using (3.11) (only the first line applies for $i = 0$) and (3.26), we get

$$\mathbb{E}_{\mathbf{q}^*}(T_1) := \sum_{j=1}^{N} j\, q_j^* = Op_{N,1}(0, \mathbf{q}^*) = Op^*_{N,1} \qquad (3.38)$$

and, using (3.37),

$$\mathbb{E}_{\mathbf{p}^*}(S) + \mathbb{E}_{\mathbf{q}^*}(T_1) = n^* \,. \tag{3.39}$$

Relation (3.38) establishes a close relation between the optimal expected interim inspection time point and the optimal payoff. This relation holds for all No-No inspection games in Parts I and II. Note that if more than one interim inspection is performed, then the first interim inspection time point T_k seems – in analogy to (3.38) – to fulfil the relation $\mathbb{E}_{\mathbf{q}^*}(T_k) = Op^*_{N,k}$ in the No-No inspection game; see p. 47. Surely this relation is true in the context of the discrete and continuous time Se-No inspection games; see Table 13.2.

Finally, let us mention that the optimal expected time point for the start of the illegal activity is smaller than the optimal expected interim inspection time point. This can be seen with (3.37) and (3.38): $\mathbb{E}_{\mathbf{p}^*}(S) < \mathbb{E}_{\mathbf{q}^*}(T_1)$ is equivalent to $n^* < 2\,Op^*_{N,1}$, which is – according to Table 3.3 and the results in Lemma 3.2 – fulfilled for all $N = 2, 3, \ldots$.

Up to now the zero-sum game with payoff matrix $A = (Op_{N,1}(i,j))_{i=0,\ldots,N,j=1,\ldots,N}$, see p. 26, is analysed. If N increases, the reference time interval is getting longer and longer. However, from a practical point of view the reference time interval has a fixed length, e.g., one year. For that reason we consider the zero-sum game with the same pure strategy sets I_N and $J_{N,1}$, see (3.10), but now with the payoff matrix $A/(N+1)$. The start of the illegal activity resp. the interim inspection take place at the time points $0, 1/(N+1), \ldots, N/(N+1)$ resp. $1/(N+1), \ldots, N/(N+1)$. Because we have only multiplied the payoffs with a positive constant, this game has the same saddle point(s) like the original game; see Karlin (1959a). Let $\tilde{n}^*(N)$ resp. $\widetilde{Op}^*_{N,1}$ be the cut-off value resp. the optimal payoff to the Operator for the zero-sum game with the payoff matrix $A/(N+1)$. Then

$$\tilde{n}^*(N) := \frac{1}{N+1}\, n^*(N) \qquad \text{and} \qquad \widetilde{Op}^*_{N,1} := \frac{1}{N+1}\, Op^*_{N,1} \,. \tag{3.40}$$

Note that in order to indicate the dependence on N, we write in the remainder of this section $p^*_i(N)$ instead of p^*_i, and $q^*_j(N)$ instead of q^*_j.

We now investigate the asymptotic behaviour of $\tilde{n}(N)$, $\widetilde{Op}^*_{N,1}$ and the saddle point strategies for the zero-sum game with payoff matrix $A/(N+1)$. Let $s \in [0,1]$ be given. Then there exists a number $\ell(s,N) \in \{0, \ldots, N+1\}$ and $\delta(s,N) \in [0, 1/(N+1))$ with $s = \ell(s,N)/(N+1) + \delta(s,N)$. We define

$$P^*_N(s) := \sum_{i=0}^{\ell(s,N)} p^*_i(N) \,. \tag{3.41}$$

The cumulative distribution function $P^*_N(s)$ of $\mathbf{p}^*$ is the probability that in the game with payoff matrix $A/(N+1)$ the start of the illegal activity is performed at time point s or earlier. The cumulative distribution function $Q^*_N(t)$ of $\mathbf{q}^*$ can be defined in a similar way: For a given $t \in [0,1]$ there exists a number $\ell(t,N) \in \{0, \ldots, N+1\}$ and $\delta(t,N) \in [0, 1/(N+1))$ with $t = \ell(t,N)/(N+1) + \delta(t,N)$. We define

$$Q^*_N(t) := \begin{cases} 0 & \text{for} \quad 0 \le t < \dfrac{1}{N+1} \\ \displaystyle\sum_{j=1}^{\ell(t,N)} q^*_j(N) & \text{for} \quad \dfrac{1}{N+1} \le t \le 1 \end{cases} \,. \tag{3.42}$$

$Q_N^*(t)$ is the probability that in the game with payoff matrix $A/(N+1)$ the interim inspection is performed at time point t or earlier.

The next Theorem, see Krieger (2007) and Krieger (2008), deals with the behaviour of the functions $\tilde{n}^*(N)$, $\widetilde{Op}^*_{N,1}$, $P_N^*(s)$ and $Q_N^*(t)$ as given by (3.40) – (3.42).

Theorem 3.2. *Given the No-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection, and with the payoff matrix $A/(N+1)$, where the matrix A is given in Table 3.2.*

Then we obtain for the cut-off value $\tilde{n}^*(N)$ and the optimal payoff $\widetilde{Op}^*_{N,1}$ to the Operator the following asymptotic behaviour

$$\lim_{N\to\infty} \tilde{n}^*(N) = \lim_{N\to\infty} \frac{1}{N+1}\, n^*(N) = 1 - \frac{1}{e} \approx 0.632121 \tag{3.43}$$

and

$$\lim_{N\to\infty} \widetilde{Op}^*_{N,1} = \lim_{N\to\infty} \frac{1}{N+1}\, Op^*_{N,1} = \frac{1}{e} \approx 0.367879\,.$$

Furthermore, $P_N^*(s)$ and $Q_N^*(t)$ are for any $s, t \in [0,1]$ pointwise convergence with the limits

$$P^*(s) := \lim_{N\to\infty} P_N^*(s) = \begin{cases} \dfrac{1}{e}\,\dfrac{1}{1-s} & s \in \left[0, 1-\dfrac{1}{e}\right) \\ 1 & s \in \left[1-\dfrac{1}{e}, 1\right] \end{cases} \tag{3.44}$$

and

$$Q^*(t) := \lim_{N\to\infty} Q_N^*(t) = \begin{cases} (-1)\ln[1-t] & t \in \left[0, 1-\dfrac{1}{e}\right) \\ 1 & t \in \left[1-\dfrac{1}{e}, 1\right] \end{cases}\,. \tag{3.45}$$

Proof. From (3.29) we get

$$\left(1-\frac{1}{e}\right)\frac{N}{N+1} < \frac{1}{N+1}\, n^*(N) < \left(1-\frac{1}{e}\right) + \frac{1}{N+1}\,,$$

and with the Sandwich Theorem[4]

$$\lim_{N\to\infty} \frac{1}{N+1}\, n^*(N) = 1 - \frac{1}{e}\,,$$

as required. Because (3.43) implies

$$\lim_{N\to\infty} \frac{N+1-n^*(N)}{N+1} = \lim_{N\to\infty} \frac{N+2-n^*(N)}{N+1} = \frac{1}{e}\,,$$

we obtain, using (3.30),

$$\frac{N+1-n^*(N)}{N+1} < \frac{1}{N+1}\, Op^*_{N,1} < \frac{N+2-n^*(N)}{N+1}\,,$$

[4] Let $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ be sequences of real numbers such that $a_k \leq b_k \leq c_k$ at least for all $k \geq k_0$. If $\lim_{k\to\infty} a_k = \lim_{k\to\infty} c_k =: \alpha$, then $\lim_{k\to\infty} b_k = \alpha$.

which leads – applying the Sandwich Theorem again – to

$$\lim_{N\to\infty} \frac{1}{N+1}\, Op^*_{N,1} = \frac{1}{e}\,.$$

We first show the asymptotic behaviour $P_N(s)$. Define for any $s \in [0, 1-1/e)$

$$N_1(s) := \frac{s}{\left(1-\frac{1}{e}\right) - s}\,. \tag{3.46}$$

Then we have $\ell(s,N) \le n^*(N) - 1$ for all $N \ge N_1(s)$. This can be seen as follows: Because $N_1(s) > 0$, we obtain for all $N \ge N_1(s)$

$$\left(\left(1-\frac{1}{e}\right) - s\right) N \ge s$$

and therewith, using (3.29) and $\delta(s,N) \ge 0$,

$$n^*(N) > \left(1-\frac{1}{e}\right) N \ge s\,(N+1) \ge (s - \delta(s,N))\,(N+1) = \ell(s,N)\,,$$

i.e., $\ell(s,N) \le n^*(N) - 1$ for all $N \ge N_1(s)$, as required. Therefore, we get, using (3.24) and (3.41), for all $N \ge N_1(s)$

$$P^*_N(s) = \sum_{i=0}^{\ell(s,N)} p^*_i(N) = \frac{N+1-n^*(N)}{N-\ell(s,N)}\,. \tag{3.47}$$

(3.47) illustrates the necessity of the requirement $N \ge N_1(s)$: In case of $N = 12$ and any $s \in (8/13, 1-1/e)$ we have, using Table 3.3, $\ell(s,12) = n^* = 8$. Thus, (3.47) would yield a number larger than one. Indeed, $N_1(s) \ge 37$ for any $s \in (8/13, 1-1/e)$.

Because of (3.43) and $\lim_{N\to\infty} \ell(s,N)/N = s$, (3.47) leads to

$$\lim_{N\to\infty} P^*_N(s) = \lim_{N\to\infty} \frac{N+1-n^*(N)}{N}\, \frac{1}{1-\dfrac{\ell(s,N)}{N}} = \frac{1}{e}\,\frac{1}{1-s}$$

for any $s \in [0, 1-1/e)$. In case of $s \in (1-1/e, 1]$ we define

$$N_2(s) := \frac{1}{s - \left(1-\frac{1}{e}\right)} - 1\,, \tag{3.48}$$

which implies $\ell(s,N) \ge n^*(N)$ for all $N \ge N_2(s)$: Because $N_2(s) > 0$ we get for all $N \ge N_2(s)$

$$\left(s - \left(1-\frac{1}{e}\right)\right)\,(N+1) \ge 1\,,$$

which is equivalent to

$$\left(s - \frac{1}{N+1}\right)\,(N+1) \ge \left(1-\frac{1}{e}\right)\,(N+1)\,.$$

Thus, (3.29) and $\delta(s,N) < 1/(N+1)$ leads to

$$\ell(s,N) = (s - \delta(s,N))\,(N+1) > \left(s - \frac{1}{N+1}\right)\,(N+1) \geq \left(1 - \frac{1}{e}\right)\,(N+1)$$
$$> n^*(N) - 1\,,$$

i.e., $\ell(s,N) \geq n^*(N)$ for all $N \geq N_2(s)$, as required. Thus, (3.41) yields for all $N \geq N_2(s)$

$$P_N^*(s) = \sum_{i=0}^{\ell(s,N)} p_i^*(N) = 1\,.$$

Thus, we have shown that $\lim_{N\to\infty} P_N^*(s) = P^*(s)$ holds for any $s \in [0, 1-1/e) \cup (1-1/e, 1]$. Furthermore, we have

$$1 = \lim_{\epsilon\to 0} P^*\left(1 - \frac{1}{e} - \epsilon\right) \leq P^*\left(1 - \frac{1}{e}\right) \leq \lim_{\epsilon\to 0} P^*\left(1 - \frac{1}{e} + \epsilon\right) = 1\,,$$

i.e., (3.44) is proven.

To show (3.45) let us assume $t \in (0, 1-1/e)$. If $N \geq 1/t - 1$ then the second line in (3.42) implies

$$\sum_{j=1}^{\ell(t,N)} q_j^*(N) = Q_N^*(t) \leq \sum_{j=1}^{\ell(t,N)+1} q_j^*(N)\,. \tag{3.49}$$

The right hand sum in (3.49) is less than one if and only if $\ell(t,N)+1 \leq n^*(N)-1$; see (3.18) and footnote 3 on p. 27. Modifying (3.46), we define

$$N_3(t) := \frac{t+1}{\left(1 - \frac{1}{e}\right) - t}\,,$$

and obtain $\ell(t,N)+1 \leq n^*(N)-1$ for all $N \geq N_3(t)$; the proof goes along the same lines as that below (3.46). Then (3.49) implies for all $N \geq \max(1/t-1, N_3(t))$

$$\sum_{j=1}^{\ell(t,N)} \frac{1}{N+1-j} = Q_N^*(t) \leq \sum_{j=1}^{\ell(t,N)+1} \frac{1}{N+1-j}$$

and, using (3.31),

$$\ln\left[\frac{N+1}{N-\ell(t,N)+1}\right] \leq Q_N^*(t) \leq \ln\left[\frac{N}{N-\ell(t,N)-1}\right]\,,$$

which is equivalent to

$$\ln\left[\frac{1+1/N}{1-\ell(t,N)/N+1/N}\right] \leq Q_N^*(t) \leq \ln\left[\frac{1}{1-\ell(t,N)/N-1/N}\right]\,.$$

Because $\lim_{N\to\infty} \ell(t,N)/N = t$ we get

$$\lim_{N\to\infty} \frac{1+1/N}{1-\ell(t,N)/N+1/N} = \lim_{N\to\infty} \frac{1}{1-\ell(t,N)/N-1/N} = \frac{1}{1-t}$$

and therefore – using the Sandwich Theorem – $\lim_{N\to\infty} Q_N^*(t) = Q^*(t) = (-1)\ln[1-t]$ for any $t \in (0, 1-1/e)$. Because $Q_N^*(0) = 0$ for any N, (3.45) holds also for $t = 0$.

Finally, let us consider the cases $t \in (1-1/e, 1]$. Define, in analogy to (3.48),

$$N_4(t) := \frac{2}{t - \left(1 - \dfrac{1}{e}\right) - 1}\,.$$

Then we obtain – like in the derivations after (3.48) – that $\ell(t,N) \geq n^*(N) + 1$ for all $N \geq N_4(t)$, which implies, using (3.18),

$$Q_N^*(t) = \sum_{j=1}^{\ell(t,N)} q_j^*(N) = 1\,.$$

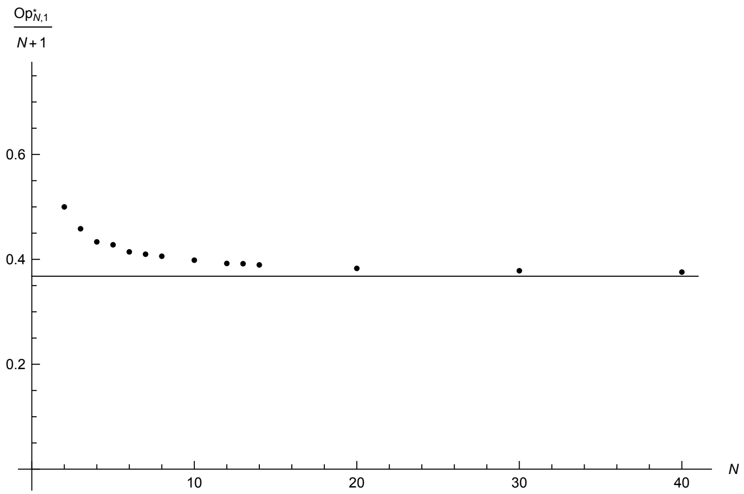
Thus, the relation $\lim_{N\to\infty} Q_N^*(t) = Q^*(t)$ for any $t \in [0, 1-1/e) \cup (1-1/e, 1]$ is shown. For the limiting case we get

$$1 = \lim_{\epsilon\to 0} Q^*\left(1 - \frac{1}{e} - \epsilon\right) \leq Q^*\left(1 - \frac{1}{e}\right) \leq \lim_{\epsilon\to 0} Q^*\left(1 - \frac{1}{e} + \epsilon\right) = 1\,,$$

which proves (3.45). □

In Figure 3.3 the dependence of $Op_{N,1}^*/(N+1)$ on N is represented using the data from Table 3.3. We see that it is indeed monotone decreasing in N, see also the right hand side of (3.34), and that it rapidly reaches the asymptotic value $1/e \approx 0.367879$.

Figure 3.3 The normalized optimal payoff $Op_{N,1}^*/(N+1)$ to the Operator as a function of N.



Result (3.43) underpins of what has already been said on p. 34: Only 63% of the possible inspections time points are eventually used for the interim inspection. Also, we observe again the property that

$$\lim_{N\to\infty} \widetilde{Op}^*_{N,1} = 1 - \lim_{N\to\infty} \tilde{n}^*(N)\,,$$

i.e., the optimal expected detection time is the time remaining after the Inspectorate does not inspect any more, except the final PIV.

Considering a large number N of possible time points for the interim inspection in a *finite* reference time interval means to approach continuous time. Models with continuous time are considered in Part II and it is no surprise that we will find corresponding results; see Lemma 9.1.

According to assumption (iv) of Chapter 2 we have considered here only the illegal game, i.e., the game, where legal behaviour of the Operator is a priori excluded. Including legal behaviour and introducing losses and gains for legal behaviour and for performing an illegal activity will lead to a formal condition for legal behaviour of the Operator. This condition then allows to determine that ratio of sanctions in case of detected and gains in case of undetected illegal behaviour of the Operator that induces the Operator to legal behaviour. We will demonstrate this in Section 7.2 for the case of $N = 3$ possible time points for $k = 1$ interim inspection.

3.2 Special numbers of inspection opportunities and two interim inspections

Looking at Theorem 3.1 it is obvious that a game theoretical solution of the No-No inspection game for any number N of possible time points for any number k of interim inspections will neither be obtained easily nor will it look simple. In fact so far it has not been possible even to find a game theoretical solution for *any* number N and $k = 2$ interim inspections. Therefore, in the following special cases of N will be considered. They show why the general case, even for $k = 2$ interim inspections, poses so many technical difficulties, but they also show how the structure for cases $k > 1$ evolve and more so – important for practical applications – how the asymptotic case $N \to \infty$ looks like.

In the following we omit the case of $N = 3$ possible time points for $k = 2$ interim inspections, because it will be considered in Chapter 6, where in addition errors of the second kind are taken into account. We formulate Lemmata for the cases $N = 4, 5, 6$ and $N = 11$, the latter one being selected for historical reasons, as will be explained then. At the end of this section we present some observations for $N > 11$ and we consider the asymptotic case $N \to \infty$.

Therefore, for any number N of possible time points the set of pure strategies I_N of the Operator is given by (3.10) and the set of pure strategies $J_{N,2}$ of the Inspectorate by

$$J_{N,2} := \left\{(j_2, j_1) \in \mathbb{N}^2 : 0 < j_2 < j_1 < N + 1\right\}\,. \tag{3.50}$$

Note that here the interim inspection time points are numbered backwards: The first interim inspection takes place at time point j_2 and the second one at time point j_1. Even though this could have been avoided in this section, it will turn out in Section 4.2 and Chapter 5, i.e., for the Se-No and the Se-Se inspection games, that backward numbering is mandatory. Thus, we apply the backward numbering for consistency reasons already here.

The payoff to the Operator, i.e., the detection time, is given by

$$Op_{N,2}(i,(j_2,j_1)) := \begin{cases} j_2 - i & \text{for} \quad 0 \le i < j_2 < j_1 < N+1 \\ j_1 - i & \text{for} \quad 1 \le j_2 \le i < j_1 < N+1 \\ N+1-i & \text{for} \quad 1 \le j_2 < j_1 \le i < N+1 \end{cases} . \tag{3.51}$$

A few comments on (3.51): The Operator can choose his time point i for the start of the illegal activity between zero and N, and the detection time depends on the position of i, i.e., before the first, between the first and the second, or after the second interim inspection. Also remember that according to assumption (x) of Chapter 2, in case that the start of the illegal activity coincides with an interim inspection it is detected only at the occasion of the next interim inspection or the PIV.

The payoff matrix of the No-No inspection game with $N = 4$ possible time points for $k = 2$ interim inspections is represented in Table 3.4.

Table 3.4 Payoff matrix of the No-No inspection game with $N = 4$ possible time points for $k = 2$ interim inspections.

	$(1,2)$	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,4)$
0	1	1	1	2	2	3
1	1	2	3	1	1	2
2	3	1	2	1	2	1
3	2	2	1	2	1	1
4	1	1	1	1	1	1

Because of the results in the last section, it is not surprising that there exist again no optimal pure strategies. Consequently we have to consider mixed strategies again: The set of mixed strategies of the Operator P_N is given by (3.13) and that for the Inspectorate, using (3.50), by

$$Q_{N,2} := \left\{ \mathbf{q} := (q_{(1,2)}, \ldots, q_{(N-1,N)})^T \in [0,1]^{\binom{N}{2}} : \sum_{j_2=1}^{N-1} \sum_{j_1=j_2+1}^{N} q_{(j_2,j_1)} = 1 \right\} . \tag{3.52}$$

The (expected) payoff to the Operator, i.e., the expected detection time, is defined in analogy to (3.15), where $\mathbf{p} \in P_N$, $\mathbf{q} \in Q_{N,2}$ and the matrix A is defined by (3.51).

The game theoretical solution of this inspection game for $N = 4, 5$ and 6, see Tschoche (2010), is presented in

Lemma 3.4. *Given the No-No inspection game with $N = 4, 5, 6$ possible time points for $k = 2$ interim inspections. The sets of mixed strategies are given by (3.13) and (3.52), and the payoff to the Operator by (3.15) using (3.51) for $N = 4, 5, 6$, respectively.*

Then an optimal strategy of the Operator and an optimal strategy of the Inspectorate together with the optimal payoff to the Operator is given in Table 3.5.

Table 3.5 Optimal strategies of the Operator (left column) and optimal strategies of the Inspectorate (middle column; the rows represent the first inspection time point j_2 and the columns the second one j_1), system quantities $\mathbb{E}_{\mathbf{p}^*}(S)$, $\mathbb{E}_{\mathbf{q}^*}(T_2)$, $\mathbb{E}_{\mathbf{q}^*}(T_1)$ and the optimal payoff $Op^*_{N,2}$ (right column) of the No-No inspection game with $N = 4, 5, 6$ possible time points (from top to bottom) for $k = 2$ interim inspections.

0	1	2	3	4
$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{3}{10}$	0

-	2	3	4
1	0	$\frac{1}{2}$	0
2	-	0	$\frac{1}{2}$
3	-	-	0

$\mathbb{E}_{\mathbf{p}^*}(S) = 13/10$

$\mathbb{E}_{\mathbf{q}^*}(T_2) = 3/2$

$\mathbb{E}_{\mathbf{q}^*}(T_1) = 7/2$

$Op^*_{4,2} = 3/2$

0	1	2	3	4	5
$\frac{4}{17}$	$\frac{2}{17}$	$\frac{3}{17}$	$\frac{5}{34}$	$\frac{11}{34}$	0

-	2	3	4	5
1	0	$\frac{4}{17}$	$\frac{3}{17}$	0
2	-	0	$\frac{11}{34}$	$\frac{2}{17}$
3	-	-	0	$\frac{5}{34}$
4	-	-	-	0

$\mathbb{E}_{\mathbf{p}^*}(S) = 75/34$

$\mathbb{E}_{\mathbf{q}^*}(T_2) = 59/34$

$\mathbb{E}_{\mathbf{q}^*}(T_1) = 137/34$

$Op^*_{5,2} = 59/34$

0	1	2	3	4	5	6
$\frac{4}{17}$	$\frac{2}{17}$	$\frac{3}{17}$	$\frac{5}{51}$	$\frac{7}{51}$	$\frac{4}{17}$	0

-	2	3	4	5	6
1	0	$\frac{3}{34}$	$\frac{14}{51}$	0	0
2	-	0	$\frac{1}{17}$	$\frac{5}{17}$	0
3	-	-	0	$\frac{7}{34}$	$\frac{4}{51}$
4	-	-	-	0	0
5	-	-	-	-	0

$\mathbb{E}_{\mathbf{p}^*}(S) = 127/51$

$\mathbb{E}_{\mathbf{q}^*}(T_2) = 98/51$

$\mathbb{E}_{\mathbf{q}^*}(T_1) = 233/51$

$Op^*_{6,2} = 98/51$

Proof. Using the results in Table 3.5, it can be shown that $Op_{4,2}(\mathbf{p}^*, (j_2, j_1)) = Op^*_{4,2}$ for any (j_2, j_1) with $\mathbf{q}^*_{(j_2,j_1)} > 0$ and $Op_{4,2}(i, \mathbf{q}^*) = Op^*_{4,2}$ for all $i = 0, \ldots, 3$, and furthermore $Op_{4,2}(\mathbf{p}^*, (j_2, j_1)) > Op^*_{4,2}$ for any (j_2, j_1) with $\mathbf{q}^*_{(j_2,j_1)} = 0$ and $Op_{4,2}(4, \mathbf{q}^*) = 1 < Op^*_{4,2}$. Therefore, the saddle point criterion (19.11) is fulfilled and thus, $\mathbf{p}^*$ and $\mathbf{q}^*$ are optimal strategies with the optimal payoff $Op^*_{4,2}$.

The proof for the cases $N = 5$ and $N = 6$ goes along the same lines. □

Let us comment the results of Lemma 3.4 and Table 3.5: First, a kind of step structure can be observed in the Inspectorate's optimal strategies. Because the pairs $(2, 4)$, $(3, 5)$ and $(3, 6)$ for $N = 4, 5, 6$, respectively, are played with positive probability, the existence of a cut-off value n^* like for the case of one interim inspection can not (yet) be observed. Therefore, an equation similar to (3.37) can also not be seen; see p. 46.

Second, as in (3.38), the relation $\mathbb{E}_{\mathbf{q}^*}(T_2) = Op^*_{N,2}$ holds for $N = 4, 5, 6$ (and also for $N = 11$; see Table 3.6), and it seems to be true for any number N of possible time points for any number $k \geq 2$ of interim inspections; see p. 47. Note that the optimal expected time point for the two interim inspections are not given by $\mathbb{E}_{\mathbf{q}^*}(T_n) = (3-n)\,(N+1)/3,\ n = 1,2$, as one might have expected according to the results for the Se-No and Se-Se inspection games; see Lemma 4.4 and Chapter 5.

Before continuing, we include, strangely enough, a fable published by Canty and Avenhaus (1991c), and which was designed to help to explain to a larger audience of practitioners the advantage of randomized interim inspection schemes.

> *A Fable: The Inspector who got Something for Nothing*
>
> *Once upon a time there was a safeguards inspector who wanted to spend more time with his family. The inspector was responsible for a power reactor in a far away land, and had to journey there once every year when the reactor was refuelled as well as three times in between because of his timeliness goal.*[5]
>
> *One day, during a refuelling inspection, the inspector went to the reactor boss and said that he would like, in future, to be allowed to perform his interim inspections on the last day of every month, instead of once every three months as had been the case up until then. The reactor boss frowned and asked the inspector if this was absolutely necessary. The inspector replied that it was, in the interest of increased safeguards effectiveness and efficiency. Upon hearing these words, the reactor boss sighed wearily and agreed to the inspector's request. The inspector then asked the reactor boss if he would be offended if he, the inspector, didn't show up for some of the eleven interim inspections. The reactor boss was puzzled, but said he would most certainly not be offended. The inspector then left the power reactor, rejoicing inwardly, saying „Now I shall only have to make two interim inspections per year, rather than three. I will still attain my timeliness goal and will be able to spend more time with my family!".*
>
> *Upon arriving back at headquarters, the inspector went to his safeguards bosses and told them of the deal he made at the reactor, and how he intended to save one interim inspection per year. At first the safeguards bosses were very angry, saying that the inspector was mad to think that he could get something for nothing and that his calculations must be incorrect. But then they consulted the literature and found a paper by two obscure but reputable safeguards experts Canty and Avenhaus (1991a) which confirmed exactly the calculations of the inspector. The safeguards bosses laughed and said that the inspector was very wise, and if he got much wiser he wouldn't have to work at all. They rewarded him by making him responsible for a second power reactor.*
>
> *Thus end the sad tale of the inspector who got something for nothing, but was not able to spend more time with his family.*

What did the inspector in the fable do? He considered, in the language of this monograph, the No-No inspection game with $N = 11$ possible time points for $k = 2$ interim inspections. Thus, the Inspectorate's set of pure strategy consists now of 55 pairs (j_2, j_1).

The game theoretical solution of this inspection game, see Canty and Avenhaus (1991a) and Tschoche (2010), is presented in

[5] The IAEA timeliness detection goal is defined as "the target detection times applicable to specific nuclear material categories. These goals are used for establishing the frequency of inspections and safeguards activities at a facility or a location outside facilities during a calendar year, in order to verify that no abrupt diversion has occurred"; see IAEA (2002).

Lemma 3.5. *Given the No-No inspection game with $N = 11$ possible time points for $k = 2$ interim inspections. The sets of mixed strategies are given by (3.13) and (3.52), and the payoff to the Operator by (3.15) using (3.51) for $N = 11$.*

Then an optimal strategy of the Operator and an optimal strategy of the Inspectorate is given in Table 3.6.

The optimal payoff to the Operator is

$$Op^*_{11,2} = \frac{15143}{5100} \approx 2.969\,.$$

Table 3.6 Optimal strategies of the Operator (top) and of the Inspectorate (bottom) of the No-No inspection game with $N = 11$ possible time points for $k = 2$ interim inspections.

-	0	1	2	3	4	5	6	7	8	9	10	11
	$\frac{4}{17}$	$\frac{4}{85}$	$\frac{24}{425}$	$\frac{36}{425}$	$\frac{9}{85}$	$\frac{5}{102}$	$\frac{7}{102}$	$\frac{31}{340}$	$\frac{107}{1020}$	$\frac{8}{51}$	0	0

-	2	3	4	5	6	7	8	9	10	11
1	0	0	0	$\frac{353}{10200}$	$\frac{1}{6}$	$\frac{2}{425}$	0	0	0	0
2	-	0	0	0	0	$\frac{83}{425}$	$\frac{1}{255}$	0	0	0
3	-	-	0	0	0	0	$\frac{1}{5}$	0	0	0
4	-	-	-	0	0	0	$\frac{47}{1020}$	$\frac{208}{1275}$	0	0
5	-	-	-	-	0	0	0	$\frac{217}{1275}$	$\frac{157}{10200}$	0
6	-	-	-	-	-	0	0	0	0	0
7	-	-	-	-	-	-	0	0	0	0
8	-	-	-	-	-	-	-	0	0	0
9	-	-	-	-	-	-	-	-	0	0
10	-	-	-	-	-	-	-	-	-	0

Proof. The results in Table 3.6 imply that $Op_{11,2}(\mathbf{p}^*, (j_2, j_1)) = Op^*_{11,2}$ for any (j_2, j_1) with $\mathbf{q}^*_{(j_2,j_1)} > 0$ and $Op_{11,2}(i, \mathbf{q}^*) = Op^*_{11,2}$ for all $i = 0, \ldots, 9$. Furthermore, we have $Op_{11,2}(\mathbf{p}^*, (j_2, j_1)) > Op^*_{11,2}$ for any (j_2, j_1) with $\mathbf{q}^*_{(j_2,j_1)} = 0$ and $Op_{11,2}(i, \mathbf{q}^*) < Op^*_{11,2}$ for $i = 10, 11$. Therefore, using the saddle point criterion (19.11), $\mathbf{p}^*$ and $\mathbf{q}^*$ are optimal strategies with the optimal payoff $Op^*_{11,2}$. □

Returning once more to the fable we see that the timeliness goal of three months is met – at least in expectation – by the optimal strategy of the Inspectorate thus, it saves indeed one interim inspection. If the Inspectorate implements the new strategy with only two interim inspections or if it stays with the three fixed interim inspections every three month, depends for instance on the risk aversion of the Inspectorate. Why? Because the probability that the detection time is *larger* than 3 month – in this case the timeliness goal is not met – is, using the optimal strategies in Table 3.6, given by

$$\mathbb{P}(\text{detection time is } \textit{larger} \text{ than } 3 \text{ month}) = \frac{589631}{1734000} \approx 0.34\,,$$

i.e., although one interim inspection is saved (cost aspect), the probability of not meeting the timeliness goal is about 34% (risk aspect).

In Figure 3.4 the ratio $Op^*_{N,2}/(N+1)$ is presented as a function of N. We see that it approaches rather rapidly an asymptotic value which is given by

$$\lim_{N\to\infty} \frac{Op^*_{N,2}}{N+1} = \frac{1}{e\,(e-1)} \approx 0.2141\,,$$

as will be shown in Chapter 9.

Figure 3.4 The normalized optimal payoff $Op^*_{N,2}/(N+1)$ to the Operator as a function of N.

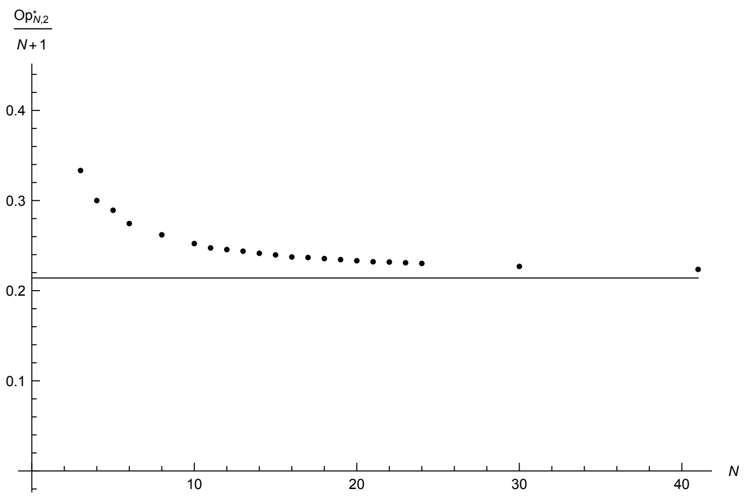


Table 3.6 shows that the step structure of the optimal strategy of the Inspectorate is even more pronounced than in the previous cases, but now a step may comprise three possibilities for the first interim inspection. Also, it is interesting that pure strategies $(j_2, j_2+1), (j_2, j_2+2)$ and (j_2, j_2+3) for $j_2 = 0, \ldots, 8$ are not played. Beyond that it is interesting that the first possible time point for the second interim inspection ($j_1 = 5$ in Table 3.6) is the last possible time point for the first interim inspection ($j_2 = 5$). The same holds for $N = 5$ and 6; see Table 3.5. Finally, the last possibility for the second interim inspection is time point 10; this indicates that there exists a cut-off value n^*, here $n^*(11) = 10$, like for the case of one interim inspection. However, (3.37) does not hold: $\mathbb{E}_{\mathbf{p}^*}(S) = 22357/5100 \neq 10 - Op^*_{11,2} = 35857/5100$.

Numerically, the following No-No inspection games have been considered additionally to the ones already discussed in this section: $N = 8, 10, 12, \ldots, 24, 30$ and 41. At least two observations are worth being reported: First, for all these N the Inspectorate plays at least a combination (j_2, j_1) with positive probability which places the first interim inspection at time point 1, i.e., $j_2 = 1$. Second, the optimal strategies of the Inspectorate have a step structure which makes the first step downward at time point $(1, \lfloor (N-1)/2 \rfloor + 2)$, where the floor function $\lfloor\ \rfloor$ maps x to the greatest integer less than or equal to x. It remains an open problem if these observations are true for any number N of possible time points.

For practical applications these results are very helpful: For moderate N, optimal strategies and the optimal expected detection times can be obtained with the help of M. Canty's Mathematica® programs; see Canty (2003). For larger N the continuous time version of this No-No inspection game, which will be discussed in Chapter 9, can be used. This holds, as one may expect, even for $k > 2$ interim inspections.

Let us conclude this section with a remark on more general cases than treated here. The complexity of the optimal strategies and optimal payoffs for $k = 2$ interim inspections and $N > 10$ possible time points indicates that it will be very difficult to get analytical solutions for these and even more general cases, i.e., any k with $2 \leq k < N$. Nevertheless, properties of their solutions can still be obtained as demonstrated by the following example.

According to (3.38) and also to the cases considered in this section let us answer the question, if in the No-No inspection game with N possible time points for $2 \leq k < N$ interim inspections the relation $\mathbb{E}_{\mathbf{q}^*}(T_k) = Op^*_{N,k}$ holds, i.e., if the optimal expected interim inspection time point T_k of the first interim inspection is equal to the optimal expected detection time. The answer is yes, which can be seen as follows: Let $q^*_{(j_k,\ldots,j_1)}$ denote the Inspectorate's optimal probability to choose the time points $j_k < \ldots < j_1$ for its k interim inspections. Under the assumption[6] that $p_0^* > 0$, the indifference principle, see Theorem 19.1, implies, making use of $Op_{N,k}(0,(j_k,\ldots,j_1)) = j_k$, that

$$\begin{aligned}
\mathbb{E}_{\mathbf{q}^*}(T_k) &:= \sum_{\substack{(j_k,\ldots,j_1):\\ 0<j_k<\ldots<j_1<N+1}} j_k\, q^*_{(j_k,\ldots,j_1)} \\
&= \sum_{\substack{(j_k,\ldots,j_1):\\ 0<j_k<\ldots<j_1<N+1}} Op_{N,k}(0,(j_k,\ldots,j_1))\, q^*_{(j_k,\ldots,j_1)} \\
&= Op_{N,k}(0,\mathbf{q}^*) = Op^*_{N,k}\,. \qquad (3.53)
\end{aligned}$$

Note that for the derivation of (3.53) the optimal strategies needed not be known.

[6] $p_0^* > 0$ is fulfilled for $k = 1$ and any $N \geq 2$, see Theorem 3.1, and for $k = 2$ and at least all cases of N considered in this section.

Chapter 4

No-Se and Se-No inspection games

In this chapter the No-Se and Se-No inspection games with discrete time and the playing for time inspection philosophy are considered. Recall that in the No-Se inspection game the Inspectorate behaves sequentially and the Operator does not, whereas in the Se-No inspection game the Operator behaves sequentially and the Inspectorate does not.

In this chapter, assumption (v) of Chapter 2 is specified as follows:

(v') During an interim inspection the Inspectorate does not commit an error of the second kind, i.e., the illegal activity, see assumption (iv), is detected with certainty during the next interim inspection or with certainty during the final PIV; see assumption (iii).

Assumptions (vii) will be specified in the following sections, while the remaining assumptions of Chapter 2 hold throughout this chapter. Note that the No-Se and Se-No inspection games with uncertain detection of an illegal activity at an interim inspection, i.e., $\beta \geq 0$, are treated in Sections 6.2 and 6.3.

The reason why the No-Se and Se-No inspection games are treated in this chapter together is that so far the No-Se inspection game has not been analysed for any number N of possible time points and for k interim inspection(s). From the application point of view, there was obviously no interest so far, and theoreticians were much more interested in the No-No, the Se-No and the Se-Se inspection games. Nevertheless, in Section 4.1 some special cases will be considered which lead to the same optimal expected detection times and the same optimal strategies of the Operator as the corresponding cases of the No-No inspection game.

In Section 4.2 the Se-No inspection game will be analysed comprehensively, where the cases of $k = 1$ and $k = 2$ interim inspection(s) are based on Krieger and Avenhaus (2014). Furthermore, important properties of the optimal strategies, the optimal expected detection time, and system quantities as well as the relations between them will be investigated. On p. 74 system quantities of the No-No and Se-No inspection games of Sections 3.1, 4.2 and 6.3 are compared.

4.1 No-Se for special numbers of inspection opportunities and interim inspections

The inspection game analysed in this section is based on the following specification:

(vii') The Operator decides at the beginning of the reference time interval, i.e., at time point 0, when to start the illegal activity.

The Inspectorate decides at the beginning of the reference time interval when to perform its first interim inspection. At the time point of its first interim inspection, it decides when to perform the second interim inspection.

Let us start with the fact that for any number N of possible time points for $k = 1$ interim inspection there is no difference between the No-No and the No-Se inspection game thus, Theorems 3.1 and 3.2 hold as well in that case. Therefore, the No-Se inspection game with $N = 3$ possible time points for $k = 2$ interim inspections should be studied next. Because this game is a special case of the No-Se inspection game analysed in Section 6.2, we consider now the case with $N = 4$ possible time points for $k = 2$ interim inspections. Before doing that let us just mention that in Sections 6.1 and 6.2 we will show that in case of $N = 3$ possible time points for $k = 2$ interim inspections and even for non-vanishing errors of the second kind, i.e., $\beta \geq 0$, the game theoretical solutions of the No-No and the No-Se inspection games are the same in the sense that the optimal strategies of the Operator and the optimal expected detection times coincide, and that the optimal strategies of the Inspectorate can be transformed uniquely into each other.

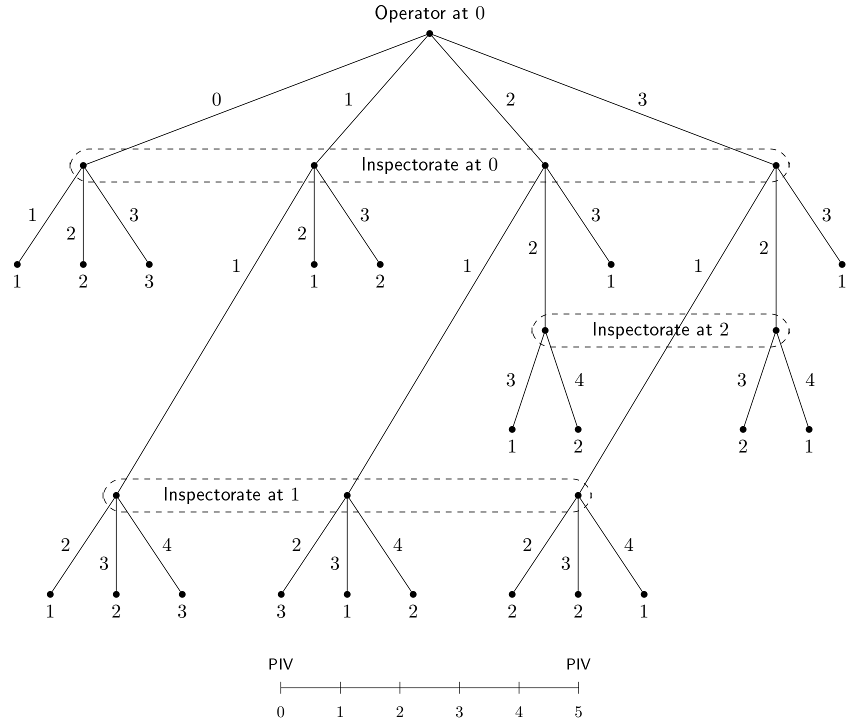
As announced we consider now the case with $N = 4$ possible time points for $k = 2$ interim inspections. The extensive form of this inspection game is represented in Figure 4.1.

The extensive form games in this monograph are represented according to the following rule: The extensive form games of the No-Se and Se-No inspection games always start with that player that behaves *no*n-sequentially, because this player decides only once. In extensive form games for the No-No (only considered in Section 6.1) and the Se-Se inspection games always the Operator starts. Note that in all extensive form games treated in this monograph chance moves are not explicitly named, but can be identified via the probabilities $1 - \beta$ and β, and $1 - \alpha$ and α, respectively.

The Operator decides at time point 0 at which of the time points 0, 1, 2 or 3 he will start the illegal activity. The possibility that he starts it at 4 is excluded here for the sake of transparency and due to the fact that it leads to the minimum possible detection time 1; see also the comment on p. 114. Formally, we deal here with a strictly dominated strategy, see the comment on p. 52, which is not played in any optimal strategy; see Myerson (1991) or Morris (1994).

Also at time point 0, the Inspectorate decides to perform its first interim inspection at time point 1, 2 or 3, not knowing the Operator's decision at time point 0. This is indicated by the first information set. In case the illegal activity is not started prior to the first interim inspection, the Inspectorate decides after the first interim inspection when to perform the second interim inspection. In case the first interim inspection takes place at 1, the Inspectorate has to choose the time point 2, 3 or 4 for its second interim inspection, and in case the first interim inspection takes place at 2, it has to choose the time point 3 or 4, in both cases not knowing what the Operator does. This leads to the second and third information set of the Inspectorate.

Figure 4.1 Extensive form of the No-Se inspection game with $N = 4$ possible time points for $k = 2$ interim inspections.



Note that here the first time in this monograph information – or the lack of information – is modelled using information sets. In subsequent sections and chapters we will use this important game theoretical concept many more times.

Like in the previous chapter, the set of pure strategies of the Operator is I_4 as given by (3.10) for $N = 4$ and thus, his set of mixed strategies is P_4 as given by (3.13) for $N = 4$. For the Inspectorate, the set of pure strategies is more complicated than that given by (3.10). Since we do not make use of it here, we consider immediately the Inspectorate's set of behavioural strategies; see van Damme (1987) and Section 19.2. Formally, we have to identify for any information set a probability distribution over the branches leaving this information set: Let $h_3(j_2)$ be the probability that the Inspectorate decides at the beginning of the reference time interval to perform the first interim inspection at time point j_2 with $j_2 = 1, 2, 3$. Furthermore, let $h_2(j_1|j_2)$ for $j_2 = 1, 2$ and $j_1 = j_2 + 1, \ldots, 4$, be the probability that the Inspectorate decides at time point j_2 to perform the second interim inspection at time point j_1. Note that in case the time point $j_2 = 3$ is chosen for the first interim inspection, necessarily j_1 has to be 4, i.e., $h_2(4|3) = 1$. Thus, $h_2(4|3)$ is not a strategic variable and is omitted in the Inspectorate's set of behavioural strategies, which is given by

$$H_{4,2} := \Big\{ \mathbf{h} := (h_3, h_2) \in [0,1]^2 : \sum_{j_2=1}^{3} h_3(j_2) = 1 \text{ and } \sum_{j_1=j_2+1}^{4} h_2(j_1|j_2) = 1 \text{ for } j_2 = 1,2 \Big\} . \quad (4.1)$$

Note that the comment made after (3.50) on backward numbering holds here as well. Also note, that we use here and in Chapter 5 the notation $(j_1|j_2)$ indicating a kind of conditional event: the choice of time point j_1 under the condition that the first interim inspection is performed at time point j_2. This notation is due to the sequential nature of the Inspectorate's behaviour. In (4.3) and also in Section 4.2, however, we will apply the notation (j_2, j_1) which indicates the Inspectorate's non-sequential behaviour.

The (expected) payoff to the Operator, i.e., the expected detection time, is, for any $\mathbf{p} \in P_4$ and any $\mathbf{h} \in H_{4,2}$, using Figure 4.1, given by

$$\begin{aligned} Op_{4,2}(\mathbf{p}, \mathbf{h}) &:= p_0\, [h_3(1) + 2\, h_3(2) + 3\, h_3(3)] \qquad (4.2) \\ &+ p_1\, [h_3(1)\, (h_2(2|1) + 2\, h_2(3|1) + 3\, h_2(4|1)) + h_3(2) + 2\, h_3(3)] \\ &+ p_2\, [h_3(1)\, (3\, h_2(2|1) + h_2(3|1) + 2\, h_2(4|1)) + h_3(2)\, (h_2(3|2) + 2\, h_2(4|2)) + h_3(3)] \\ &+ p_3\, [h_3(1)\, (2\, h_2(2|1) + 2\, h_2(3|1) + h_2(4|1)) + h_3(2)\, (2\, h_2(3|2) + h_2(4|2)) + h_3(3)] \\ &+ p_4 \, . \end{aligned}$$

It was mentioned on p. 50 that "starting the illegal activity at time point 4" is a strictly dominated strategy. To confirm this, we consider the Operator's strategies $\mathbf{p}_1 := (0,0,0,0,1)^T$ and $\mathbf{p}_2 := (0, 1/3, 1/3, 1/3, 0)^T$. Using (4.2), we get

$$\begin{aligned} Op_{4,2}(\mathbf{p}_2, \mathbf{h}) &= \frac{1}{3} \Big(h_3(1)\, [6\, h_2(2|1) + 5\, h_2(3|1) + 6\, h_2(4|1)] + 4\, h_3(2) + 4\, h_3(3) \Big) \\ &\geq \frac{1}{3} \Big(5\, h_3(1) + 4\, h_3(2) + 4\, h_3(3) \Big) \geq \frac{4}{3} > 1 = Op_{4,2}(\mathbf{p}_1, \mathbf{h}) \end{aligned}$$

for any $\mathbf{h} = (h_3, h_2) \in H_{4,2}$, i.e., $\mathbf{p}_1$ is a strictly dominated strategy.

Now let us consider the corresponding No-No inspection game the normal form of which, i.e., the payoff matrix A, has already been given in Table 3.4. This time, however, we present the formula of the expected detection time explicitly and call it $Op_{No-No}(\mathbf{p}, \mathbf{q})$, because the notation $Op_{4,2}(\cdot,\cdot)$ is reserved for the payoff to the Operator in the No-Se inspection game of this section. We get for any $\mathbf{p} := (p_0, p_1, p_2, p_3, p_4)^T \in P_4$ and any $\mathbf{q} := (q_{(1,2)}, \ldots, q_{(3,4)})^T \in Q_{4,2}$

$$\begin{aligned} Op_{No-No}(\mathbf{p}, \mathbf{q}) &:= \mathbf{p}^T A\, \mathbf{q} \\ &= p_0\, (q_{(1,2)} + q_{(1,3)} + q_{(1,4)} + 2\, q_{(2,3)} + 2\, q_{(2,4)} + 3\, q_{(3,4)}) \\ &+ p_1\, (q_{(1,2)} + 2\, q_{(1,3)} + 3\, q_{(1,4)} + q_{(2,3)} + q_{(2,4)} + 2\, q_{(3,4)}) \qquad (4.3) \\ &+ p_2\, (3\, q_{(1,2)} + q_{(1,3)} + 2\, q_{(1,4)} + q_{(2,3)} + 2\, q_{(2,4)} + q_{(3,4)}) \\ &+ p_3\, (2\, q_{(1,2)} + 2\, q_{(1,3)} + q_{(1,4)} + 2\, q_{(2,3)} + q_{(2,4)} + q_{(3,4)}) + p_4 \, . \end{aligned}$$

Again, the last pure strategy of the Operator is dropped as it is a strictly dominated strategy: Using the strategies $\mathbf{p}_1$ and $\mathbf{p}_2$ defined above, we get $Op_{No-No}(\mathbf{p}_2, \mathbf{q}) \geq 4/3 > 1 = Op_{No-No}(\mathbf{p}_1, \mathbf{q})$ for any $\mathbf{q} \in Q_{4,2}$.

Let us look again at the extensive form of the No-Se inspection game given in Figure 4.1. The probabilities $q_{(j_2,j_1)}$ that the Inspectorate chooses time points j_2 for the first, and j_1 for the second interim inspection are given as follows:

$$\begin{aligned} &q_{(1,2)} = h_3(1)\, h_2(2|1)\,, \qquad q_{(1,3)} = h_3(1)\, h_2(3|1)\,, \qquad q_{(1,4)} = h_3(1)\, h_2(4|1)\,, \\ &q_{(2,3)} = h_3(2)\, h_2(3|2)\,, \qquad q_{(2,4)} = h_3(2)\, h_2(4|2)\,, \\ &q_{(3,4)} = h_3(3)\,. \end{aligned} \tag{4.4}$$

Due to the normalization of the $\mathbf{h} = (h_3, h_2)$, see (4.1), we see that $\mathbf{q}$ is a mixed strategy of the Inspectorate. Therefore, if we replace in (4.3) the probabilities $q_{(j_2,j_1)}$ by (4.4), then we obtain the expected detection time $Op_{4,2}(\mathbf{p}, \mathbf{h})$ as given by (4.2).

If one defines on the other hand, provided the appropriate ratios exist,

$$\begin{aligned} &h_3(1) = q_{(1,2)} + q_{(1,3)} + q_{(1,4)}\,, \quad h_3(2) = q_{(2,3)} + q_{(2,4)}\,, \quad h_3(3) = q_{(3,4)} \\ &h_2(2|1) = \frac{q_{(1,2)}}{h_3(1)}\,, \quad h_2(3|1) = \frac{q_{(1,3)}}{h_3(1)}\,, \quad h_2(4|1) = \frac{q_{(1,4)}}{h_3(1)} \\ &h_2(3|2) = \frac{q_{(2,3)}}{h_3(2)}\,, \quad h_2(4|2) = \frac{q_{(2,4)}}{h_3(2)}\,, \end{aligned} \tag{4.5}$$

then one easily sees that $\mathbf{h} = (h_3, h_2)$ is a behavioural strategy of the Inspectorate. If we replace in (4.2) the probabilities (h_3, h_2) by (4.5), we arrive at (4.3), i.e., we get the expected detection time $Op_{No-No}(\mathbf{p}, \mathbf{q})$.

Since we have the same number of five free variables in the strategy sets of the Inspectorate, see $H_{4,2}$ and $Q_{4,2}$, we can determine the behavioural strategies of the Inspectorate of the No-Se inspection game with the help of those of the No-No inspection game. Thus, and following from the result we just have obtained, we have shown that any optimal strategy of the No-Se inspection game is also an optimal strategy of the No-No inspection game and vice versa.

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Lemma 4.1. *Given the No-Se inspection game with $N = 4$ possible time points for $k = 2$ interim inspections. The sets of mixed resp. behavioural strategies are given by (3.13) for $N = 4$ and (4.1), and the payoff to the Operator by (4.2).*

Then an optimal strategy of the Operator is given by

$$\mathbf{p}^* = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{10}, \frac{3}{10}, 0\right)^T \,,$$

and an optimal strategy of the Inspectorate by

$$h_3^*(1) = h_3^*(2) = \frac{1}{2}\,, h_3^*(3) = 0 \ \textit{and} \ h_2^*(j_1|j_2) = \begin{cases} 1 & \textit{for} \ (j_2, j_1) \in \{(1,3), (2,4)\} \\ 0 & \textit{otherwise} \end{cases} \,.$$

The optimal payoff to the Operator is

$$Op^*_{4,2} := Op_{4,2}(\mathbf{p}^*, \mathbf{h}^*) = \frac{3}{2}\,.$$

Proof. We apply the results of Lemma 3.4: According to what has been said before, the optimal strategy of the Operator and the optimal payoff are taken from Table 3.5. The optimal strategy of the Inspectorate is determined with the help of Table 3.5 and (4.5). □

It may not be too difficult to extend this consideration to the case of any number N of possible time points for $k = 2$ interim inspections, because there seem to exist a payoff equivalent transformation in analogy to (4.4) and (4.5) between the elements of both Inspectorate's strategy sets, i.e., between $Q_{N,2}$ defined by (3.52) and the Inspectorate's set of behavioural strategies, as generalization of (4.1),

$$H_{N,2} := \Big\{ \mathbf{h} := (h_3, h_2) \in [0,1]^2 : \sum_{j_2=1}^{N-1} h_3(j_2) = 1 \text{ and } \sum_{j_1=j_2+1}^{N} h_2(j_1|j_2) = 1 \text{ for all } j_2 = 1, \ldots, N-2 \Big\}\,, \quad (4.6)$$

where the case $(j_2, j_1) = (N-1, N)$ leading to $h_2(N|N-1) = 1$ is – in analogy to the explanations on p. 51 – excluded from the Inspectorate's strategy set. Also note that the Inspectorate's strategy sets have the same number of free variables in both inspection games: $Q_{N,2}$ has $\binom{N}{2} - 1$ free variables, and $H_{N,2}$ has

$$(N-2) + \sum_{j_2=1}^{N-2} [(N - (j_2+1) + 1) - 1] = (N-2)\,\frac{N+1}{2} = \binom{N}{2} - 1$$

free variables. Thus, and in sum, the optimal strategies obtained in Section 3.2 for $N = 4, 5, 6, 11$ for the No-No inspection game can be used here to obtain optimal strategies for No-Se inspection game, provided that the payoff transformation between Inspectorate's strategies can be proven to be valid; see also p. 81.

At present it looks infeasible to prove – or to disprove – that these results hold also for any number N of possible time points and $k < N$ interim inspections, and eventually also for $\beta \geq 0$. Since, however, there exist at present no game theoretical solutions for these general cases either for the No-No nor for the No-Se inspection game, this does not mean very much. We discuss the question of the equivalence of the Se-No and the Se-Se inspection games in Chapter 5 and of all four variants of the inspection game in the Chapter 6.

4.2 Se-No for any number of inspection opportunities and interim inspections: Krieger-Avenhaus model

We saw in Section 4.1 that there are good reasons to suppose that the No-Se inspection game is equivalent to the No-No inspection game in the sense that the optimal strategy of the Operator and the optimal payoff coincide between both variants, whereas the optimal strategies of the

Inspectorate can be transformed into each other. The Se-No inspection game, however, which will be considered now turns out to be different in many respects but – and this can be said already now – it is strongly related to the Se-Se inspection game as will be shown in Chapter 5.

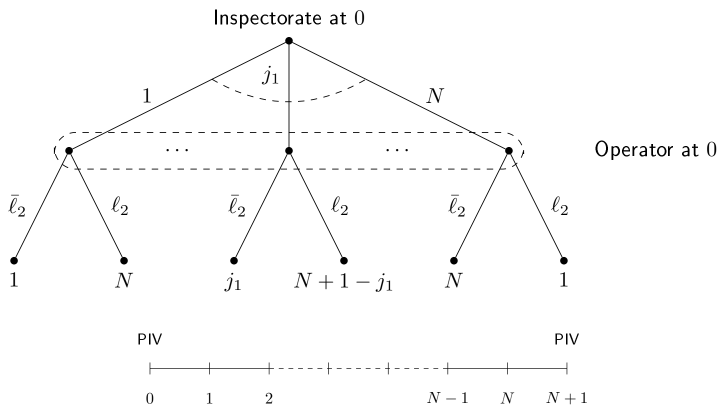
The inspection game analysed in this section is based on the following specification:

(vii') The Inspectorate decides at the beginning of the reference time interval, i.e., at time point 0, at which of the possible time points $1, \dots, N$ it will perform its k interim inspection(s).

The Operator decides at the beginning of the reference time interval whether to start the illegal activity immediately at time point 0 or to postpone the start; in the latter case he decides again after the first interim inspection, whether to start the illegal activity immediately at that time point or to postpone the start again; and so on. Because of assumption (iv), the Operator starts the illegal activity latest immediately at the time point of the last interim inspection.

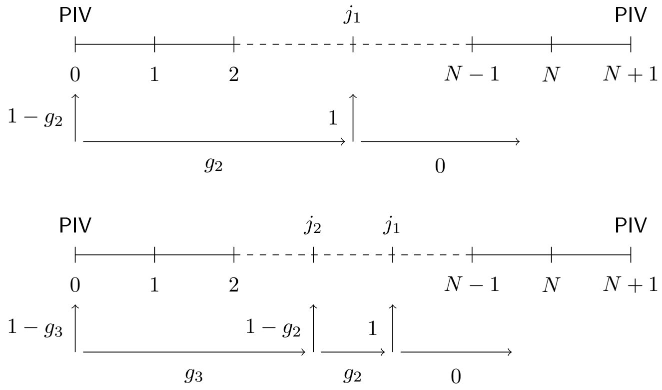
For the purpose of illustration we start with the cases of $k = 1$ and $k = 2$ interim inspection(s). The extensive form of the Se-No inspection game with $k = 1$ interim inspection is represented in Figure 4.2. Because the Inspectorate plays non-sequentially in this section, the extensive form games in Figures 4.2 and 4.4 start with the Inspectorate's decision at time point 0 due to the comment on p. 50.

Figure 4.2 Extensive form of the Se-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection.



In Figure 4.2 the Inspectorate chooses at time point 0 the time point j_1, $j_1 = 1, \dots, N$, for its only interim inspection with probabilities q_{j_1}, i.e., its set of pure strategies is given, using (3.10), by $J_{N,1}$, and its set of mixed strategies, using (3.14), by $Q_{N,1}$. Also at time point 0, the Operator decides with probability $1 - g_2$ to start the illegal activity immediately ($\bar{\ell}_2$) or to postpone it (ℓ_2) with probability g_2 which means that in this latter case he starts the illegal activity immediately after the interim inspection at time point j_1 with certainty, i.e., $g_1(j_1) = 0$; see Figure 4.3. Because this probability is fixed it is excluded from the Operator's strategy set.

Figure 4.3 Time line of the interim inspections and probabilities for starting or postponing the illegal activity for the Se-No inspection game with $N > k$ possible time points for $k = 1$ (top) resp. $k = 2$ (below) interim inspections. For reasons of clarity we write g_2 instead of $g_2(j_2)$.



Formally, the set of behavioural strategies of the Operator is given by

$$G_1 := \{g_2 : g_2 \in [0, 1]\} \ . \tag{4.7}$$

The detection times, i.e., the times between the start of the illegal activity and its detection are given at the end nodes. Note that the oval represents the information set of the Operator at time point 0 which models the information he has at the beginning of the game: Since he does not know when the interim inspection is taken place all nodes following a decision of the Inspectorate at 0 have to be in the information set.

A word on the notation: In Chapter 3 where the Operator is playing non-sequentially, he chooses a time point i, $i = 0, 1, \ldots, N$, for starting the illegal activity. Thus, his set of pure and mixed strategies is indexed by N; see I_N and P_N as given by (3.10) and (3.13), respectively. If he plays sequentially, like in this section, he only decides at the beginning PIV and at the interim inspection(s) whether to start the illegal activity. Thus, his behavioural strategy set is related to the number of interim inspections k, and is therefore indexed by k.

Using Figure 4.2, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $N \geq 2$, any $g_2 \in G_1$ and any $\mathbf{q} \in Q_{N,1}$, given by

$$Op_{N,1}(g_2, \mathbf{q}) := \sum_{j_1=1}^{N} q_{j_1} \Big((1 - g_2)\, j_1 + g_2\, (N + 1 - j_1) \Big) \ . \tag{4.8}$$

The game theoretical solution of this inspection game, see Krieger and Avenhaus (2014), is presented in

Lemma 4.2. *Given the Se-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection. The sets of behavioural resp. mixed strategies are given by (4.7) and (3.14), and the payoff to the Operator by (4.8).*

Then an optimal strategy of the Operator is given by

$$g_2^* = \frac{1}{2}\,, \tag{4.9}$$

and an optimal strategy $\mathbf{q}^* := (q_1^*, \ldots, q_N^*)^T$ *of the Inspectorate fulfils the conditions*

$$\sum_{j_1=1}^{N} j_1\, q_{j_1}^* = \frac{N+1}{2} \qquad \textit{with} \qquad \sum_{j_1=1}^{N} q_{j_1}^* = 1\,. \tag{4.10}$$

The optimal payoff to the Operator is

$$Op_{N,1}^* := Op_{N,1}(g_2^*, \mathbf{q}^*) = \frac{N+1}{2}\,. \tag{4.11}$$

Proof. Inserting (4.9) and (4.10) into (4.8) yields

$$Op_{N,1}(g_2^*, \mathbf{q}) = \frac{1}{2} \sum_{j_1=1}^{N} q_{j_1} \Big(j_1 + N + 1 - j_1\Big) = \frac{N+1}{2} = Op_{N,1}^*$$

for any $\mathbf{q} \in Q_{N,1}$, and

$$Op_{N,1}(g_2, \mathbf{q}^*) = (1 - 2\, g_2) \sum_{j_1=1}^{N} j_1\, q_{j_1}^* + g_2\,(N+1) \sum_{j_1=1}^{N} q_{j_1}^* = \frac{N+1}{2} = Op_{N,1}^*$$

for any $g_2 \in G_1$, i.e., the saddle point criterion

$$Op_{N,1}(g_2, \mathbf{q}^*) \le Op_{N,1}^* \le Op_{N,1}(g_2^*, \mathbf{q})\,,$$

see (19.10), is fulfilled as equality for any $g_2 \in G_1$ and any $\mathbf{q} \in Q_{N,1}$. □

Let us comment the results of Lemma 4.2: First, we see that the optimal strategies of the Inspectorate are only unique for $N = 2$ resulting in $q_1^* = q_2^* = 1/2$. For $N > 2$ only the optimal expected interim inspection time point is fixed, namely in the middle of the reference time interval. We can, however, provide three special cases: For any N

$$q_{j_1}^* = \frac{1}{N} \qquad \text{for} \qquad 1 \le j_1 \le N\,, \tag{4.12}$$

for $N+1$ being an even number

$$q_{j_1}^* = \begin{cases} 1 & \text{for } j_1 = (N+1)/2 \\ 0 & \text{otherwise} \end{cases}\,, \tag{4.13}$$

and, for $N+1$ being an odd number

$$q_{j_1}^* = \begin{cases} 1/2 & \text{for } j_1 \in \{N/2, N/2+1\} \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

are optimal strategies of the Inspectorate. In (4.12) the equal distribution over the Inspectorate's set of pure strategies is an optimal strategy, in (4.13) the one point distribution, i.e., a deterministic strategy is optimal, and in (4.14) an equal distribution over two pure strategies is optimal. At first sight these special optimal strategies may be guessed with the help of common sense arguments. On the other hand the lack of uniqueness of the Inspectorate's optimal strategy can hardly be guessed, and it may be used, e.g., for taking organisational or economic aspects into account.

This last statement leads to a general question which is elaborated here in more detail: Why do we discuss different inspection strategies knowing that they all lead to the same optimal expected detection time; see Chapter 19? The answers to this question comes from an application point of view: the Inspectorate might want to know which optimal strategy should be recommended for implementation. (4.12) on one hand and (4.13) and (4.14) on the other represent two extreme case: While in the former one each of the pure strategies is mixed, in the latter ones only one or two are mixed. The authors have gained different experiences with this problem, since mixing of only two pure strategies is often seen to be "not enough random" and mixing all of them is "too random". However, the set of optimal strategies allows the Inspectorate to a certain extend to take organisational or economic as well as randomization aspects into account.

Second, on p. 142 conditions for the legal behaviour of the Operator and corresponding optimal inspection strategies are discussed.

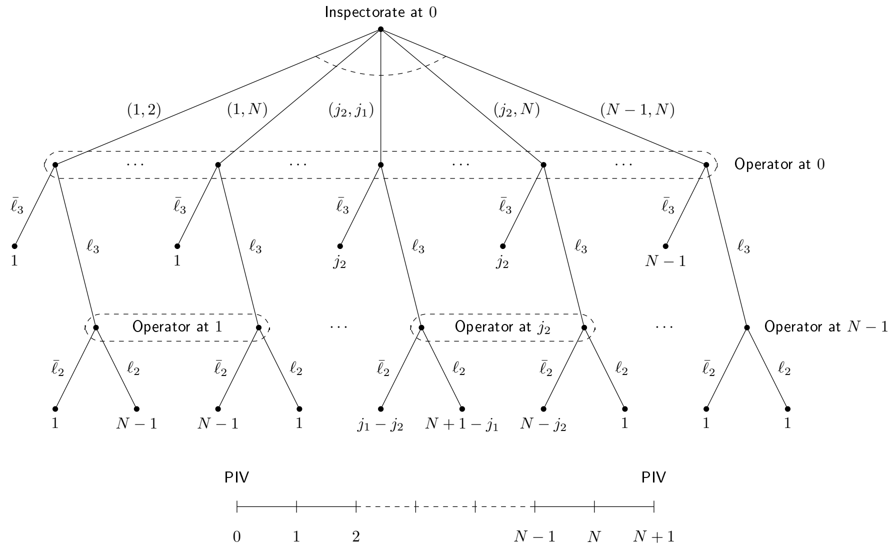
Finally, note that further properties of the optimal strategies and the optimal payoff of the Se-No inspection game are given in Lemma 4.4, in the comments after its proof, and in Table 4.2. Also note that in Lemma 6.4 the Se-No inspection game with $N \geq 2$ possible time points for $k = 1$ interim inspection and with errors of the second kind is analysed.

Consider now the Se-No inspection game with $N \geq 3$ possible time points for $k = 2$ interim inspections, the extensive form of which is represented in Figure 4.4. Again, the detection times are given at the end nodes.

The Inspectorate chooses at time point 0 a pair of time points (j_2, j_1) with $0 < j_2 < j_1 < N+1$ for its two interim inspections, which means that its set of pure strategies is given by $J_{N,2}$; see (3.50). A pair (j_2, j_1) is hereby chosen with probabilities $q_{(j_2,j_1)}$, i.e., the Inspectorate's set of mixed strategies is $Q_{N,2}$ as given by (3.52). Note that we write here (j_2, j_1) to indicate that both j_2 and j_1 are chosen at the beginning of the game, i.e., non-sequential behaviour of the Inspectorate, in contrast to the notation $(j_1|j_2)$ in (4.1) which indicates that j_1 is chosen under the condition that j_2 is chosen before, i.e., sequential behaviour of the Inspectorate; see also the comment on p. 52. Also note, that the comment after (3.50) made on backward numbering applies here as well.

Also at the beginning of the reference time interval, i.e., at time point 0, the Operator decides with probability $1 - g_3$ to start the illegal activity immediately ($\bar{\ell}_3$) or to postpone its start (ℓ_3) with probability g_3. In this latter case he decides after the first interim inspection at time point j_2 with probability $1 - g_2(j_2)$ to start the illegal activity at j_2 ($\bar{\ell}_2$) or to postpone its start again (ℓ_2) with probability $g_2(j_2)$. In this latter case he starts the illegal activity immediately after the second interim inspection at time point j_1 with certainty, i.e., $g_1(j_1) = 0$. For the same reason as in the game with $k = 1$ interim inspection, this probability is excluded from the Operator's strategy set. Note that if $j_2 = N - 1$ then $j_1 = N$, and it does not make any difference for the Operator to start the illegal activity right at time point j_2 or at j_1, because the detection time is 1 in both cases. Therefore, the value of $g_2(N - 1)$ does not play any role in any optimal strategy and is also excluded from the Operator's strategy set. Thus, the

Figure 4.4 Extensive form of the Se-No inspection game with $N > 2$ possible time points for $k = 2$ interim inspections.



Operator's set of behavioural strategies is given by

$$G_2 := \{\mathbf{g} := (g_3, g_2) : g_3 \in [0,1], g_2 : \{1, \ldots, N-2\} \to [0,1]\} . \tag{4.15}$$

The information sets in Figure 4.4 can be explained as follows: The information set "Operator at 0" results from the fact that the players decide independently of each other at the beginning of the reference time interval; see assumption (viii) of Chapter 2. In case he does not start the illegal activity at time point 0, there exist $N-1$ additional information sets because the Operator does not know at time point j_2, $j_2 = 1, \ldots, N-1$, when the second interim inspection will be performed.

Using Figure 4.4, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $N \geq 3$, any $\mathbf{g} \in G_2$ and any $\mathbf{q} \in Q_{N,2}$, given by

$$\begin{aligned} Op_{N,2}(\mathbf{g}, \mathbf{q}) := \sum_{j_2=1}^{N-2} \sum_{j_1=j_2+1}^{N} q_{(j_2,j_1)} \Big[(1-g_3)\, j_2 + g_3 \\ \Big((1-g_2(j_2))\,(j_1-j_2) + g_2(j_2)\,(N+1-j_1) \Big) \Big] \\ + q_{(N-1,N)} \Big[(1-g_3)\,(N-1) + g_3\, 1 \Big] . \end{aligned} \tag{4.16}$$

Define the marginal probabilities of $q_{(j_2,j_1)}$ by

$$q_{j_2 \cdot} := \sum_{j_1=j_2+1}^{N} q_{(j_2,j_1)} \qquad \text{and} \qquad q_{\cdot j_1} := \sum_{j_2=1}^{j_1-1} q_{(j_2,j_1)} \tag{4.17}$$

for $j_2 = 1, \ldots, N-1$ and $j_1 = 2, \ldots, N$, respectively.

The game theoretical solution of this inspection game, see Krieger and Avenhaus (2014), is presented in

Lemma 4.3. *Given the Se-No inspection game with $N > 2$ possible time points for $k = 2$ interim inspections. The sets of behavioural resp. mixed strategies are given by (4.15) and (3.52), and the payoff to the Operator by (4.16).*

Then an optimal strategy of the Operator is given by

$$g_3^* = \frac{2}{3}\,, \qquad g_2^*(j_2) = \frac{1}{2} \quad \textit{for} \quad j_2 = 1, \ldots, N-2\,, \tag{4.18}$$

and an optimal strategy $\mathbf{q}^* := (q^*_{(1,2)}, \ldots, q^*_{(N-1,N)})^T$ of the Inspectorate fulfils the conditions

$$\sum_{j_1=j_2+1}^{N} j_1\, q^*_{(j_2,j_1)} = \frac{N+1+j_2}{2}\, q^*_{j_2\cdot} \quad \textit{for} \quad j_2 = 1, \ldots, N-2\,, \tag{4.19}$$

$$\sum_{j_2=1}^{N-1} j_2\, q^*_{j_2\cdot} = \frac{N+1}{3} \quad \textit{with} \quad \sum_{j_2=1}^{N-1} \sum_{j_1=j_2+1}^{N} q^*_{(j_2,j_1)} = 1\,. \tag{4.20}$$

The optimal payoff to the Operator is

$$Op^*_{N,2} := Op_{N,2}(\mathbf{g}^*, \mathbf{q}^*) = \frac{N+1}{3}\,. \tag{4.21}$$

Proof. We show that the saddle point criterion

$$Op_{N,2}(\mathbf{g}, \mathbf{q}^*) \le Op^*_{N,2} \le Op_{N,2}(\mathbf{g}^*, \mathbf{q}) \tag{4.22}$$

see (19.10), is fulfilled as equalities for any $\mathbf{g} \in G_2$ and any $\mathbf{q} \in Q_{N,2}$. From (4.16) and (4.18) we get

$$\begin{aligned} Op_{N,2}(\mathbf{g}^*, \mathbf{q}) &= \frac{1}{3} \sum_{j_2=1}^{N-1} \sum_{j_1=j_2+1}^{N} q_{(j_2,j_1)} \Big[j_2 + \Big(j_1 - j_2 + (N+1-j_1) \Big) \Big] = \frac{N+1}{3} \\ &= Op^*_{N,2} \end{aligned} \tag{4.23}$$

for any $\mathbf{q} \in Q_{N,2}$. For the proof of the left hand inequality of (4.22) we show that the coefficient of $g_2(j_2)$ is zero for all $j_2 = 1, \ldots, N-2$. By (4.19) we obtain for all $j_2 = 1, \ldots, N-2$

$$\sum_{j_1=j_2+1}^{N} q^*_{(j_2,j_1)}\, [N+1+j_2 - 2\, j_1] = (N+1+j_2)\, q^*_{j_2\cdot} - 2\, \frac{N+1+j_2}{2}\, q^*_{j_2\cdot} = 0\,,$$

which leads, using (4.16), to

$$Op_{N,2}(\mathbf{g}, \mathbf{q}^*) = \sum_{j_2=1}^{N-2} \sum_{j_1=j_2+1}^{N} q^*_{(j_2,j_1)} \Big[(1-g_3)\, j_2 + g_3\, (j_1 - j_2) \Big]$$

$$+\sum_{j_2=1}^{N-2} g_3\, g_2(j_2) \sum_{j_1=j_2+1}^{N} q^*_{(j_2,j_1)}\, [N+1+j_2-2\, j_1]$$

$$+\, q^*_{(N-1,N)} \Big[(1-g_3)\,(N-1) + g_3\, 1\Big]$$

$$= \sum_{j_2=1}^{N-1} \sum_{j_1=j_2+1}^{N} q^*_{(j_2,j_1)} \Big[(1-g_3)\, j_2 + g_3\,(j_1-j_2)\Big]\,. \qquad (4.24)$$

Now the coefficient of g_3 turns out to be zero: Making use of (4.19) and (4.20) we get

$$\sum_{j_2=1}^{N-1} \sum_{j_1=j_2+1}^{N} q^*_{(j_2,j_1)}\, [j_1 - 2\, j_2] = \sum_{j_2=1}^{N-1} \left[\frac{N+1+j_2}{2}\, q^*_{j_2\cdot} - 2\, j_2\, q^*_{j_2\cdot}\right]$$
$$= \frac{N+1}{2} - \frac{3}{2} \sum_{j_2=1}^{N-1} j_2\, q^*_{j_2\cdot} = 0\,. \qquad (4.25)$$

Therefore, (4.24) and (4.25) finally yield

$$Op_{N,2}(\mathbf{g}, \mathbf{q}^*) = \sum_{j_2=1}^{N-1} \sum_{j_1=j_2+1}^{N} j_2\, q^*_{(j_2,j_1)} + \sum_{j_2=1}^{N-1} \sum_{j_1=j_2+1}^{N} q^*_{(j_2,j_1)}\, [j_1 - 2\, j_2]$$
$$= \sum_{j_2=1}^{N-1} \sum_{j_1=j_2+1}^{N} j_2\, q^*_{(j_2,j_1)} = Op^*_{N,2}$$

for any $\mathbf{g} \in G_2$, i.e., (4.22) is together with (4.23) fulfilled as equality. □

Let us comment the results of Lemma 4.3: First, similar to the case of $k=1$ interim inspection, the optimal strategies of the Inspectorate are only unique for $N=3$ resulting in $q^*_{(1,2)} = q^*_{(1,3)} = q^*_{(2,3)} = 1/3$. However, it can be seen that for any $N > 3$

$$q^*_{(j_2,j_1)} = \binom{N}{2}^{-1} \qquad \text{for all} \qquad (j_2, j_1) \qquad \text{with} \qquad 0 < j_2 < j_1 < N+1\,, \qquad (4.26)$$

and for $N+1$ being a multiple of 3

$$q^*_{(j_2,j_1)} = \begin{cases} 1 & \text{for } (j_2,j_1) = ((N+1)/3, 2\,(N+1)/3) \\ 0 & \text{otherwise} \end{cases}$$

are optimal strategies for the Inspectorate, i.e., they fulfil (4.19) and (4.20). Again, in the first case the equal distribution over the set of pure strategies is an optimal strategy, while in the second case a one point distribution is optimal. Using (4.17), (4.19) and (4.20), we see that

$$\sum_{j_1=2}^{N} j_1\, q^*_{\cdot j_1} = \sum_{j_1=2}^{N} j_1 \sum_{j_2=1}^{j_1-1} q^*_{(j_2,j_1)} = \sum_{j_2=1}^{N-1} \sum_{j_1=j_2+1}^{N} j_1\, q^*_{(j_2,j_1)} = \sum_{j_2=1}^{N-1} \frac{N+1+j_2}{2}\, q^*_{j_2\cdot}$$
$$= 2\, \frac{N+1}{3} = 2 \sum_{j_2=1}^{N-1} j_2\, q^*_{j_2\cdot} \qquad (4.27)$$

holds for each of the Inspectorate's optimal strategies $\mathbf{q}^*$, which means that the optimal expected interim inspection time points are fixed; they are at $1/3$ and $2/3$ of the length of the reference time interval.

Second, having found the equal distribution over all feasible pairs (j_2, j_1) to be an optimal strategy, see (4.26), one might ask what the optimal strategy is which mixes least pure strategies. For the case of $k = 1$ interim inspection and in case that $N+1$ is an odd number we have seen in (4.14) that one gets an optimal strategy if one chooses the two time points $N/2$ and $N/2+1$ with equal probabilities. Following this idea in case of $k = 2$ interim inspections one would have to choose two integer time points adjacent to $(N+1)/3$ and adjacent to $2\,(N+1)/3$ with probabilities each such that the optimal expected interim inspection time points are just $(N+1)/3$ and $2\,(N+1)/3$. Does this procedure lead to an optimal strategy? For the sake of demonstration we choose $N = 4$. Since the first resp. the second optimal expected interim inspection time point is $5/3$ resp. $10/3$, we place the first inspection at time point 1 or 2 and the second one at time points 3 or 4. The joint distribution of (j_2, j_1) is defined to be $q_{(j_2,j_1)} = q_{j_2\cdot}\, q_{\cdot j_1}$ (independent compound):

(j_2, j_1)	2	3	4	$q_{j_2\cdot}$
1	0	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
2	0	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{2}{3}$
3	0	0	0	0
$q_{\cdot j_1}$	0	$\frac{2}{3}$	$\frac{1}{3}$	

Due to the construction it is clear that (4.20) and (4.27) are fulfilled. However, (4.19) is violated. This example shows that it is not sufficient to specify the marginal distributions such that the optimal expected interim inspection time points are $(N+1)/3$ and $2\,(N+1)/3$. If $(N-1)/3$ or $N/3$ is an integer, then an optimal strategy exists which mixes only *three* pure strategies of the Inspectorate:

- If $(N-1)/3$ is an integer, then $q^*_{(j_2,j_1)}$ given by

(j_2, j_1)	$2\,\frac{N-1}{3}+1$	$2\,\frac{N-1}{3}+2$
$\frac{N-1}{3}$	$\frac{1}{3}$	0
$\frac{N-1}{3}+1$	$\frac{1}{3}$	$\frac{1}{3}$

is an optimal strategy of the Inspectorate: We get for $j_2 = (N-1)/3$

$$\sum_{j_1=j_2+1}^{N} j_1\, q^*_{(j_2,j_1)} = \frac{1}{3}\left(2\,\frac{N-1}{3}+1\right) = \frac{N+1+(N-1)/3}{2}\,\frac{1}{3}\,,$$

for $j_2 = (N-1)/3+1$

$$\sum_{j_1=j_2+1}^{N} j_1\, q^*_{(j_2,j_1)} = \frac{1}{3}\left(4\,\frac{N-1}{3}+3\right) = \frac{N+1+(N-1)/3+1}{2}\,\frac{2}{3}\,,$$

and for any $j_2 \in \{1, \ldots, N-1\} \setminus \{(N-1)/3, (N-1)/3+1\}$

$$\sum_{j_1=j_2+1}^{N} j_1\, q^*_{(j_2,j_1)} = \sum_{j_1=j_2+1}^{N} j_1\, 0 = 0 = \frac{N+1+j_2}{2}\, 0\,.$$

Thus, (4.19) is fulfilled. Furthermore, we have

$$\sum_{j_2=1}^{N-1} j_2\, q^*_{j_2\cdot} = \frac{1}{3}\,\frac{N-1}{3} + \frac{2}{3}\left(\frac{N-1}{3}+1\right) = \frac{N+1}{3}\,,$$

i.e., (4.20) is valid.

- If $N/3$ is an integer, then $q^*_{(j_2,j_1)}$ given by

(j_2, j_1)	$2\,\frac{N}{3}$	$2\,\frac{N}{3}+1$
$\frac{N}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{N}{3}+1$	0	$\frac{1}{3}$

is an optimal strategy of the Inspectorate: We obtain for $j_2 = N/3$

$$\sum_{j_1=j_2+1}^{N} j_1\, q^*_{(j_2,j_1)} = \frac{1}{3}\left(4\,\frac{N}{3}+1\right) = \frac{N+1+N/3}{2}\,\frac{2}{3}\,,$$

for $j_2 = N/3+1$

$$\sum_{j_1=j_2+1}^{N} j_1\, q^*_{(j_2,j_1)} = \frac{1}{3}\left(2\,\frac{N}{3}+1\right) = \frac{N+1+N/3+1}{2}\,\frac{1}{3}\,,$$

and for any $j_2 \in \{1, \ldots, N-1\} \setminus \{N/3, N/3+1\}$

$$\sum_{j_1=j_2+1}^{N} j_1\, q^*_{(j_2,j_1)} = \sum_{j_1=j_2+1}^{N} j_1\, 0 = 0 = \frac{N+1+j_2}{2}\, 0\,,$$

i.e., (4.19) is fulfilled. (4.20) holds as well:

$$\sum_{j_2=1}^{N-1} j_2\, q^*_{j_2\cdot} = \frac{2}{3}\,\frac{N}{3} + \frac{1}{3}\left(\frac{N}{3}+1\right) = \frac{N+1}{3}\,.$$

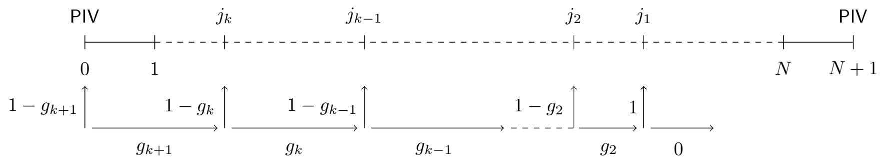
A general conjecture about the minimal number of pure strategies to be mixed is given in the remarks after Lemma 4.4.

Third, as mentioned on p. 25, in general it is easier to prove that optimal strategies satisfy the saddle point criteria than to find them. In the case of $k=1$ interim inspection it is not difficult to find the solution since the game in Figure 4.2 can be transformed into a $2 \times N$ matrix game for which simple solution recipes exist. For the case of $k=2$ interim inspections, for which the extensive form is the appropriate representation – of course it also can be transformed into a normal form, i.e., a matrix game – a combination of backward induction and indifference principle, see Theorem 19.1, is helpful for finding optimal strategies: If we put $g_2^*(j_2) = 1/2$ for all $j_2 = 1, \ldots, N-1$, then we get in case of legal behaviour at time point 0 (ℓ_3) for all j_2, $j_2 = 1, \ldots, N-1$, the detection time $(N+1-j_2)/2$; see Figure 4.4. If we then put $1 - g_3^* = 1/3$, then we get the payoff $(N+1)/3$ for all choices (j_2, j_1) with $0 < j_2 < j_1 < N+1$ of the Inspectorate.

Finally, again note that further properties of the optimal strategies and the optimal payoff of the Se-No inspection game are given in Lemma 4.4, in the comments after its proof, and in Table 4.2.

We now turn to the general case of k interim inspections. According to assumption (vii'), the Inspectorate chooses k interim inspection time points $(j_k, \ldots, j_1)$ with $j_{k+1} := 0 < j_k < \ldots < j_1 < j_0 := N+1$ at the beginning of the reference time interval $[0, N+1]$; see Figure 4.5. Again we apply the backward numbering, see the comment after (3.50): The $(k-n+1)$-th interim inspection is performed at time point j_n, $n = 2, \ldots, k$.

Figure 4.5 Time line of the interim inspections and probabilities for starting or postponing the illegal activity for the Se-No inspection game with $N > k$ possible time points for k interim inspections. For reasons of clarity we write g_n instead of $g_n(j_n)$, $n = 2, \ldots, k$.



If we define for all $n = 1, \ldots, k$

$$S_n := \left\{ (j_k, \ldots, j_n) \in \mathbb{N}^{k-n+1} : 0 < j_k < \ldots < j_n < N - n + 2 \right\}, \tag{4.28}$$

then $J_{N,k} := S_1$ is the set of pure strategies of the Inspectorate and – as a generalization of (3.14) and (3.52) –

$$Q_{N,k} := \left\{ \mathbf{q} := (q_{(1,\ldots,k)}, \ldots, q_{(N-k+1,\ldots,N)})^T \in [0,1]^{\binom{N}{k}} : \sum_{(j_k,\ldots,j_1) \in S_1} q_{(j_k,\ldots,j_1)} = 1 \right\} \tag{4.29}$$

its set of mixed strategies.

According to assumption (vii'), the Operator starts the illegal activity at time point 0 with probability $1 - g_{k+1}$ or he postpones its start with probability g_{k+1}, in the latter case he starts it at j_k with probability $1 - g_k(j_k)$ which depends on j_k or he postpones its start again with probability $g_k(j_k)$. The $(k-n+1)$-th interim inspection is performed at time point j_n, $n = 2, \ldots, k$; see Figure 4.5. Then the Operator starts the illegal activity at time point j_n with probability $1 - g_n(j_n)$ and postpones its start again with probability $g_n(j_n)$. If he does not start the illegal activity before time point j_1, he has to do it at j_1, i.e., $g_1(j_1) = 0$. Again, we exclude $g_1(j_1)$ from the Operator's strategy set and define – as a generalization of (4.15) – the set of behavioural strategies G_k of the Operator by

$$G_k := \Big\{ \mathbf{g} := (g_{k+1}, \ldots, g_2) : g_{k+1} \in [0,1]\,,$$
$$g_n : \{k-n+1, \ldots, N-n\} \to [0,1]\,, \quad n = 2, \ldots, k \Big\}\,. \tag{4.30}$$

The time line of the interim inspections and probabilities for starting or postponing the illegal activity is represented in Figure 4.5. Note that like in the case of $k = 2$ interim inspections on p. 58 the probabilities $g_n(N-n+1)$, $n = 2, \ldots, k$, are excluded from the Operator's strategy set, because if $j_n = N-n+1$ for some n, then $j_{n-1} = N-n+2, \ldots, j_1 = N$ and the payoff to the Operator is 1 no matter at which of the time points $N-n+1, \ldots, N$ he starts the illegal activity.

A word on modelling: If the Operator behaves legally until time point j_n, then the probability $1 - g_n$ to start the illegal activity at j_n is modelled as a function of j_n only, because the Operator's payoff in the remaining game, i.e., the game starting at time point j_n, depends only on the time points $j_{n-1}, \ldots, j_1$ and not on $j_k, \ldots, j_{n+1}$. Thus, we model $g_n = g_n(j_n)$ for all $n = 2, \ldots, k$ and for all $j_n = k-n+1, \ldots, N-n+1$. Furthermore, it can be seen in the proof of Theorem 4.1 that a dependence of g_n also on the time points $j_k, \ldots, j_{n+1}$, i.e., $g_n = g_n(j_n | j_k, \ldots, j_{n+1})$, leads to the same optimal strategies.

Like in Lemmata 4.2 and 4.3, we assume in the following that $k < N$, because if $k = N$ then the fixed detection time 1 is achieved independently of the Operator's behaviour. In Sections 6.1 and 6.3, however, also the case $k = N$ for special values of N is considered, because the introduction of an error of the second kind implies that the detection time in case of $k = N$ depends on the Operator's behaviour.

For $2 \le k < N$ the (expected) payoff to the Operator, i.e., the expected detection time, is given as follows: For any fixed vector of interim inspection time points $(j_k, \ldots, j_1) \in S_1$, three types of detection times, i.e., differences between interim inspection time points, occur, namely

- j_k in case the illegal activity is started at the beginning of the reference time interval (with probability $1 - g_{k+1}$);
- $(j_{n-1} - j_n)$, $n = 2, \ldots, k$, in case the illegal activity is only started at time point j_n (with probability $g_{k+1} \prod_{\ell=n+1}^{k} g_\ell(j_\ell)\,(1 - g_n)$);
- $(N + 1 - j_1)$ in case the start of the illegal activity is postponed until time point j_1 (with probability $g_{k+1} \prod_{\ell=2}^{k} g_\ell(j_\ell)$). According to assumption (iv) of Chapter 2, the Operator must start the illegal activity then at time point j_1.

Thus, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $N > k$, any $\mathbf{g} \in G_k$ and any $\mathbf{q} \in Q_{N,k}$, given by

$$Op_{N,k}(\mathbf{g}, \mathbf{q}) := \sum_{(j_k,\ldots,j_1) \in S_1} q_{(j_k,\ldots,j_1)} \Big[(1 - g_{k+1})\, j_k \tag{4.31}$$
$$+\; g_{k+1} \sum_{n=2}^{k} (1 - g_n(j_n))\,(j_{n-1} - j_n) \prod_{\ell=n+1}^{k} g_\ell(j_\ell)$$
$$+\; g_{k+1} \prod_{\ell=2}^{k} g_\ell(j_\ell)\,(N + 1 - j_1) \Big]\,,$$

with $\prod_{\ell=k+1}^{k} g_\ell(j_\ell) := 1$. Let the random variable $T_n, n = 1, \ldots, k$, be the $(k-n+1)$-th interim inspection time point. Then the realization of T_n is j_n, and the expected interim inspection time point $\mathbb{E}_{\mathbf{q}}(T_n)$ is, for a mixed strategy $\mathbf{q} \in Q_{N,k}$, given by

$$\mathbb{E}_{\mathbf{q}}(T_n) := \sum_{(j_k,\ldots,j_1)\in S_1} q_{(j_k,\ldots,j_1)}\, j_n\,, \qquad n = 1, \ldots, k\,. \tag{4.32}$$

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Theorem 4.1. *Given the Se-No inspection game with $N > k$ possible time points for k interim inspections. The sets of behavioural resp. mixed strategies are given by (4.30) and (4.29), and the payoff to the Operator by (4.31).*

Then an optimal strategy of the Operator is given by

$$g_{k+1}^* = \frac{k}{k+1} \quad \textit{and} \tag{4.33}$$

$$g_n^*(j_n) = \frac{n-1}{n} \quad \textit{for all } j_n = k-n+1, \ldots, N-n \textit{ and all } n = 2, \ldots, k\,, \tag{4.34}$$

and any mixed strategy from the set

$$Q_{N,k}^* = \Big\{ \mathbf{q}^* \in Q_{N,k} : \sum_{j_1=j_2+1}^{N} q^*_{(j_k,\ldots,j_1)}\,(N+1-2\,j_1+j_2) = 0 \textit{ for any } (j_k,\ldots,j_2) \in S_2\,,$$

$$\textit{for all } n = 2,\ldots,k-1 \textit{ and any } (j_k,\ldots,j_{n+1}) \in S_{n+1} :$$

$$\sum_{j_n=j_{n+1}+1}^{N-n+1} \cdots \sum_{j_1=j_2+1}^{N} q^*_{(j_k,\ldots,j_1)}\,(j_{n-1} - 2\,j_n + j_{n+1}) = 0\,,$$

$$\sum_{j_k=1}^{N-k+1} \cdots \sum_{j_1=j_2+1}^{N} q^*_{(j_k,\ldots,j_1)}\,(j_{k-1} - 2\,j_k) = 0 \Big\} \neq \emptyset \tag{4.35}$$

is an optimal strategy of the Inspectorate.

The optimal payoff to the Operator is

$$Op_{N,k}^* := Op_{N,k}(\mathbf{g}^*, \mathbf{q}^*) = \frac{N+1}{k+1}\,. \tag{4.36}$$

Proof. We have to show that the saddle point criterion

$$Op_{N,k}(\mathbf{g}, \mathbf{q}^*) \leq Op_{N,k}^* \leq Op_{N,k}(\mathbf{g}^*, \mathbf{q})\,, \tag{4.37}$$

see (19.10), is fulfilled for any $\mathbf{g} \in G_k$ and any $\mathbf{q} \in Q_{N,k}$. Using (4.33) and (4.34), we get for all $n = 1, \ldots, k-1$

$$g_{k+1}^* \prod_{\ell=n+1}^{k} g_\ell^*(j_\ell) = \frac{k}{k+1}\,\frac{k-1}{k}\,\cdots\,\frac{n}{n+1} = \frac{n}{k+1}\,, \tag{4.38}$$

and therefore, using (4.29) and (4.31),

$$Op_{N,k}(\mathbf{g}^*,\mathbf{q}) = \frac{1}{k+1} \sum_{(j_k,\dots,j_1)\in S_1} q_{(j_k,\dots,j_1)} \Big[j_k + \sum_{n=2}^{k} (j_{n-1}-j_n) + (N+1-i_1) \Big]$$

$$= \frac{N+1}{k+1}\,,$$

i.e., (4.36) for any $\mathbf{q} \in Q_{N,k}$. That is, the right hand inequality of (4.37) is fulfilled as equality.

For the proof of the left hand inequality of (4.37) we first note that $Q^*_{N,k} \neq \emptyset$, which is proven in Lemma 4.4. Let $\mathbf{q}^* \in Q^*_{N,k}$ be arbitrary. We show by induction that the terms of $Op_{N,k}(\mathbf{g},\mathbf{q}^*)$ containing g_{k+1} and $g_n(j_n)$ are all equal to zero. Thus, g_{k+1} and $g_n(j_n)$ in $Op_{N,k}(\mathbf{g},\mathbf{q}^*)$ can be put to zero.

We start the induction with $g_2(j_2)$, i.e., $n=2$: For the terms of $Op_{N,k}(\mathbf{g},\mathbf{q}^*)$ containing $g_2(j_2)$ we obtain by (4.31)

$$\begin{aligned}
&\sum_{(j_k,\dots,j_1)\in S_1} q^*_{(j_k,\dots,j_1)} \Big[g_{k+1} \prod_{\ell=2}^{k} g_\ell(j_\ell) \Big(-(j_1-j_2) + (N+1-j_1) \Big) \Big] \\
&= \sum_{j_k=1}^{N-k+1} \cdots \sum_{j_2=j_3+1}^{N-1} \Big[g_{k+1} \prod_{\ell=2}^{k} g_\ell(j_\ell) \sum_{j_1=j_2+1}^{N} q^*_{(j_k,\dots,j_1)} (N+1-2\,j_1+j_2) \Big] \qquad (4.39) \\
&= 0\,,
\end{aligned}$$

because of (4.35). Because $g_2(j_2)$ appears in the equations above only in multiplicative form and because all the terms containing $g_2(j_2)$ are zero, see (4.39), we can simply put $g_2(j_2)=0$ for any $(j_k,\dots,j_2)\in S_2$.

Suppose that for an index $n \le k-1$ we have $g_2(j_2) = g_3(j_3) = \dots = g_n(j_n) = 0$. For the terms of $Op_{N,k}(\mathbf{g};\mathbf{q}^*)$ containing $g_{n+1}(j_{n+1})$ we obtain again with (4.31)

$$\begin{aligned}
&\sum_{(j_k,\dots,j_1)\in S_1} q^*_{(j_k,\dots,j_1)} \Big[g_{k+1} \prod_{\ell=n+1}^{k} g_\ell(j_\ell) \Big((j_n-j_{n+1}) + (j_{n-1}-j_n) \Big) \Big] \\
&= \sum_{j_k=1}^{N-k+1} \cdots \sum_{j_{n+1}=j_{n+2}+1}^{N-(n+1)+1} \Big[g_{k+1} \prod_{\ell=n+1}^{k} g_\ell(j_\ell) \sum_{j_n=j_{n+1}+1}^{N-n+1} \cdots \sum_{j_1=j_2+1}^{N} \\
&\qquad\qquad q^*_{(j_k,\dots,j_1)} (j_{n-1} - 2\,j_n + j_{n+1}) \Big] = 0\,,
\end{aligned}$$

again due to (4.35). Therefore, we have $g_2(j_2) = g_3(j_3) = \dots = g_k(j_k) = 0$, and (4.31) and (4.35) yields for $n=k$

$$\begin{aligned}
Op_{N,k}(\mathbf{g},\mathbf{q}^*) &= \sum_{(j_k,\dots,j_1)\in S_1} q^*_{(j_k,\dots,j_1)} \Big[(1-g_{k+1})\, j_k + g_{k+1}\,(j_{k-1}-j_k) \Big] \\
&= \sum_{(j_k,\dots,j_1)\in S_1} q^*_{(j_k,\dots,j_1)}\, j_k \qquad (4.40)
\end{aligned}$$

for any $\mathbf{g} \in G_k$. Since the linear equations in (4.35) are fulfilled for all $n = 1, \ldots, k$, we obtain, using (4.32),

$$\mathbb{E}_{\mathbf{q}^*}(T_{n-1}) - 2\,\mathbb{E}_{\mathbf{q}^*}(T_n) + \mathbb{E}_{\mathbf{q}^*}(T_{n+1}) = 0 \tag{4.41}$$

for all $n = 1, \ldots, k$ with $\mathbb{E}_{\mathbf{q}^*}(T_0) := N + 1$ and $\mathbb{E}_{\mathbf{q}^*}(T_{k+1}) := 0$. It can now be seen that

$$\mathbb{E}_{\mathbf{q}^*}(T_n) = (k - n + 1)\,\frac{N+1}{k+1}\,, \qquad n = 1, \ldots, k \tag{4.42}$$

is the only solution of (4.41). Relation (4.42) holds self-evidently for any $\mathbf{q}^* \in Q^*_{N,k}$. Thus (4.40) and (4.42) finally imply

$$Op_{N,k}(\mathbf{g}, \mathbf{q}^*) = \sum_{(j_k,\ldots,j_1) \in S_1} j_k\, q^*_{(j_k,\ldots,j_1)} = \mathbb{E}_{\mathbf{q}^*}(T_k) = \frac{N+1}{k+1} \tag{4.43}$$

for any $\mathbf{g} \in G_k$, i.e., the left hand inequality of (4.37) is also fulfilled as equality. □

Using (4.41), the rather abstract set (4.35) of optimal strategies of the Inspectorate becomes meaningful: The set of optimal strategies of the Inspectorate can be fully characterized by the uniquely determined expected interim inspection time points (4.42): Any $\mathbf{q}^* \in Q^*_{N,k}$ fulfils (4.42), and for any $\mathbf{q}^* \in Q_{N,k}$ which fulfils (4.42), we get $\mathbf{q}^* \in Q^*_{N,k}$. We will return to this important result in Chapters 10 to 12; see pp. 189 and 240.

Important properties of the optimal strategies and the optimal expected detection time are given in the Lemma 4.4, which is published in this monograph for the first time. As mentioned in Table 1.1 we are able – in contrast to abstract matrix games – to consider system quantities like expected interim inspections time points and the expected time point for the start of the illegal activity, which are both interesting from a practitioner's point of view.

Lemma 4.4. *Given the Se-No inspection game with $N > k$ possible time points for k interim inspections analysed in Theorem 4.1.*

Then the following assertions hold:

1. *The mixed strategy*

$$q^*_{(j_k,\ldots,j_1)} = \binom{N}{k}^{-1} \qquad \text{for any} \qquad (j_k, \ldots, j_1) \in S_1 \tag{4.44}$$

is an optimal strategy of the Inspectorate, i.e., an element of $Q^*_{N,k}$.

2. *If $(N + 1)/(k + 1)$ is an integer, then the mixed strategy*

$$q^*_{(j_k,\ldots,j_1)} = \begin{cases} 1 & \text{for} \quad (j_k, \ldots, j_1) = \left(\frac{N+1}{k+1}, 2\,\frac{N+1}{k+1}, \ldots, k\,\frac{N+1}{k+1}\right) \\ 0 & \text{otherwise} \end{cases} \tag{4.45}$$

is an optimal strategy of the Inspectorate, i.e., an element of $Q^*_{N,k}$.

3. *The optimal expected interim inspection time points $\mathbb{E}^*_{\mathbf{q}^*}(T_n)$ fulfil the relation*

$$\mathbb{E}_{\mathbf{q}^*}(T_n) = (k - n + 1)\,\frac{N+1}{k+1} = (k - n + 1)\,Op^*_{N,k} \tag{4.46}$$

for all $n = 1, \ldots, k$ *and any optimal strategy* $\mathbf{q}^* \in Q^*_{N,k}$*, and the relation*

$$\mathbb{E}_{\mathbf{q}^*}(T_n) - \mathbb{E}_{\mathbf{q}^*}(T_{n+1}) = \frac{N+1}{k+1}\,, \tag{4.47}$$

for all $n = 0, \ldots, k$*, where with* $\mathbb{E}_{\mathbf{q}^*}(T_0) := N+1$ *and* $\mathbb{E}_{\mathbf{q}^*}(T_{k+1}) := 0$*.*

4. *The optimal payoff to the Operator* $Op^*_{N,k}$ *and the optimal expected time point of the last interim inspection* $\mathbb{E}_{\mathbf{q}^*}(T_1)$ *fulfil the relation*

$$Op^*_{N,k} + \mathbb{E}_{\mathbf{q}^*}(T_1) = N+1 \tag{4.48}$$

for any optimal strategy $\mathbf{q}^* \in Q^*_{N,k}$*.*

5. *The optimal expected time point for the start of the illegal activity* $\mathbb{E}_{(\mathbf{g}^*,\mathbf{q}^*)}(S)$ *is given by*

$$\mathbb{E}_{(\mathbf{g}^*,\mathbf{q}^*)}(S) = \frac{k}{k+1}\,\frac{N+1}{2} = \frac{k}{2}\,Op^*_{N,k} \tag{4.49}$$

for $\mathbf{g}^*$ *given by (4.33) and (4.34), and any optimal strategy* $\mathbf{q}^* \in Q^*_{N,k}$*.*

Proof. 1. We use the following binomial formula

$$\sum_{i=a}^{b} \binom{i}{a} = \binom{b+1}{a+1} \qquad \text{for all } a, b \in \mathbb{N}_0 \text{ with } a < b\,, \tag{4.50}$$

to prove

$$\sum_{(j_k,\ldots,j_1)\in S_1} j_n = (k-n+1)\binom{N+1}{k+1} \qquad \text{for all } n = 1, \ldots, k\,. \tag{4.51}$$

The proof of (4.50) can be found in van Lint and Wilson (1992), or directly shown by induction. In fact, the left hand side of (4.51) is for $n = 1, \ldots, k-1$, using (4.50) and S_1 given by (4.28), equivalent to

$$\begin{aligned}
\sum_{(j_k,\ldots,j_1)\in S_1} j_n &= \sum_{j_1=k}^{N} \cdots \sum_{j_n=k-n+1}^{j_{n-1}-1} j_n \sum_{j_{n+1}=k-n}^{j_n-1} \cdots \sum_{j_{k-2}=3}^{j_{k-3}-1} \sum_{j_{k-1}=2}^{j_{k-2}-1} \sum_{j_k=1}^{j_{k-1}-1} 1 \\
&= \sum_{j_1=k}^{N} \cdots \sum_{j_n=k-n+1}^{j_{n-1}-1} j_n \sum_{j_{n+1}=k-n}^{j_n-1} \cdots \sum_{j_{k-2}=3}^{j_{k-3}-1} \sum_{j_{k-1}=2}^{j_{k-2}-1} (j_{k-1}-1) \\
&= \sum_{j_1=k}^{N} \cdots \sum_{j_n=k-n+1}^{j_{n-1}-1} j_n \sum_{j_{n+1}=k-n}^{j_n-1} \cdots \sum_{j_{k-2}=3}^{j_{k-3}-1} \binom{j_{k-2}-1}{2} \\
&= \ldots = \sum_{j_1=k}^{N} \cdots \sum_{j_n=k-n+1}^{j_{n-1}-1} j_n \binom{j_n-1}{k-n}\,.
\end{aligned} \tag{4.52}$$

Note that (4.52) is also valid for $n = k$ due to the definition of S_1 and without utilizing (4.50). Since we have for all $n = 1, \dots, k$

$$j_n \binom{j_n - 1}{k - n} = (k - n + 1) \binom{j_n}{k - n + 1},$$

we get, using (4.50) and (4.52), for all $n = 1, \dots, k$

$$\begin{aligned}
\sum_{(j_k,\dots,j_1)\in S_1} j_n &= (k - n + 1) \sum_{j_1=k}^{N} \dots \sum_{j_n=k-n+1}^{j_{n-1}-1} \binom{j_n}{k - n + 1} \\
&= (k - n + 1) \sum_{j_1=k}^{N} \dots \sum_{j_{n-1}=k-n+2}^{j_{n-2}-1} \binom{j_{n-1}}{k - n + 2} \\
&= \dots = (k - n + 1) \sum_{j_1=k}^{N} \binom{j_1}{k} = (k - n + 1) \binom{N + 1}{k + 1},
\end{aligned}$$

i.e., (4.51) for all $n = 1, \dots, k$.

Now, the system of linear equations given by (4.35) is, using (4.44), equivalent to the equations

$$\sum_{(j_k,\dots,j_1)\in S_1} (N + 1 - 2\, j_1 + j_2) = 0\,,$$

$$\sum_{(j_k,\dots,j_1)\in S_1} (j_{n-1} - 2\, j_n + j_{n+1}) = 0\,,$$

$$\sum_{(j_k,\dots,j_1)\in S_1} (j_{k-1} - 2\, j_k) = 0\,,$$

the validity of which can be seen immediately, if we use (4.51) with appropriately chosen n: For $n = 1$ we get

$$\begin{aligned}
&\sum_{(j_k,\dots,j_1)\in S_1} (N + 1 - 2\, j_1 + j_2) \\
&= \binom{N + 1}{k + 1} ((k - 2 + j) - 2\,(k - 1 + 1)) + \binom{N}{k} (N + 1) \\
&= -\binom{N + 1}{k + 1} (k + 1) + \binom{N}{k} (N + 1) = 0\,.
\end{aligned}$$

For $n = 2, \dots, k - 1$ we obtain

$$\begin{aligned}
&\sum_{(j_k,\dots,j_1)\in S_1} (j_{n-1} - 2\, j_n + j_{n+1}) \\
&= \binom{N + 1}{k + 1} ((k - (n + 1) + 1) - 2\,(k - n + 1) + k - (n - 1) + 1) = 0\,,
\end{aligned}$$

and finally for $n = k$

$$\sum_{(j_k,\dots,j_1)\in S_1} (j_{k-1} - 2\,j_k)$$

$$= \binom{N+1}{k+1} \left(-2\,(k-k+1) - 2\,(k-n+1) + k - (k-1) + 1\right) = 0\,,$$

which completes the proof of the first assertion.

2. The vector $(j_k, \dots, j_1)$ given by (4.45) fulfills

$$N + 1 - 2\,j_1 + j_2 = 0 \qquad \text{and} \qquad j_{k-1} - 2\,j_k = 0$$

and for all $n = 2, \dots, k-1$

$$j_{n-1} - 2\,j_n + j_{n+1} = 0\,.$$

Therefore, the linear equations in (4.35) are fulfilled and $q^*_{(j_k,\dots,j_1)}$ given by (4.45) is an element of $Q^*_{N,k}$.

3. Relation (4.46) has already been shown in the proof of Theorem 4.1; see the derivations of (4.41) and (4.42). (4.47) follows directly from (4.46).

4. (4.48) follows directly from (4.43) and (4.46).

5. The expected time point for the start of the illegal activity $\mathbb{E}_{(\mathbf{g},\mathbf{q})}(S)$ is, using Figure 4.5, for any $\mathbf{g} \in G_k$ and any $\mathbf{q} \in Q_{N,k}$ given by (recall $\prod_{\ell=k+1}^{k} g_\ell(j_\ell) := 1$)

$$\mathbb{E}_{(\mathbf{g},\mathbf{q})}(S) = (1 - g_{k+1})\,0$$

$$+ \, g_{k+1} \sum_{(j_k,\dots,j_1)\in S_1} \left(\sum_{n=2}^{k} j_n\,(1 - g_n(j_n)) \prod_{\ell=n+1}^{k} g_\ell(j_\ell) + j_1 \prod_{\ell=2}^{k} g_\ell(j_\ell) \right)\,.$$

Because of (4.34), $g^*_n(j_n)$, $n = 2, \dots, k$, is independent of j_n, and we get, using (4.32) and (4.38), with $g^*_n := g^*_n(j_n)$ for any $\mathbf{q} \in Q_{N,k}$

$$\mathbb{E}_{(\mathbf{g}^*,\mathbf{q})}(S) = \frac{1}{k+1} \sum_{(j_k,\dots,j_1)\in S_1} \sum_{n=1}^{k} j_n = \frac{1}{k+1} \sum_{n=1}^{k} \mathbb{E}_{\mathbf{q}^*}(T_n)\,,$$

which in itself is an interesting result. By (4.46) we get

$$\mathbb{E}_{(\mathbf{g}^*,\mathbf{q}^*)}(S) = \frac{1}{k+1}\,\frac{N+1}{k+1} \sum_{n=1}^{k} (k - n + 1) = \frac{k}{k+1}\,\frac{N+1}{2}\,,$$

which completes the proof. □

Let us conclude this chapter with a few remarks on Theorem 4.1 and Lemma 4.4, and with some general observations. The reader is also referred to Table 13.2 on p. 271 for a comparison of the discrete time and continuous time Se-No inspection game.

First, there is no cut-off value for optimal strategies like in the No-No inspection game. We will observe the same feature when we analyse the continuous time Se-No inspection game in Chapter 10.

Second, looking at the optimal strategy of the Operator, see (4.33) and (4.34), we see that $1 - g_2^*(j_2), \ldots, 1 - g_k^*(j_k), 1 - g_{k+1}^*$ form a harmonic progression

$$\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k-1}, \frac{1}{k}, \frac{1}{k+1}, \tag{4.53}$$

since their reciprocals form an arithmetic progression. Because property (4.53) is a characteristic of many Se-No and Se-Se inspection games treated in this monograph, we present in Table 4.1 an overview for easier reference. Also note that the optimal probabilities (4.33) and (4.34) depend only on the number of remaining interim inspections.

Table 4.1 Overview of Se-No and Se-Se inspection games treated in this monograph in which the components of the Operator's optimal/equilibrium strategy form a harmonic progression.

Discrete time	
Se-No inspection game	Theorem 4.1, p. 66
Se-Se inspection game	Theorems 5.2 and 5.3 pp. 82 and 88
Continuous time	
Se-No inspection game, $\beta \geq 0$ one facility	Theorem 10.1, p. 194
Se-Se inspection game, $\alpha \geq 0, \beta \geq 0$	Theorem 12.1, p. 253 only for $\alpha = 0$
Critical time	
Se-Se inspection game, $\beta \geq 0$	Theorem 16.2, p. 332
Generalized Thomas-Nisgav inspection game, $\beta \geq 0$	Theorem 17.1, p. 366
Baston-Bostock inspection game, $w_1, w_2, w \in (0, 1]$	Theorem 17.2, p. 372
Se-No inspection game with an *expected* number of inspections, $\beta \geq 0$	Lemma 24.2, p. 434

Third, the equal distribution is according to (4.44) an optimal strategy of the Inspectorate, other than in the No-No inspection game; see the remarks on pp. 25 and 35. Also, extending the results on p. 62, we dare to formulate the conjecture that in case of k interim inspections there exists an optimal strategy which mixes exactly $k+1$ pure strategies. Let us add that we do not emphasize these results per se, however, we think it is important to know that there are more optimal strategies which might be interesting from a practitioner's point of view: On p. 62, e.g., we argued that it might be the better, the fewer pure strategies are mixed. Quite generally we think that optimal strategies should be selected this way and not with the methods of Nash equilibrium selection theory; see van Damme (1987) or Harsanyi and Selten (1988).

Fourth, the pure strategy (4.45) can be an optimal strategy of the Inspectorate, other than in the No-No inspection game. This is not so surprising since for the continuous time Se-No and

Se-Se inspection games in Part II there always exist an optimal pure strategy resp. a pure Nash equilibrium strategy of the Inspectorate; see Chapters 10 and 12 and Table 13.2.

Fifth, the optimal expected detection time $(N+1)/(k+1)$ for the Se-No inspection game is surprisingly simple, other than that for the No-No inspection games in Chapters 3 and 9; see Tables 4.2 and 13.1: The length of the reference time interval is just related to the number of interim inspections. Also, we will obtain similar results for the continuous time Se-No and Se-Se inspection games in Part II, at least for $\beta = 0$.

Sixth, it is interesting to see that the optimal expected interim inspection time points and the optimal expected time point for the start of the illegal activity are closely related to the optimal expected detection time, thus being the central figure in the game; see (4.46) and (4.49). The optimal expected time point for the start of the illegal (4.49) increases with increasing k and tends towards $(N+1)/2$, i.e., the middle of the reference time interval.

Seventh, Table 4.2 represents an overview of the system quantities of discrete time No-No and Se-No inspection games; see Sections 3.1, 4.2, 6.1 and 6.3.

A final remark: We found a solution for the No-No inspection game for any number N of possible time points only for $k = 1$ interim inspection, whereas for the Se-No inspection game we were able to present a solution for any number $N > k$ of possible time points for k interim inspections. This is surprising since at first sight the differences between these inspection games do not appear so essential. The only consolidation is that this happens frequently in Mathematics. Why, e.g., is the formula for the area within an ellipse so close to that within a circle, but that for its circumference so different?

Table 4.2 System quantities of the No-No and Se-No inspection games of Sections 3.1, 4.2 and 6.3.

	Cut-off value	Optimal expected detection time	Optimal expected time point for the start of the illegal activity	Optimal expected interim inspection time point(s)
No-No for any N, $k=1$ and $\beta \geq 0$	$n^* = \min \left\{ n : \sum_{j=1}^{n} \frac{1}{N-j+1} \geq 1-\beta \right\}$	$Op_{N,1}^* = \sum_{j=1}^{n^*} \frac{N+1-n^*}{N+1-j} + \beta\,(N+1-n^*)$	$\mathbb{E}_{\mathbf{p}^*}(S) = n^* - Op_{N,1}^* + \beta\,(N+1-n^*)$	$\mathbb{E}_{\mathbf{q}^*}(T_1) = \frac{1}{1-\beta}\,(Op_{N,1}^* - \beta\,(N+1))$
Se-No for any N, $k<N$ and $\beta = 0$	Does not exist	$Op_{N,k}^* = \frac{N+1}{k+1}$	$\mathbb{E}_{(\mathbf{g}^*,\mathbf{q}^*)}(S) = \frac{k}{2}\,Op_{N,k}^*$	$\mathbb{E}_{\mathbf{q}^*}(T_n) = (k-n+1)\,Op_{N,k}^*$, $n=1,\ldots,k$
Se-No for any N, $k=1$ and $\beta \geq 0$	Does not exist	$Op_{N,1}^* = \frac{N+1}{2-\beta}$	$\mathbb{E}_{(g_2^*,\mathbf{q}^*)}(S) = \frac{(1-\beta)^2}{2-\beta}\,Op_{N,1}^*$	$\mathbb{E}_{\mathbf{q}^*}(T_1) = (1-\beta)\,Op_{N,1}^*$

Chapter 5

Se-Se inspection game

In this chapter the last of the four variants of the playing for time inspection game with discrete time, which have been introduced in Table 2.1, is considered. Let us repeat that in the Se-Se inspection game both the Operator and the Inspectorate behave sequentially.

In this chapter, assumption (v) of Chapter 2 is specified as follows:

(v') During an interim inspection the Inspectorate does not commit an error of the second kind, i.e., the illegal activity, see assumption (iv), is detected with certainty during the next interim inspection or with certainty during the final PIV; see assumption (iii).

Assumptions (vii) will be specified in the following sections, while the remaining assumptions of Chapter 2 hold throughout this chapter. Note that the Se-Se inspection game with $N = 3$ possible time points for $k = 2$ interim inspections and uncertain detection of an illegal activity at an interim inspection, i.e., $\beta \geq 0$, is treated in Section 6.4.

In Section 5.1 we first present the case of $N > 2$ possible time points for $k = 2$ interim inspections, since the case of $k = 1$ interim inspection is identical to that of the Se-No inspection game which is solved in Lemma 4.2 of Section 4.2. Thereafter, the case of any number $N > k$ of possible time points for k interim inspections is analysed and it is shown that it leads to the same optimal payoff as that of the Se-No inspection game treated in Section 4.2. In Section 5.2 the same Se-Se inspection game as that of Section 5.1 is solved using a recursive approach. Section 5.3 focuses on a step by step Se-Se inspection game. The game theoretical solution of this game is compared to that of the Se-Se inspection game treated in Section 5.1.

5.1 Any number of inspection opportunities and interim inspections

The inspection game analysed in this section is based on the following specification:

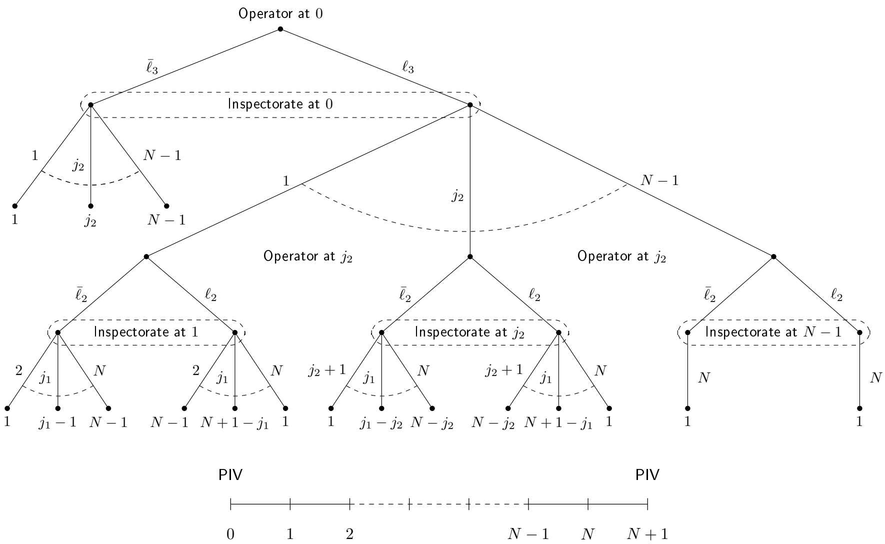
(vii') The Operator decides at the beginning of the reference time interval, i.e., at time point 0, whether to start the illegal activity immediately at time point 0 or to postpone the start; in the latter case he decides again after the first interim inspection, whether to start the illegal activity immediately at that time point or to postpone the start again; and so on.

Because of assumption (iv), the Operator starts the illegal activity latest immediately at the time point of the last interim inspection.

The Inspectorate decides at the beginning of the reference time interval when to perform its first interim inspection. At the time point of its first interim inspection, it decides when to perform the second interim inspection, and so on.

Let us start with the Se-Se inspection game with N possible time points for $k = 2$ interim inspections, the extensive form of which is given in Figure 5.1. Because both players decide sequentially in this chapter, all extensive form games start with the Operator's decision at 0; see the comment on p. 50.

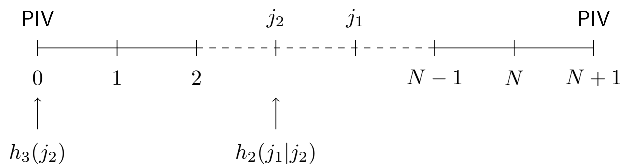
Figure 5.1 Extensive form of the Se-Se inspection game with $N > 2$ possible time points for $k = 2$ interim inspections.



Let us explain Figure 5.1 in detail. The Operator behaves in the same way as in the Se-No inspection game discussed in Section 4.2; see also Figure 4.3: At the beginning of the reference time interval he decides with probability $1 - g_3$ to start the illegal activity immediately ($\bar{\ell}_3$) or to postpone its start (ℓ_3) with probability g_3. In this latter case he decides after the first interim inspection at time point j_2 with probability $1 - g_2(j_2)$ to start the illegal activity now ($\bar{\ell}_2$) or to postpone its start again (ℓ_2) with probability $g_2(j_2)$. Because of assumption (iv) of Chapter 2 he starts the illegal activity in the latter case after the second interim inspection at time point j_1 with certainty, i.e., $g_1(j_2, j_1) = 0$. Because this probability is fixed it is excluded from the Operator's strategy set. Thus, the set G_2 of behavioural strategies of the Operator is again given by (4.15).

The Inspectorate, not knowing the Operator's decision at $j_3(= 0)$, chooses at the beginning of the reference time interval the time point $j_2 \in \{1, \ldots, N-1\}$ for its first interim inspection with probability $h_3(j_2)$; see Figure 5.2. At time point j_2 and in case the illegal activity is not started at 0 (ℓ_3), the time point j_1, $j_1 \in \{j_2+1,\ldots,N\}$, for the second interim inspection is chosen with probability $h_2(j_1|j_2)$. Thus, the Inspectorate's set $H_{N,2}$ of behavioural strategies is given by (4.6). As mentioned on p. 52, we use the notation $(j_1|j_2)$ to indicate the sequential nature of the Inspectorate's behaviour.

Figure 5.2 Time line of the interim inspections and the Inspectorate's probabilities for the Se-Se inspection game with $N > k$ possible time points for $k = 2$ interim inspections.



According to Figure 5.1, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $N \geq 3$, any $\mathbf{g} \in G_2$ and any $\mathbf{h} \in H_{N,2}$, given by

$$Op_{N,2}(\mathbf{g},\mathbf{h}) := \sum_{j_2=1}^{N-2} h_3(j_2)\Big[(1-g_3)\,j_2 + g_3 \sum_{j_1=j_2+1}^{N} h_2(j_1|j_2)\Big((1-g_2(j_2))\,(j_1-j_2) + g_2(j_2)\,(N+1-j_1)\Big)\Big] + h_3(N-1)\,[(1-g_3)\,(N-1)+g_3]\,. \tag{5.1}$$

The game theoretical solution of this inspection game, see Krieger and Avenhaus (2014), is presented in

Lemma 5.1. *Given the Se-Se inspection game with $N > 2$ possible time points for $k = 2$ interim inspections. The sets of behavioural strategies are given by (4.15) and (4.6), and the payoff to the Operator by (5.1).*

Then an optimal strategy of the Operator is given by

$$g_3^* = \frac{2}{3} \quad \textit{and} \quad g_2^*(j_2) = \frac{1}{2} \quad \textit{for} \quad j_2 = 1,\ldots,N-2\,, \tag{5.2}$$

and an optimal strategy of the Inspectorate by

$$\sum_{j_2=1}^{N-1} j_2\, h_3^*(j_2) = \frac{N+1}{3} \quad \textit{and} \tag{5.3}$$

$$\sum_{j_1=j_2+1}^{N} j_1\, h_2^*(j_1|j_2) = \frac{N+j_2+1}{2} \quad \textit{for} \quad j_2 = 1,\ldots,N-2\,. \tag{5.4}$$

The optimal payoff to the Operator is

$$Op_{N,2}^* := Op_{N,2}(\mathbf{g}^*,\mathbf{h}^*) = \frac{N+1}{3}\,.$$

Proof. (5.1) together with (5.2) – (5.4) imply

$$Op_{N,2}(\mathbf{g}^*, \mathbf{h}) = Op_{N,2}(\mathbf{g}^*, \mathbf{h}^*) = Op_{N,2}(\mathbf{g}, \mathbf{h}^*)$$

for any $\mathbf{g} \in G_2$ and any $\mathbf{h} \in H_{N,2}$, i.e., the saddle point inequality $Op_{N,2}(\mathbf{g}, \mathbf{h}^*) \leq Op^*_{N,2} \leq Op_{N,2}(\mathbf{g}^*, \mathbf{h})$ is fulfilled as equality. □

Let us comment the results of Lemma 5.1: First, we see that the optimal strategies of the Operator and the optimal payoff to the Operator are the same as that of the Se-No inspection game as given by Lemma 4.3. At first sight this is very surprising, since the extensive forms of the two inspection games are so different; compare Figures 4.4 and 5.1. However, a closer look at the two payoff functions (4.16) and (5.1) shows that they can be transformed into each other: Define for any $(h_3, h_2) \in H_{N,2}$

$$q_{(j_2,j_1)} := h_3(j_2)\, h_2(j_1|j_2) \qquad \text{for all} \qquad 0 < j_2 < j_1 < N+1\,. \tag{5.5}$$

Then we have $\mathbf{q} \in Q_{N,2}$; see (3.52). If we replace in (4.16) the $q_{(j_2,j_1)}$ by (5.5) then we get (5.1). Conversely, defining for any $\mathbf{q} \in Q_{N,2}$ the marginal probability of the first interim inspection time point by

$$q_{j_2\cdot} := \sum_{j_1=j_2+1}^{N} q_{(j_2,j_1)} \qquad \text{for all} \qquad j_2 = 1, \ldots, N-1\,,$$

and setting

$$h_3(j_2) := q_{j_2\cdot} \qquad \text{and} \qquad h_2(j_1|j_2) := \begin{cases} \dfrac{q_{(j_2,j_1)}}{q_{j_2\cdot}} & \text{for } q_{j_2\cdot} > 0 \\ [0,1] & \text{for } q_{j_2\cdot} = 0 \end{cases}\,, \tag{5.6}$$

we have $(h_3, h_2) \in H_{N,2}$. Replacing (h_3, h_2) in (5.1) by (5.6) leads to (4.16). Because the $q_{(j_2,j_1)}$'s, $h_3(j_2)$'s and $h_2(j_1|j_2)$'s sum up to one, we get – as on p. 54 – that there are $\binom{N}{2} - 1$ independent $q_{(j_2,j_1)}$'s, and

$$\underbrace{(N-1)-1}_{h_3's} + \underbrace{\sum_{j_2=1}^{N-2} \Big[(N-j_2)-1\Big]}_{h_2's} = \binom{N}{2} - 1$$

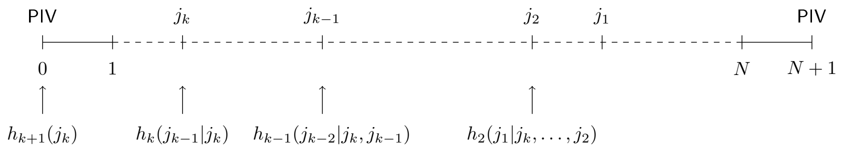
independent $h_3(j_2)$'s and $h_2(j_1|j_2)$'s. Probabilistically speaking, the h's are conditional, and the $\mathbf{q}$'s joint probabilities.

Second, also we see that here not only the optimal expected interim inspection time point of the first, but also that of the second, depending on the first one, is fixed; see (5.3) and (5.4). Furthermore, it is interesting that contrary to the situation in the Se-No inspection game, see (4.26), the equal distribution $h_3^*(j_2) = 1/(N-1)$ on the set $\{1, \ldots, N-1\}$, is *not* part of an optimal strategy, because it does not fulfil (5.3). On the other hand the equal distribution $h_2^*(j_1|j_2) = 1/(N-j_2)$ on the set $\{j_2+1, \ldots, N\}$ fulfils (5.4), i.e., it can be chosen as a component of an optimal strategy.

Let us now turn to the general case of any number $k < N$ of interim inspections. Again, the Operator behaves in the same way as in the Se-No inspection game thus, his set G_k of behavioural strategies is given by (4.30). Note that we model again g_n as a function of j_n only, i.e., $g_n = g_n(j_n)$ for all $n = 2, \ldots, k$ and for all $j_n = k - n + 1, \ldots, N - n + 1$; see p. 65 for the justification.

The Inspectorate, not knowing the Operator's decision at 0, chooses at the beginning of the reference time interval the time point $j_k \in \{1, \ldots, N - k + 1\}$ for its first interim inspection with probability $h_{k+1}(j_k)$. The $(k - n + 1)$-th interim inspection is performed at time point j_n, $n = 2, \ldots, k$, and the Inspectorate chooses at j_n the time point $j_{n-1} \in \{j_n + 1, \ldots, N - n + 1\}$ for its next interim inspection with probability $h_n(j_{n-1}|j_k, \ldots, j_n)$; see Figure 5.3. Why do we assume that h_n depends on the whole history $j_k, \ldots, j_{n-1}$? Intuitively one would assume that h_n depends only on j_n and j_{n-1} for the following reason: Suppose the start of the illegal activity is postponed until time point j_n. Then the payoff to the Inspectorate for the remaining game depends only on j_n and the remaining time points $j_{n-1}, \ldots, j_1$. Because the Inspectorate chooses at time point j_n only j_{n-1} – due to the sequential decision making –, the choice $h_n(j_{n-1}|j_n)$ would be appropriate. The reason for the assumption $h_n = h_n(j_{n-1}|j_k, \ldots, j_n)$ is that we want to use the results of Theorem 4.1 from the Se-No inspection game. To apply this Theorem we need to transform the probabilities $q_{(j_k,\ldots,j_1)}$ into $h_{k+1}(j_k), h_k(j_{k-1}|j_k), \ldots, h_2(j_1|j_k, \ldots, j_2)$, and for that purpose h_n needs to be conditioned on the whole history $j_k, \ldots, j_n$; see (5.9).

Figure 5.3 Time line of the interim inspections and the Inspectorate's probabilities for the Se-Se inspection game with $N > k$ possible time points for k interim inspections.



As a generalization of (4.6), the set of the Inspectorate's behavioural strategies is given by

$$H_{N,k} := \Big\{ \mathbf{h} := (h_{k+1}, h_k, \ldots, h_2) \in [0,1]^k : \sum_{j_k=1}^{N-k+1} h_{k+1}(j_k) = 1 \,,$$

$$\sum_{j_{n-1}=j_n+1}^{N-n+2} h_n(j_{n-1}|j_k, \ldots, j_n) = 1 \quad \text{for all} \quad n = 2, \ldots, k \tag{5.7}$$

$$\text{and all} \quad (j_k, \ldots, j_n) \in \mathbb{N}^{k-n+1} : 0 < j_k < \ldots < j_n < N - n + 1 \Big\} \,.$$

Similar to the derivations of the expected detection time for the Se-No inspection game on p. 65, we derive that of the Se-Se inspection game. Again, three types of detection times, i.e., differences between interim inspection time points, are distinguished, namely

- j_k in case the illegal activity is started at the beginning of the reference time interval (with probability $1 - g_{k+1}$), and the first interim inspection is performed at time point j_k (with probability $h_{k+1}(j_k)$). Thus, we get for the j_k component of the payoff to the Operator, using (4.28),

$$\sum_{j_k=1}^{N-k+1} h_{k+1}(j_k)\,(1 - g_{k+1})\,j_k$$

$$= \sum_{(j_k,\ldots,j_1)\in S_1} h_{k+1}(j_k) \prod_{m=2}^{k} h_m(j_{m-1}|j_k,\ldots,j_m)\,(1-g_{k+1})\,j_k\,,$$

because

$$\sum_{j_{k-1}=j_k+1}^{N-k+2} \cdots \sum_{j_2=j_3+1}^{N-1} \sum_{j_1=j_2+1}^{N} \prod_{m=2}^{k} h_m(j_{m-1}|j_k,\ldots,j_m) = 1$$

holds for all $j_k = 1,\ldots,N-k+1$.

- $(j_{n-1}-j_n)$, $n=2,\ldots,k$, in case the illegal activity is only started at time point j_n (with probability $g_{k+1}\ \prod_{\ell=n+1}^{k} g_\ell(j_\ell)\,(1-g_n(j_n))$). Similar to the derivations above, we get for the $j_{n-1}-j_n$ component of the payoff to the Operator

$$\sum_{j_k=1}^{N-k+1} \sum_{j_{k-1}=j_k+1}^{N-k+2} \cdots \sum_{j_n=j_{n+1}+1}^{N-n+1} h_{k+1}(j_k) \prod_{m=n}^{k} h_m(j_{m-1}|j_k,\ldots,j_m)$$
$$\cdot\, g_{k+1}\,(1-g_n(j_n))\,(j_{n-1}-j_n) \prod_{\ell=n+1}^{k} g_\ell(j_\ell)\,,$$

which can be written as

$$\sum_{(j_k,\ldots,j_1)\in S_1} h_{k+1}(j_k) \prod_{m=2}^{k} h_m(j_{m-1}|j_k,\ldots,j_m)$$
$$\cdot\, g_{k+1}\,(1-g_n(j_n))\,(j_{n-1}-j_n) \prod_{\ell=n+1}^{k} g_\ell(j_\ell)\,,$$

because of the convention $\prod_{m=2}^{1} \ldots =: 1$, and

$$\sum_{j_{n-1}=j_n+1}^{N-n+2} \cdots \sum_{j_2=j_3+1}^{N-1} \sum_{j_1=j_2+1}^{N} \prod_{m=2}^{n-1} h_m(j_{m-1}|j_k,\ldots,j_m) = 1$$

holds for all $j_n = k-n+1,\ldots,N-n+1$.

- $(N+1-j_1)$ in case the start of the illegal activity is postponed until time point j_1 (with probability $g_{k+1}\ \prod_{\ell=2}^{k} g_\ell(j_\ell)$). According to assumption (iv) of Chapter 2, the Operator must start the illegal activity then at time point j_1. Thus, we get for the $N+1-j_1$ component of the payoff to the Operator

$$\sum_{j_k=1}^{N-k+1} \cdots \sum_{j_2=j_3+1}^{N-1} \sum_{j_1=j_2+1}^{N} h_{k+1}(j_k) \prod_{m=2}^{k} h_m(j_{m-1}|j_k,\ldots,j_m)$$
$$\cdot\, g_{k+1} \prod_{\ell=2}^{k} g_\ell(j_\ell)\,(N+1-j_1)$$

$$= \sum_{(j_k,\ldots,j_1)\in S_1} h_{k+1}(j_k) \prod_{m=2}^{k} h_m(j_{m-1}|j_k,\ldots,j_m)\,g_{k+1} \prod_{\ell=2}^{k} g_\ell(j_\ell)\,(N+1-j_1)\,.$$

Thus, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $2 \leq k < N$, any $\mathbf{g} \in G_k$ and any $\mathbf{h} \in H_{N,k}$, given by

$$Op_{N,k}(\mathbf{g},\mathbf{h}) := \sum_{(j_k,\ldots,j_1)\in S_1} h_{k+1}(j_k) \prod_{m=2}^{k} h_m(j_{m-1}|j_k,\ldots,j_m) \Big[(1-g_{k+1})\, j_k + g_{k+1} \sum_{n=2}^{k} (1-g_n(j_n))\,(j_{n-1}-j_n) \prod_{\ell=n+1}^{k} g_\ell(j_\ell) + g_{k+1} \prod_{\ell=2}^{k} g_\ell(j_\ell)\,(N+1-j_1)\Big]\,, \tag{5.8}$$

which corresponds to the expected detection time for the Se-No inspection game as given by (4.31).

Before we turn to the solution of the Se-Se inspection game we show that the Inspectorate's strategies in the Se-No and the Se-Se inspection game are payoff equivalent, and that these strategies can be transformed into each other. Let us note that Kuhn's Theorem, see Kuhn (1953), cannot be applied here, because it deals with the transformation of behavioural into mixed strategies and vice versa within a *single* game. Here, however, we consider *two* inspection games which are different from the modelling point of view, namely the Se-No and the Se-Se inspection game, the payoffs (4.31) and (5.8) of which have to be shown to be equivalent after appropriate transformation of the Inspectorate's strategies. This is done in

Theorem 5.1. *Given the Se-No and the Se-Se inspection games with $N > k$ possible time points for k interim inspections. The Operator's set of behavioural strategies is given by (4.30) and the Inspectorate's strategy sets by (4.29) and (5.7) for the Se-No and the Se-Se inspection game, respectively. The payoffs to the Operator are given by (4.31) and (5.8).*

(i) Define for a strategy $\mathbf{q} = (q_{(1,\ldots,k)},\ldots,q_{(N-k+1,\ldots,N)})^T \in Q_{N,k}$

$$h_{k+1}(j_k) := q_{(j_k,\cdots)}\,, \quad h_k(j_{k-1}|j_k) := \frac{q_{(j_k,j_{k-1},\cdots)}}{q_{(j_k,\cdots)}}\,, \quad \ldots$$
$$h_n(j_{n-1}|j_k,\ldots,j_n) := \frac{q_{(j_k,\ldots,j_n,j_{n-1},\cdots)}}{q_{(j_k,\ldots,j_n,\cdots)}}\,,\ \ldots\,,\ h_2(j_1|j_k,\ldots,j_2) := \frac{q_{(j_k,\ldots,j_2,j_1)}}{q_{(j_k,\ldots,j_2,\cdot)}}\,, \tag{5.9}$$

where we assume that the appropriate ratios exist and that the points replacing the indices indicate their summation.

Then $\mathbf{h} := (h_{k+1}(j_k), h_k(j_{k-1}|j_k),\ldots,h_2(j_1|j_k,\ldots,j_2)) \in H_{N,k}$ and $\mathbf{h}$ is payoff equivalent to $\mathbf{q}$, i.e., we have with (4.31) and (5.8): $Op_{N,k}(\mathbf{g},\mathbf{h}) = Op_{N,k}(\mathbf{g},\mathbf{q})$ for any $\mathbf{g} \in G_k$.

(ii) Define for a strategy $\mathbf{h} = (h_{k+1}(j_k), h_k(j_{k-1}|j_k),\ldots,h_2(j_1|j_k,\ldots,j_2)) \in H_{N,k}$

$$q_{(j_k,\ldots,j_1)} := h_{k+1}(j_k) \prod_{m=2}^{k} h_m(j_{m-1}|j_m) \quad \text{for any} \quad (j_k,\ldots,j_1) \in S_1\,. \tag{5.10}$$

Then $\mathbf{q} := (q_{(1,\ldots,k)},\ldots,q_{(N-k+1,\ldots,N)})^T \in Q_{N,k}$ and $\mathbf{q}$ is payoff equivalent to $\mathbf{h}$, i.e., we have with (4.31) and (5.8): $Op_{N,k}(\mathbf{g},\mathbf{q}) = Op_{N,k}(\mathbf{g},\mathbf{h})$ for any $\mathbf{g} \in G_k$.

Proof. The assertion follows immediately from the payoffs (4.31) and (5.8) and from the normalization of the $\mathbf{q}$'s on one hand and the $\mathbf{h}$'s on the other. □

Note that with the same methods as those used at the end of Section 4.1 it can be shown that there is the same number $\binom{N}{k} - 1$ of independent $\mathbf{q}$'s and $\mathbf{h}$'s.

Now with the help of Theorems 5.1 and 4.1, the game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Theorem 5.2. *Given the Se-Se inspection game with $N > k$ possible time points for k interim inspections. The sets of behavioural strategies are given by (4.30) and (5.7), and the payoff to the Operator by (5.8).*

Then an optimal strategy of the Operator is given by (4.33) and (4.34). The set $H^*_{N,k}$ of the optimal strategies of the Inspectorate is obtained from (4.35) by taking any element $\mathbf{q}^* \in Q^*_{N,k}$ and derive $\mathbf{h}^*$ according to (5.9).

The optimal payoff to the Operator is

$$Op^*_{N,k} := Op_{N,k}(\mathbf{g}^*, \mathbf{h}^*) = \frac{N+1}{k+1}\,. \tag{5.11}$$

Proof. The statement follows directly from the Theorems 4.1 and 5.1. □

To illustrate the transformations (5.9) and (5.10) we consider the case of $N = 7$ possible time points for $k = 3$ interim inspections. Then (4.44) implies that

$$q^*_{(j_3,j_2,j_1)} = \binom{7}{3}^{-1} = \frac{1}{35} \qquad \text{for all} \qquad 0 < j_3 < j_2 < j_1 < 8\,, \tag{5.12}$$

and furthermore, (4.45) implies that

$$q^*_{(j_3,j_2,j_1)} = \begin{cases} 1 & \text{for} \quad (j_3,j_2,j_1) = (2,4,6) \\ 0 & \text{otherwise} \end{cases} \tag{5.13}$$

are optimal strategies of the Inspectorate. Let us consider (5.12) first. Because

$$q^*_{(j_3,j_2,\cdot)} = \frac{7-j_2}{35} \qquad \text{and} \qquad q^*_{(j_3,\cdot\cdot)} = \frac{1}{35}\binom{7-j_3}{2}$$

we get, using (5.9),

$$\begin{aligned} h^*_4(j_3) &= q^*_{(j_3,\cdot\cdot)} = \frac{1}{35}\binom{7-j_3}{2}\,, && 0 < j_3 < 6 \\ h^*_3(j_2|j_3) &= \frac{q^*_{(j_3,j_2,\cdot)}}{q^*_{(j_3,\cdot\cdot)}} = \frac{7-j_2}{\binom{7-j_3}{2}}\,, && 0 < j_3 < j_2 < 6 \\ h^*_2(j_1|j_3,j_2) &= \frac{q^*_{(j_3,j_2,j_1)}}{q^*_{(j_3,j_2,\cdot)}} = \frac{1}{7-j_2}\,, && 0 < j_3 < j_2 < j_1 < 8\,, \end{aligned} \tag{5.14}$$

which means that indeed $h^*_2(j_1|j_3,j_2)$ is independent of j_3. We will come back to this result in the next section on p. 90.

From (5.13) on the other hand, we get

$$q^*_{(j_3,j_2,\cdot)} = \begin{cases} 1 & \text{for } (j_3,j_2)=(2,4) \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad q^*_{(j_3,\cdot\cdot)} = \begin{cases} 1 & \text{for } j_3=2 \\ 0 & \text{otherwise} \end{cases} ,$$

and therefore,

$$h^*_4(j_3) = q^*_{(j_3,\cdot\cdot)} = \begin{cases} 1 & \text{for } j_3=2 \\ 0 & \text{otherwise} \end{cases}$$

$$h^*_3(j_2|j_3) = \frac{q^*_{(j_3,j_2,\cdot)}}{q^*_{(j_3,\cdots)}} = \begin{cases} 1 & \text{for } (j_3,j_2)=(2,4) \\ 0 & \text{otherwise} \end{cases}$$

$$h^*_2(j_1|j_3,j_2) = \frac{q^*_{(j_3,j_2,j_1)}}{q^*_{(j_3,j_2,\cdot)}} = \begin{cases} 1 & \text{for } \ (j_3,j_2,j_1)=(2,4,6) \\ 0 & \text{otherwise} \end{cases} ,$$

which means, e.g., that

$$h^*_2(6|2,4) = 1 \qquad \text{and} \qquad h^*_2(6|1,4) = 0\,.$$

Thus, the optimal probability for choosing $j_1 = 6$ depends not only on the choice of j_2 but also on the choice of j_3, which is an unexpected and counter-intuitive result; see the comment on p. 79.

Note that all what has been said following Theorem 4.1, in particular the results of Lemma 4.4, hold here as well with the appropriate change of $\mathbf{q}$'s and $\mathbf{h}$'s and therefore, they are not repeated here. Also note that because the Operator's optimal strategy are given by (4.33) and (4.34), i.e., it coincides with the one of the Se-No inspection game, $1-g^*_2(j_2),\ldots,1-g^*_k(j_k),1-g^*_{k+1}$ form a harmonic progression; see also Table 4.1 on p. 72 for an overview of inspection games with this property.

5.2 Any number of inspection opportunities and interim inspections: a recursive approach

Having determined in Section 5.1 the game theoretical solution of the Se-Se inspection game based on the solution of the Se-No inspection game, we will now present a different approach to the solution of this game which is based on its representation in *recursive form*. In this monograph, quite a few inspections games are described as recursive games; see Chapters 16 and 17. Thus, we will introduce them here in some detail including the backward induction procedure as a solution technique for finding optimal strategies and optimal payoffs.

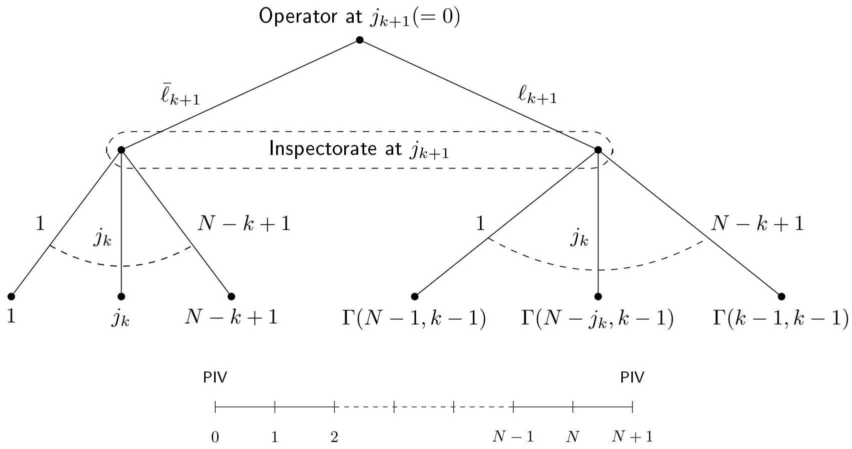
The inspection game analysed in this section is based on the specification (vii') on p. 75.

Let us denote the extensive form of the Se-Se inspection game with N possible time points for $k < N$ interim inspections by $\Gamma(N,k)$.[1] The recursive form of the first stage of this game

[1] We did not give names to the games considered so far. Here, and in the recursive games analysed in Chapters 16 and 17 it turns out that it is useful to do this.

is represented in Figure 5.4. Note that we intentionally use the term *stage* in this section in contrast to the term *step* which is used in Section 5.3 and Chapters 16 and 17. The term step refers to the situation in which a decision is made at *any* inspection opportunity, e.g., at time points $1, 2, \dots, N$ in the game discussed in Section 5.3. The term stage, on contrary, refers to the situation in which only at some inspection opportunities a decision is made. In this section, for instance, the first stage includes the Operator's decision at $j_{k+1}(= 0)$ and the time point j_k of the first interim inspection. Thus, if $j_k > 1$ then no decision is made at the intermediate time point(s) $1, \dots, j_k - 1$, and therefore the term step would not be appropriate in this case.

Figure 5.4 Recursive extensive form of the first stage of the Se-Se inspection game $\Gamma(N, k)$ with $N > k$ possible time points for k interim inspections.



At the beginning of the reference time interval $j_{k+1}(= 0)$ the Operator decides to start the illegal activity immediately ($\bar{\ell}_{k+1}$) or to postpone its start (ℓ_{k+1}), whereas the Inspectorate decides to place its first interim inspection at time point j_k with $j_k = 1, \dots, N - k + 1$. If the Operator starts the illegal activity at j_{k+1}, then the payoff to the Operator is j_k. If the Operator postpones the start of the illegal activity, then the Se-Se inspection game with $N - j_k$ possible time points for $k-1$ interim inspections starts. According to the terminology introduced above, this game is called $\Gamma(N - j_k, k - 1)$.

The exciting point is that the $N - k + 1$ new games $\Gamma(N - 1, k - 1), \dots, \Gamma(N - j_k, k - 1), \dots, \Gamma(k - 1, k - 1)$ are *proper* subgames of the game $\Gamma(N, k)$, i.e., their nodes do not share information sets of other subgames. This fact allows us to use a *backward induction procedure* which means that we can replace in Figure 5.4 the subgames by their optimal payoffs $Op^*_{N-1,k-1}, \dots, Op^*_{N-j_k,k-1}, \dots, 1$ to the Operator, i.e., the optimal expected detection times; see Owen (1988). The payoff matrix of the normal form game which is obtained this way, and which corresponds to the extensive form game given in Figure 5.4, is represented in Table 5.1.

We determine the solution of this game by rendering the Inspectorate indifferent between the choices j_k and $j_k + 1$; see Theorem 19.1. Suppose there exist an optimal strategy of the Inspectorate with $h^*_{k+1}(j_k) > 0$ and $h^*_{k+1}(j_k + 1) > 0$ for an j_k with $j_k = 1, \dots, N - k$. Then

Table 5.1 Payoff matrix of extensive form game in Figure 5.4.

	1	$\dots$	j_k	j_k+1	$\dots$	$N-k+1$
$\bar{\ell}_{k+1}$	1	$\dots$	j_k	j_k+1	$\dots$	$N-k+1$
ℓ_{k+1}	$Op^*_{N-1,k-1}$	$\dots$	$Op^*_{N-j_k,k-1}$	$Op^*_{N-j_k-1,k-1}$	$\dots$	$Op^*_{k-1,k-1}(=1)$

we obtain, using Table 5.1,

$$\begin{aligned} Op^*_{N,k} &= (1-g^*_{k+1})\, j_k + g^*_{k+1}\, Op^*_{N-j_k,k-1} \\ &= (1-g^*_{k+1})\,(j_k+1) + g^*_{k+1}\, Op^*_{N-j_k-1,k-1}\,, \end{aligned} \tag{5.15}$$

which implies

$$g^*_{k+1} = \frac{1}{1+Op^*_{N-j_k,k-1} - Op^*_{N-j_k-1,k-1}} \tag{5.16}$$

and

$$Op^*_{N,k} = \frac{(Op^*_{N-j_k,k-1} - Op^*_{N-j_k-1,k-1})\, j_k + Op^*_{N-j_k,k-1}}{1+Op^*_{N-j_k,k-1} - Op^*_{N-j_k-1,k-1}}\,. \tag{5.17}$$

This is a recursive relation for the Operator's optimal payoff $Op^*_{N,k}$. In order to be able to give a unique solution, boundary conditions have to be fixed. For the inspection game in this section they are for all $n = 1,\dots,N$ given by

$$Op^*_{n,0} = n+1 \qquad \text{and} \qquad Op^*_{n,n} = 1\,, \tag{5.18}$$

and can be explained as follows: If n possible time points but no interim inspection are left, then the Operator will start the illegal activity immediately and it is detected only at the final PIV resulting in the payoff $n+1$. If the number of interim inspections coincides with the number of possible time points, then the Inspectorate has to inspect at any time point and it does not matter at which of these the Operator starts it illegal activity because it will always be detected one time unit later and therefore, the payoff is 1.

Now, even though (5.17) together with (5.18) looks complicated, it can be easily seen that

$$Op^*_{N,k} = \frac{N+1}{k+1} \tag{5.19}$$

fulfils (5.17) and (5.18). Because $Op^*_{N,k}$ given by (5.19) does not depend on j_k, (5.15) holds for all j_k with $j_k = 1,\dots,N-k$, which implies that the right hand side of the saddle point inequality (19.11) is fulfilled as equality. Inserting (5.19) into (5.16) leads to

$$g^*_{k+1} = \frac{k}{k+1}\,. \tag{5.20}$$

To determine the optimal probabilities $h^*_{k+1}(j_k)$, $j_k = 1,\dots,N-k+1$, we apply the indifference principle again: Because the Operator plays both pure strategies with positive probability in his optimal strategy, the Inspectorate chooses its probabilities $h^*_{k+1}(j_k)$ as to make the Operator indifferent between the choices $\bar{\ell}_{k+1}$ and ℓ_{k+1}: Using Table 5.1 and (5.19) with $N \to N - j_k$ and $k \to k-1$, Theorem 19.1 yields

$$\frac{N+1}{k+1} = \sum_{j_k=1}^{N-k+1} j_k \, h^*_{k+1}(j_k) \qquad \text{and} \qquad \frac{N+1}{k+1} = \sum_{j_k=1}^{N-k+1} \frac{N-j_k+1}{k} \, h^*_{k+1}(j_k) \, ,$$

which is equivalent to

$$\sum_{j_k=1}^{N-k+1} j_k \, h^*_{k+1}(j_k) = \frac{N+1}{k+1} \, . \tag{5.21}$$

Note that the Inspectorate's first stage optimal strategies as given by (5.21) coincide for $k=2$ interim inspections with (5.3) in Lemma 5.1. The relation between $h^*_{k+1}(j_k) = q^*_{(j_k,\cdots)}$ as given by Theorem 5.2 and the one given by (5.21), however, remains an open question for $k>2$ interim inspections. All we can say is that the optimal expected interim inspection time point of the first interim inspection coincide: In fact, according to (4.43) and (5.9), we have

$$\begin{aligned} \sum_{j_k=1}^{N-k+1} j_k \, h^*_{k+1}(j_k) &= \sum_{j_k=1}^{N-k+1} j_k \, q^*_{(j_k,\cdots)} \\ &= \sum_{(j_k,\ldots,j_1)\in S_1} j_k \, q^*_{(j_k,\ldots,j_1)} = \mathbb{E}_{\mathbf{q}^*}(T_k) = \frac{N+1}{k+1} \, , \end{aligned} \tag{5.22}$$

i.e., (5.21). Summing up, the optimal payoff to the Operator $Op^*_{N,k}$ and the optimal first stage probabilities of the Operator g^*_{k+1} are the same as that given by Theorem 5.2.

In order to determine the Inspectorate's second stage optimal strategies, we extend Figure 5.4 such that we consider explicitly the Inspectorate's possibilities for choosing the time point j_{k-1} of the second interim inspection; see Figure 5.5. We see that there are $N-k+1$ different subgames of the original game $\Gamma(N,k)$. For $k=2$ interim inspections we obtain Figure 5.1, if we replace $\Gamma(N-j_{k-1},k-2)$ by $N+1-j_1$ for $j_1 = j_2+1,\ldots,N$.

We see immediately that these subgames are structurally the same as the one given by Table 5.1. Thus, we obtain for all j_k with $j_k = 1,\ldots,N-k+1$ the payoff matrix in Table 5.2.

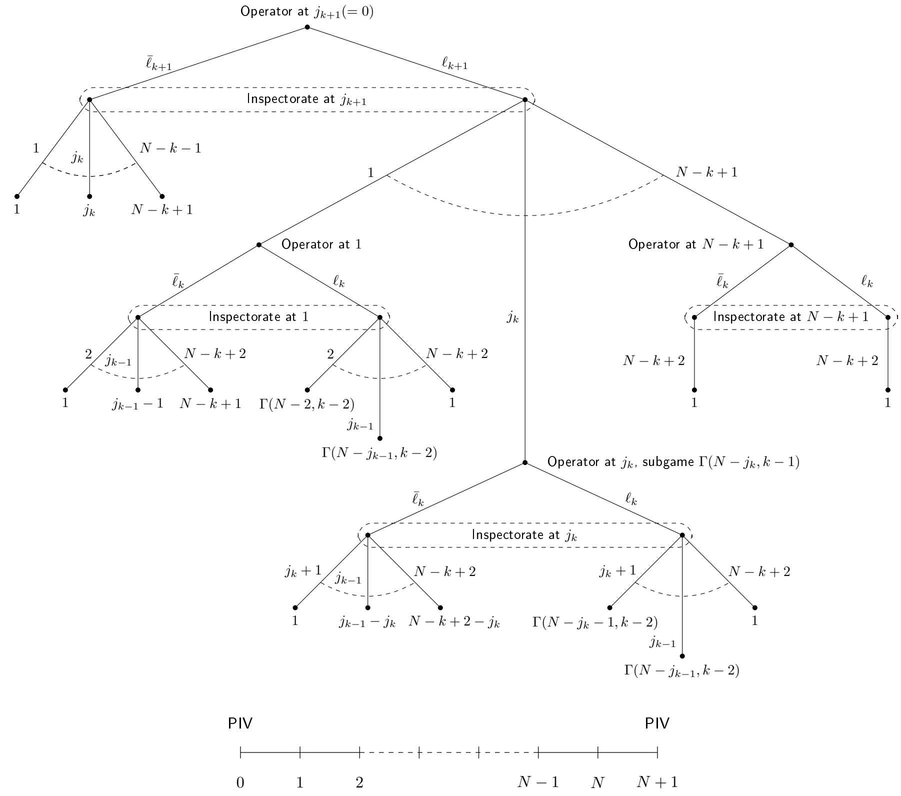
Table 5.2 Payoff matrix of the subgame $\Gamma(N-j_k,k-1)$ starting at time point j_k of the recursive extensive form game in Figure 5.5.

	j_k+1	$\ldots$	j_{k-1}	$\ldots$	$N-k+2$
$\bar{\ell}_k$	j_k+1-j_k	$\ldots$	$j_{k-1}-j_k$	$\ldots$	$N-k+2-j_k$
ℓ_k	$Op^*_{N-j_k-1,k-2}$	$\ldots$	$Op^*_{N-j_{k-1},k-2}$	$\ldots$	$Op^*_{k-2,k-2}(=1)$

It will be shown in Theorem 5.3, that the Inspectorate's optimal strategy at the second stage is given by

$$\sum_{j_{k-1}=j_k+1}^{N-k+2} j_{k-1} \, h^*_k(j_{k-1}|j_k) = \frac{N+(k-1)\,j_k+1}{k} \tag{5.23}$$

Figure 5.5 Recursive extensive form of the first and second stage of the Se-Se inspection game $\Gamma(N, k)$ with $N > k$ possible time points for k interim inspections.



for all $j_k = 1, \ldots, N-k+1$ which simplifies for $k = 2$ interim inspections to (5.4) keeping in mind that the case $j_k = N-k+1$ can be excluded from (5.23) because it implies $h_k^*(N-k+2|N-k+1) = 1$; see also the comment after (5.24).

Furthermore, the optimal expected interim inspection time point of the second interim inspection is, using (5.22) and (5.23), given by

$$\frac{N + (k-1)\,\mathbb{E}_{\mathbf{q}^*}(T_k) + 1}{k} = \frac{N + (k-1)\,\dfrac{N+1}{k+1} + 1}{k} = 2\,\frac{N+1}{k+1}\,,$$

which is the same as that of the Se-No inspection game; see (4.46) for $n = k-1$.

Of course, the optimal probability for the Operator to postpone the illegal activity at time point j_k again, is, using (5.20), given by

$$g_k^*(j_k) = \frac{k-1}{k} \qquad \text{for all} \qquad 1 \le j_k \le N-k\,,$$

since only $k-1$ interim inspections are left. Again, this result coincides for $k=2$ interim inspections with (5.2).

In contrast to the procedure in Section 5.1, see the remark on p. 79, we now assume that the probabilities h_n are only conditioned on j_n, i.e., $h_n(j_{n-1}|j_n)$, which reflects – as mentioned there – the fact that the payoff to the Inspectorate for the remaining game depends only on j_n and the remaining time points $j_{n-1},\dots,j_1$, and that the Inspectorate chooses at time point j_n only j_{n-1}. We will return to this issue on p. 90. In sum, the set of the Inspectorate's behavioural strategies is given by

$$\tilde{H}_{N,k} := \Big\{ \mathbf{h} := (h_{k+1}, h_k, \dots, h_2) \in [0,1]^k : \sum_{j_k=1}^{N-k+1} h_{k+1}(j_k) = 1\,, \\ \sum_{j_{n-1}=j_n+1}^{N-n+2} h_n(j_{n-1}|j_n) = 1 \quad \text{for all} \quad n = 2,\dots,k \\ \text{and all} \quad j_n \in \{k-n+1,\dots,N-n\} \Big\}\,. \tag{5.24}$$

Note that as in the definition of $H_{N,k}$ in (5.7) the case $j_n = N-n+1$ is excluded, because it implies that $j_{n-1} = N-n+2$ and because $1 \le j_{n-1} \le N-n+2$ the conditional probability $h_n(j_{n-1}|j_n)$ has to be one.

Because the payoff to the Operator is defined recursively in this game, we present in Table 5.3 its recursive normal form, where $Op_{N-j_{n-1},n-2}$ denotes the payoff (not necessarily the optimal payoff) to the Operator in the subgame with $N-j_{n-1}$ possible time points for $n-2$ interim inspections.

Table 5.3 Recursive normal form of the subgame $\Gamma(N-j_n, n-1)$ starting at time point j_n, and in case the Operator behaves legally at $j_{k+1}(=0),\dots,j_{n+1}$, $2 \le n \le k+1$.

	j_n+1	...	j_{n-1}	...	$N-n+2$
$\bar{\ell}_n$	j_n+1-j_n	...	$j_{n-1}-j_n$	...	$N-n+2-j_n$
ℓ_n	$Op_{N-(j_n+1),n-2}$	...	$Op_{N-j_{n-1},n-2}$	...	$Op_{N-(N-n+2),n-2}(=1)$

We now apply the recursive approach to the entire game $\Gamma(N,k)$. The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Theorem 5.3. *Given the recursive form of the Se-Se inspection game with $N > k$ possible time points for k interim inspections, i.e., $\Gamma(N,k)$. The sets of behavioural strategies are given by (4.30) and (5.24). The payoff to the Operator is defined recursively using the recursive normal form representation in Table 5.3, and the optimal payoff to the Operator fulfils the boundary conditions (5.18).*

Then an optimal strategy of the Operator is given by

$$g^*_{k+1} = \frac{k}{k+1} \qquad \textit{and} \\ g^*_n(j_n) = \frac{n-1}{n} \qquad \textit{for all} \quad n=2,\dots,k \quad \textit{and} \quad j_n = k-n+1,\dots,N-n\,, \tag{5.25}$$

and an optimal strategy of the Inspectorate by

$$\sum_{j_k=1}^{N-k+1} j_k\, h^*_{k+1}(j_k) = \frac{N+1}{k+1} \tag{5.26}$$

and, for all $n = 2, \ldots, k$ and all $j_n = k-n+1, \ldots, N-n$, by

$$\sum_{j_{n-1}=j_n+1}^{N-n+2} j_{n-1}\, h^*_n(j_{n-1}|j_n) = \frac{N+(n-1)\,j_n+1}{n}\,. \tag{5.27}$$

The optimal payoff to the Operator is

$$Op^*_{N,k} := Op_{N,k}(\mathbf{g}^*, \mathbf{h}^*) = \frac{N+1}{k+1}\,. \tag{5.28}$$

Proof. We prove the result with the help of the induction principle. At time point j_2 with $1 \le j_2 \le N-1$, i.e., the time point of the $(k-1)$th interim inspection, the Inspectorate decides about the time point j_1 of the last interim inspection. That means the game has reached the subgame $\Gamma(N-j_2, 1)$ in which the payoff to the Operator is denoted by $Op_{N-j_2,1}(g_2, h_2)$. For brevity reasons we write h_2 instead of $h_2(j_1|j_2)$. Note that because the case $j_2 = N-1$ implies $j_1 = N$, and therefore the Inspectorate has no strategic alternative, we confine ourselves to $j_2 = k-1, \ldots, N-2$. The payoff matrix of the game $\Gamma(N-j_2, 1)$ is given in Table 5.4.

Table 5.4 Payoff matrix of the subgame $\Gamma(N-j_2, 1)$ starting at time point j_2.

	j_2+1	$\ldots$	j_1	$\ldots$	N
$\bar{\ell}_2$	j_2+1-j_2	$\ldots$	j_1-j_2	$\ldots$	$N-j_2$
ℓ_2	$N-(j_2+1)+1$	$\ldots$	$N-j_1+1$	$\ldots$	1

Using (5.25) and Table 5.4, we get for all $j_1 = j_2+1, \ldots, N$

$$Op_{N-j_2,1}(g^*_2, j_1) = \frac{1}{2}\,(j_1 - j_2 + N - j_1 + 1) = \frac{N-j_2+1}{2}\,, \tag{5.29}$$

i.e., $Op_{N-j_2,1}(g^*_2, h^*_2) = Op_{N-j_2,1}(g^*_2, h_2)$ for all h_2 and therefore, the right hand saddle point inequality (19.11) is fulfilled as equality. Because $g^*_2 > 0$, the indifference principle of Theorem 19.1 yields by Table 5.4 and (5.29)

$$\frac{N-j_2+1}{2} = \sum_{j_1=j_2+1}^{N} (j_1-j_2)\, h^*_2(j_1|j_2) \qquad \text{and}$$

$$\frac{N-j_2+1}{2} = \sum_{j_1=j_2+1}^{N} (N-j_1+1)\, h^*_2(j_1|j_2)$$

which is equivalent to

$$\sum_{j_1=j_2+1}^{N} j_1\, h^*_2(j_1|j_2) = \frac{N+j_2+1}{2}\,,$$

i.e., (5.27) for $n = 2$.

We now distinguish the two cases 1) $n = 3, \ldots, k$ and 2) $n = k + 1$.

Ad 1) If $n = 3, \ldots, k$, then already $(k + 1 - n)$ interim inspections have been performed, i.e., the inspection game has reached time point j_n with $k - n + 1 \le j_n \le N - n$. Again we exclude $j_n = N - n + 1$. Then the Inspectorate decides about the time point j_{n-1} with $j_n{+}1 \le j_{n-1} \le N{-}n{+}2$ of the next interim inspection, i.e., the game has reached the subgame $\Gamma(N - j_n, n - 1)$ in which the optimal payoff to the Operator is denoted by $Op^*_{N-j_n,n-1}$. The induction hypothesis is

$$Op^*_{N-j_{n-1},n-2} = \frac{(N - j_{n-1}) + 1}{(n-2) + 1} \,. \tag{5.30}$$

The payoff matrix of the subgame $\Gamma(N - j_n, n - 1)$ is given in Table 5.5.

Table 5.5 Payoff matrix of the subgame $\Gamma(N - j_n, n - 1)$ starting at time point j_n.

	$j_n + 1$	$\ldots$	j_{n-1}	$\ldots$	$N - n + 2$
$\bar{\ell}_n$	$j_n + 1 - j_n$	$\ldots$	$j_{n-1} - j_n$	$\ldots$	$N - n + 2 - j_n$
ℓ_n	$Op^*_{N-(j_n+1),n-2}$	$\ldots$	$Op^*_{N-j_{n-1},n-2}$	$\ldots$	$Op^*_{N-(N-n+2),n-2}(=1)$

Then using (5.25), (5.30) and Table 5.5, we get for all $j_{n-1} = j_n + 1, \ldots, N - n + 2$

$$\begin{aligned} Op_{N-j_n,n-1}(g^*_n(j_n), j_{n-1}) &= \frac{1}{n}\,(j_{n-1} - j_n) + \frac{n-1}{n}\,Op^*_{N-j_{n-1},n-2} \\ &= \frac{1}{n}\,(j_{n-1} - j_n) + \frac{n-1}{n}\,\frac{N - j_{n-1} + 1}{n-1} \\ &= \frac{N - j_n + 1}{n}\,, \end{aligned}$$

i.e., (5.30) for $N - j_{n-1} \to N - j_n$ and $n - 2 \to n - 1$. Again the indifference principle yields by Table 5.5 and (5.30)

$$\sum_{j_{n-1}=j_n+1}^{N-n+2} j_{n-1}\, h^*_n(j_{n-1}|j_n) = \frac{N + (n-1)\,j_n + 1}{n}\,,$$

i.e., (5.27).

Ad 2) If $n = k + 1$ we consider the entire game $\Gamma(N, k)$, the payoff matrix of which is given in Table 5.1. The optimal strategies (5.20) and (5.21), and the optimal payoff to the Operator (5.19) coincide with (5.25), (5.26) and (5.28). □

Let us now return to the example of $N = 7$ possible time points for $k = 3$ interim inspections which we considered at the end of the Section 5.1. By definition, the strategy (5.13) cannot be optimal in the recursive game because $h^*_2(j_1|j_2)$ depends on j_3. The strategy (5.12), however, is optimal: In addition to the normalization of the h's, and according to (5.26) and (5.27) the following equations need to be fulfilled:

$$\sum_{j_3=1}^{5} j_3\, h_4^*(j_3) = 2 \quad \text{and}$$

$$\sum_{j_2=j_3+1}^{6} j_2\, h_3^*(j_2|j_3) = \frac{8+2\,j_3}{3} \qquad \text{and} \qquad \sum_{j_1=j_2+1}^{7} j_1\, h_2^*(j_1|j_2) = \frac{8+j_2}{2}\,.$$

The proof that the h's given by (5.14) fulfil this system of equations can be carried out using elementary summation formulae and eventually some patience.

Summing up, the Operator's optimal strategies and the optimal payoffs to the Operator coincide in the original Se-Se inspection game in Section 5.1 and the recursive Se-Se inspection game of this section. The above examples leads to the conjecture that if we consider only those optimal strategies h_n^* of the Inspectorate from Theorem 5.2 which only depend on the current interim inspection time point j_n and not on $j_k, \ldots, j_{n+1}$, then they are also optimal strategies of the recursive Se-Se inspection game treated in this section.

Like in Section 5.1 we see that $1 - g_2^*(j_2), \ldots, 1 - g_k^*(j_k), 1 - g_{k+1}^*$ as given by (5.25) form a harmonic progression; see also Table 4.1 on p. 72 for an overview of inspection games with this property.

5.3 Any number of inspection opportunities and interim inspections: step by step inspection game

Consider the following *modified* version of the Se-Se inspection game which has been proposed and analysed in a preliminary way by Canty and Avenhaus (1991b), and which is based on the following specification:

(vii') The Operator decides *at any step*, i.e., at any of the time points $0, 1, \ldots, N$, whether he will start the illegal activity immediately in case he did not do so before, or not. Because of assumption (iv), the Operator starts the illegal activity latest immediately at the time point of the last interim inspection.

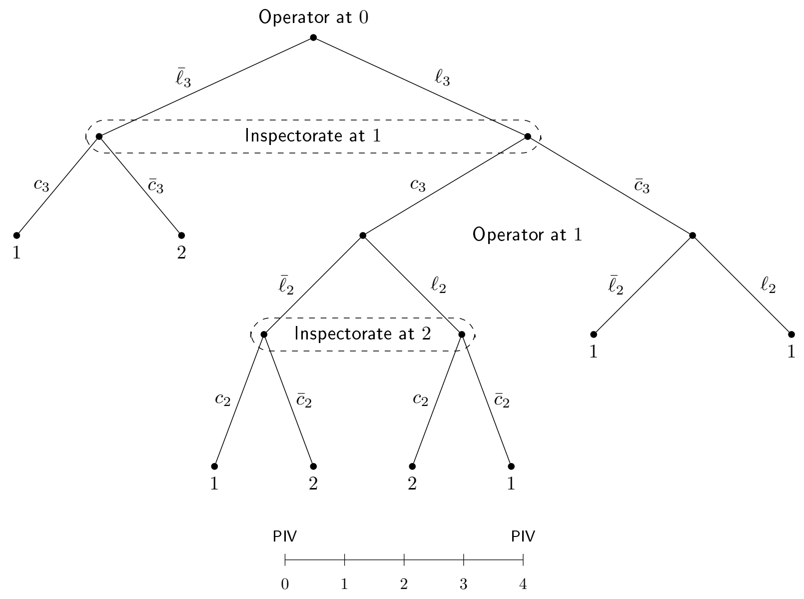
The Inspectorate decides *at any step*, i.e., at any of the time points $1, \ldots, N$, if it will perform an interim inspection as long as it has interim inspections left.

Note that assumption (vii') in Sections 5.1 and 5.2 means for example that the Inspectorate decides at the beginning of the reference time interval at which time point it performs its first interim inspection, while here it decides only whether to perform its first interim inspection at time point 1 or not.

Before dealing with the general step by step version of the Se-Se inspection game – in the following called step by step inspection game – with N possible time points for k interim inspections we consider several special cases. Figure 5.6 represents the extensive form of the step by step inspection game with $N = 3$ possible time points for $k = 2$ interim inspections.

In Figure 5.6, the Operator decides at time point 0 to start the illegal activity immediately ($\bar{\ell}_3$) with probability $1 - g_3$ or to postpone its start (ℓ_3) with probability g_3. In the latter case he

Figure 5.6 Extensive form of the step by step inspection game with $N = 3$ possible time points for $k = 2$ interim inspections.



decides at time point 1 to start the illegal activity at this time point ($\bar{\ell}_2$) with probability $1 - g_2$ or to postpone its start again (ℓ_2) with probability g_2. In the latter case he starts it at time point 2 because then the expected detection time is from the interval $[1, 2]$, while in case of postponing the start of the illegal activity to time point 3 the detection time is 1, the shortest possible one. Formally speaking, waiting until time point 3 for starting the illegal activity is a weakly dominated strategy, which is excluded from the model. Thus, the Operator's set of behavioural strategies is given by

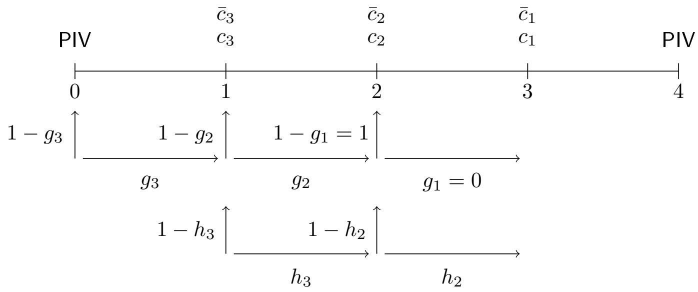
$$G_2 := \{\mathbf{g} := (g_3, g_2) : g_3, g_2 \in [0, 1]\} \, . \tag{5.31}$$

The time line of the interim inspections together with the Operator's probabilities in the step by step inspection game with $N = 3$ possible time points for $k = 2$ interim inspections are represented in Figure 5.7.

The Inspectorate decides at time point 1, not knowing the Operator's decision at time point 0, to perform the first interim inspection at time point 1 (c_3) with probability $1 - h_3$ or to postpone it ($\bar{c}_3$) with probability h_3.[2] In the first case it decides at time point 2, not knowing the Operator's decision at time point 1, to perform the second interim inspection at time point 2 (c_2) with probability $1 - h_2$ or to postpone it with probability h_2. In the latter case the Inspectorate has to perform it at time point 3 (c_1). If the Inspectorate does not perform the first interim inspection at time point 1 ($\bar{c}_3$), it must perform the interim inspections at time

[2] Even though in this section interim inspections are considered, we use here the symbol c indicating a control which corresponds to the notation used in Chapters 16 and 17.

Figure 5.7 Time line of the interim inspections and probabilities of the Operator and Inspectorate in the step by step inspection game with $N = 3$ possible time points for $k = 2$ interim inspections. c_n and $\bar{c}_n$, $n = 1, 2, 3$, indicate the actions of the Inspectorate at time point $4-n$.



points 2 and 3. Therefore, the Inspectorate's set of behavioural strategies is given by

$$H_3 := \Big\{ \mathbf{h} := (h_3, h_2) : h_3, h_2 \in [0,1] \Big\} , \tag{5.32}$$

see also Figure 5.7. Note that the index in G_2 refers to the number k of interim inspections while the index in H_3 refers to the number N of possible time points.

Also note that there are more informations sets of the Inspectorate than presented in Figure 5.6 which, however, have no alternatives at their nodes and therefore, can be omitted if optimal strategies are sought.[3] We will come back to this subtle issue on p. 317.

Using Figure 5.6, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $\mathbf{g} \in G_2$ and any $\mathbf{h} \in H_3$, given by

$$\begin{aligned} Op_{3,2}(\mathbf{g}, \mathbf{h}) &:= (1-g_3) \Big[(1-h_3)\,1 + h_3\,2 \Big] \\ &\quad + g_3 \Big[(1-h_3)\,[(1-g_2)\,((1-h_2)\,1 + h_2\,2) + g_2\,((1-h_2)\,2 + h_2\,1)] + h_3\,1 \Big] . \end{aligned} \tag{5.33}$$

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Lemma 5.2. *Given the step by step version of the Se-Se inspection game with $N = 3$ possible time points for $k = 2$ interim inspections, the extensive form of which is represented in Figure 5.6. The sets of behavioural strategies are given by (5.31) and (5.32), and the payoff to the Operator by (5.33).*

Then an optimal strategy of the Operator is given by

$$g_3^* = \frac{2}{3} \qquad \textit{and} \qquad g_2^* = \frac{1}{2} , \tag{5.34}$$

[3] For instance: After the moves $\ell_3 \bar{c}_3$ the Inspectorate does not know at time point 2 whether the Operator has started the illegal activity at time point 1 ($\bar{\ell}_2$) or postponed its start again (ℓ_2), i.e., the two nodes after the moves $\ell_3 \bar{c}_3 \bar{\ell}_2$ and $\ell_3 \bar{c}_3 \ell_2$ belong to an information set which is not presented in Figure 5.6, because the Inspectorate must inspect (c_2) and has no other strategic alternatives in this situation.

and an optimal strategy of the Inspectorate by

$$h_3^* = \frac{1}{3} \qquad \textit{and} \qquad h_2^* = \frac{1}{2}\,. \tag{5.35}$$

The optimal payoff to the Operator is

$$Op_{3,2}^* := Op_{3,2}(\mathbf{g}^*, \mathbf{h}^*) = \frac{4}{3}\,. \tag{5.36}$$

Proof. Using (5.33), (5.34) – (5.36) imply $Op_{3,2}(\mathbf{g}^*, \mathbf{h}) = Op_{3,2}^* = Op_{3,2}(\mathbf{g}, \mathbf{h}^*)$ for any $\mathbf{g} \in G_2$ and any $\mathbf{h} \in H_3$, i.e., the saddle point criterion is fulfilled as equality. □

For later purposes, see p. 98, we determine the optimal expected interim inspection time points $\mathbb{E}_{\mathbf{h}^*}(T_2)$ and $\mathbb{E}_{\mathbf{h}^*}(T_1)$ of the two interim inspections at time points T_2 and T_1. Using (5.35) and Figure 5.7 we get

$$\mathbb{E}_{\mathbf{h}^*}(T_2) = (1 - h_3^*)\,1 + h_3^*\,2 = \frac{4}{3} = Op_{3,2}^* \qquad \text{and}$$

$$\mathbb{E}_{\mathbf{h}^*}(T_1) = (1 - h_3^*)\,(1 - h_2^*)\,2 + (h_3^* + (1 - h_3^*)\,h_2^*)\;3 = \frac{8}{3} = 2\,Op_{3,2}^*\,,$$

this means that the optimal expected interim inspection time points divide the reference time interval into three sections of equal lengths.

Looking once more at Figure 5.6 we see that after the moves $\ell_3 c_3$ a proper subgame with $N = 2$ possible time points for $k = 1$ interim inspection is reached. This permits a recursive method for solving this game: If we replace this subgame by its optimal payoff $Op_{2,1}^* = 3/2$, then we have a reduced game in form of a 2×2 matrix game which can be solved easily and, of course, leads to the same solution as the standard method. In fact, the existence of a proper subgame in the step by step inspection game with $N = 3$ possible time points for $k = 2$ interim inspections was the reason for starting the analysis with this game.

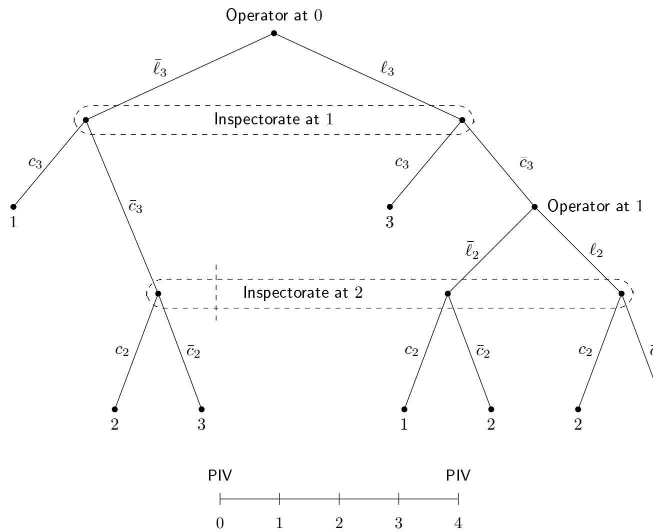
Let us consider next the step by step inspection game with $N = 3$ possible time points for $k = 1$ interim inspection, the extensive form of which is represented in Figure 5.8. We see that the Operator's and the Inspectorate's decisions at 0 and 1 are the same as in the game of Figure 5.6. At 2, however, the Inspectorate has a choice between c_2 and $\bar{c}_2$ if it chose $\bar{c}_3$ at 1. This, in turn, creates a larger information set for the Inspectorate at 1 than that given in the game of Figure 5.6.

Let g_3, g_2 and h_3, h_2 be the Operator's and the Inspectorate's probabilities as introduced for the step by step inspection game with $N = 3$ possible time points for $k = 2$ interim inspections. It will be shown in Theorem 5.4 that this inspection game has the game theoretical solution

$$g_3^* = \frac{1}{2}\,, \quad g_2^* = 1\,, \qquad h_3^* = \frac{2}{3}\,, \qquad h_2^* = \frac{1}{2} \qquad \text{and} \qquad Op_{3,1}^* = 2\,. \tag{5.37}$$

Contrary to the step by step inspection game with $N = 3$ possible time points for $k = 2$ interim inspections, see Figure 5.6, the step by step inspection game with $N = 3$ possible time points for $k = 1$ interim inspection does not contain any proper subgame. The fact, however, that the game appears already to be finished after the Operator acts illegally at time point 0 ($\bar{\ell}_3$), motivates the attempt to cut the information set of the Inspectorate at time point 2, as indicated in the Figure 5.8. This way, we obtain on the right hand side a subgame starting

Figure 5.8 Extensive form of the step by step inspection game with $N = 3$ possible time points for $k = 1$ interim inspection.



after the moves $\ell_3 \bar{c}_3$ with $N = 2$ possible time points for $k = 1$ interim inspection, and we proceed in a similar way as before: We replace the subgame by its optimal expected detection time $Op^*_{2,1} = 3/2$, and use at the left hand branch in Figure 5.8 the same $h^*_2 = 1/2$ as that obtained from the subgame. Thus, we obtain a reduced extensive form game which can be solved easily. The game theoretical solution of this *auxiliary* step by step inspection game with $N = 3$ possible time points for $k = 1$ interim inspection, i.e., the game of Figure 5.8 with the cutted information set, is given by

$$\tilde{g}^*_3 = \frac{1}{2}\,, \quad \tilde{g}^*_2 = \frac{1}{2} \qquad \text{and} \qquad \tilde{h}^*_3 = \frac{2}{3}\,, \quad \tilde{h}^*_2 = \frac{1}{2} \qquad \text{and} \qquad \widetilde{Op}^*_{3,1} = 2. \tag{5.38}$$

Thus, comparing the game theoretical solution of the step by step inspection game with $N = 3$ possible time points for $k = 1$ interim inspection with those of the corresponding auxiliary step by step inspection game, i.e., comparing (5.37) with (5.38), the surprising result is that the optimal strategy of the Inspectorate and the optimal payoff to the Operator coincide, while the optimal strategies of the Operator are different.

Since it has not been successful to solve the step by step inspection game with any number N of possible time points for any number k of interim inspections, it is tempting to solve the corresponding auxiliary step by step inspection game as illustrated above. In fact, this has been done in Canty and Avenhaus (1991b) leading to $\widetilde{Op}^*_{N,k} = (N+1)/(k+1)$ as the optimal payoff to the Operator, i.e., the same as that for the Se-Se inspection game; see (5.11). We do not present the derivations for the auxiliary step by step inspection game here because we cannot compare its game theoretical solution with that of the step by step inspection game.

Instead let us turn to the step by step inspection game with N possible time points for $k = 1$ interim inspection. Let $1 - g_N$ resp. g_N denote the probability that the Operator starts the illegal activity at time point 0 resp. to postpone its start. In the latter case he will start it immediately at the time point of the interim inspection because only then he maximizes the time between start and detection at the final PIV; see assumption (vii'). Thus, the Operator's set of behavioural strategies is given by

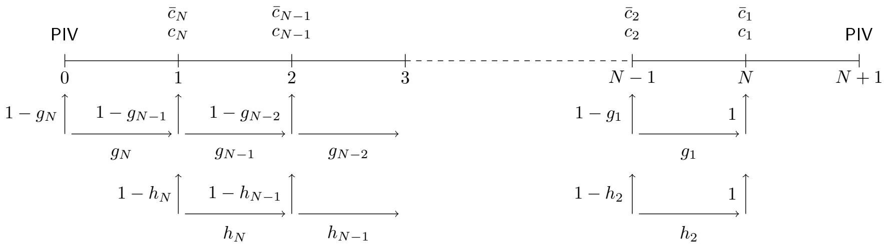
$$G_1 := \{g_N : g_N \in [0,1]\} \ . \tag{5.39}$$

Note that a formal description of the Operator's behaviour in the step by step game would require the introduction of the probabilities $g_{N-1}, \ldots, g_1$ of postponing the start of the illegal activity at any step; see also Figure 5.9. However, any strategy $(g_N, g_{N-1}, \ldots, g_1), g_n \in [0,1]$, $n = 1, \ldots, N$, is weakly dominated by the strategy $(g_N, g^*_{N-1}, \ldots, g^*_1)$ with

$$g^*_n(c) = \begin{cases} 0 & \text{if } c = c_{n+1} \\ 1 & \text{if } c = \bar{c}_{n+1} \end{cases} , \qquad n = 1, \ldots, N-1 \,, \tag{5.40}$$

and thus, only g_N needs to be considered as strategic variable. The meaning of (5.40) is that in case the Operator postpones the start of the illegal activity at time point 0, he will wait until the only interim inspection is performed, and starts then the illegal activity immediately.

Figure 5.9 Time line of the interim inspections and probabilities of the Operator and Inspectorate in the step by step inspection game with $N > k$ possible time points for $k = 1$ interim inspection. c_n and $\bar{c}_n$, $n = 1, \ldots, N$, indicate the actions of the Inspectorate at time point $(N+1) - n$.



At any step n, $n = 1, \ldots, N$, and in case the Inspectorate has not performed the interim inspections yet, it performs the interim inspections (c_n) with probability $1 - h_n$ and postpones it again ($\bar{c}_n$) with probability h_n; see Figure 5.9. In case the Inspectorate postpones its interim inspection until time point N, i.e., $h_N = h_{N-1} = \ldots = h_2 = 1$, it has to inspect at time point N which means $h_1 = 0$, and therefore, h_1 is not a strategic variable. Thus, the sets of behavioural strategies of the Inspectorate is given by

$$H_N := \Big\{ \mathbf{h} := (h_N, h_{N-1}, \ldots, h_2) : h_n \in [0,1] \,, n = 2, \ldots, N \Big\} \,. \tag{5.41}$$

The (expected) payoff to the Operator, i.e., the expected detection time, is, for any $g_N \in G_1$ and any $\mathbf{h} \in H_N$, given by

$$\begin{aligned} Op_{N,1}(g_N, \mathbf{h}) := (1 - g_N) \Big[& 1\,(1 - h_N) + 2\,h_N\,(1 - h_{N-1}) + \ldots \\ & + \ (N-1)\,h_N \ldots h_3\,(1 - h_2) + N\,h_N \ldots h_2 \Big] \end{aligned}$$

$$+ g_N \Big[N (1 - h_N) + (N-1) h_N (1 - h_{N-1}) + \ldots$$
$$+ 2 h_N \ldots h_3 (1 - h_2) + 1 h_N \ldots h_2 \Big]. \qquad (5.42)$$

Payoff (5.42) can be explained, using Figure 5.9, as follows: If the Operator starts the illegal activity at time point 0 (with probability $1 - g_N$), then the detection time is $1, 2, \ldots, N$ with the probabilities $1 - h_N, h_N (1 - h_{N-1}), \ldots, h_N \ldots h_2$, respectively. In case he postpones the start of the illegal activity (with probability g_N), the detection time is $N, N-1, \ldots, 1$ again with the probabilities $1 - h_N, h_N (1 - h_{N-1}), \ldots, h_N \ldots h_2$, respectively.

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Theorem 5.4. *Given the step by step version of the Se-Se inspection game with $N > 1$ possible time points for $k = 1$ interim inspection. The sets of behavioural strategies are given by (5.39) and (5.41), and the payoff to the Operator by (5.42).*

Then an optimal strategy of the Operator is given by

$$g_N^* = \frac{1}{2}, \qquad g_n^*(c) = \begin{cases} 0 & \text{if } c = c_{n+1} \\ 1 & \text{if } c = \bar{c}_{n+1} \end{cases}, \qquad n = 1, \ldots, N-1, \qquad (5.43)$$

and an optimal strategy of the Inspectorate by

$$h_n^* = \frac{n-1}{n}, \qquad n = 2, \ldots, N. \qquad (5.44)$$

The optimal payoff to the Operator is

$$Op_{N,1}^* := Op_{N,1}(g_N^*, \mathbf{h}^*) = \frac{N+1}{2}. \qquad (5.45)$$

Proof. Using (5.42) we obtain with (5.43)

$$\frac{2\, Op_{N,1}(g_N^*, \mathbf{h})}{N+1} = (1 - h_N) + h_N (1 - h_{N-1}) + \ldots + h_N \ldots h_3 (1 - h_2) + h_N \ldots h_2 = 1$$

for any $\mathbf{h} \in H_N$. (5.44) implies for all $n = 2, \ldots, N-1$

$$h_N^* h_{N-1}^* \ldots h_{n+1}^* (1 - h_n^*) = \frac{N-1}{N} \frac{N-2}{N-1} \cdots \frac{n}{n+1} \frac{1}{n} = \frac{1}{N} \qquad (5.46)$$

and

$$h_N^* h_{N-1}^* \ldots h_3^* h_2^* = \frac{N-1}{N} \frac{N-2}{N-1} \cdots \frac{2}{3} \frac{1}{2} = \frac{1}{N}. \qquad (5.47)$$

Thus, (5.42) gives

$$N\, Op_{N,1}(g_N, \mathbf{h}^*) = (1 - g_N) \frac{N(N+1)}{2} + g_N \frac{N(N+1)}{2} = \frac{N(N+1)}{2}$$

for any $g_N \in G_1$. Therefore, we have $Op_{N,1}(g_N, \mathbf{h}^*) = Op_{N,1}(g_N^*, \mathbf{h}^*) = Op_{N,1}(g_N^*, \mathbf{h})$ for any $g_N \in G_1$ and any $\mathbf{h} \in H_N$, i.e., the saddle point inequality is fulfilled as equality. □

Using Figure 5.9, (5.44), (5.46) and (5.47), the optimal expected interim inspection time point is given by

$$\mathbb{E}_{\mathbf{h}^*}(T_1) = 1\,(1-h_N^*) + 2\,h_N^*\,(1-h_{N-1}^*) + 3\,h_N^*\,h_{N-1}^*\,(1-h_{N-2}^*) + \dots + N\,h_N^*\,h_{N-1}^* \dots h_2^*$$

$$= \frac{1}{N}\sum_{i=1}^{N} i = \frac{N+1}{2} = Op_{N,1}^*\,,$$

which corresponds to (4.46) for $k=1$ interim inspection.

Furthermore, the optimal strategy of the Operator is to decide at the beginning of the reference time interval with probabilities $1/2$ to start the illegal activity immediately or to postpone its start, and in the latter case to wait for the interim inspection.

Together with the result of Lemma 5.2, this encourages us to formulate a conjecture for the general game which only makes a structural statement on optimal strategies but does not give any advice on how to calculate them.

Conjecture 5.1. *Given the step by step version of the Se-Se inspection game with $N > k$ possible time points for k interim inspections.*

Then an optimal strategy of the Operator is to postpone the start of the illegal activity at time point 0 with probability $g_N^* = k/(k+1)$. In case the Operator postpones the illegal activity he waits for the next interim inspection.

An optimal strategy of the Inspectorate is to choose the time points for the interim inspections such that the optimal expected interim inspection time points are

$$\mathbb{E}_{\mathbf{h}^*}(T_k) = \frac{N+1}{k+1}\,,\ \mathbb{E}_{\mathbf{h}^*}(T_{k-1}) = 2\,\frac{N+1}{k+1}\,, \dots,\ \mathbb{E}_{\mathbf{h}^*}(T_1) = k\,\frac{N+1}{k+1}\,.$$

The optimal payoff to the Operator is given by (5.45).

Suppose Conjecture 5.1 were true, then the step by step version of the Se-Se inspection game and the original Se-Se inspection game treated in Sections 5.1 and 5.2 share common features: They lead to the same optimal expected detection time and corresponding optimal strategies of both players. In our view the Se-Se inspection game describes the inspection problem in a more natural way since both players make their decisions at the beginning of the reference time interval and after an interim inspection: In between the players do not gain any information which might be useful for them.

Finally, let us note that von Stengel (1991) has considered a model which is very similar to the step by step model considered here. However, he assumes inter alia that an illegal activity is discovered instantly if it coincides with an interim inspection and that in this case the payoff to the Operator is zero, i.e., the same as in case of legal behaviour. Even though v. Stengel was able to determine the game theoretical solution for the general case of any number N of steps for any number k of interim inspections, we do not describe it here in detail since we cannot imagine a real situation which is met by the above mentioned assumption. Nevertheless, any researcher who attempts to prove or disprove Conjecture 5.1 is encouraged to study von Stengel (1991) first, because his solution method could provide a path for the solution of the step by step version of the Se-Se inspection game with N possible time points for k interim inspections.

Chapter 6

Models with errors of the second kind

In Chapters 3 to 5 we have assumed that an illegal activity of the Operator will be detected by the Inspectorate during the next interim inspection or at the ending Physical Inventory Verification (PIV) with certainty. In this chapter we assume that a detection will happen during an interim inspection only with a detection probability $1 - \beta$, i.e., assumption (v) of Chapter 2 is specified as follows:

(v') During an interim inspection the Inspectorate may commit an error of the second kind with probability $\beta \geq 0$, i.e., the illegal activity, see assumption (iv), is not detected during the next interim inspection with probability β. Note that if there is no interim inspection left, then it is detected with certainty at the final PIV; see assumption (iii). This non-detection probability is the same for all interim inspections.

The remaining assumptions of Chapter 2 hold throughout this chapter. Note that there are practical cases where different interim inspections may lead to different detection probabilities; an example is given in Section 6.6. As in Chapters 3 to 5, assumption (vii) of Chapter 2 needs to be appropriately modified. We do not do this in Sections 6.1 to 6.4 explicitly, but refer the reader to (vii') on pp. 21, 50, 55 and 75 in the respective chapters.

Let us comment assumption (v'): Statistical problems where only errors of the second kind are possible, also called attribute sampling problems usually occur when random sampling schemes are used, where items are counted, and where errors arise only when falsified or wrong items are not contained in the sample; see Thyregod (1988). If quantitative measurements are performed by the Inspectorate, then errors of the first kind – false alarm – may occur; see p. 4 and Section 7.4. The possibility that the Inspectorate may commit an error of the first kind is not considered in this chapter. The reason simply is that game theoretical solutions of discrete time inspection games with errors of the first and second kind do not exist, contrary to the continuous time case which will be analysed in Part II.

This chapter is based on Avenhaus et al. (2010) and Avenhaus and Krieger (2013a). In Sections 6.1 to 6.4 we analyse all four variants given in Table 2.1 for the cases of $N = 3$ possible time points for $k = 1$ resp. $k = 2$ interim inspection(s) and, for $\beta \geq 0$. Also, in Sections 6.1 and 6.3 the case of any number N of possible time points for $k = 1$ interim inspection is treated. In Section 6.5 we compare the optimal expected detection times derived in the former sections.

At first sight the inspection games with $N = 3$ possible time points for $k = 1$ resp. $k = 2$ interim inspection(s) appear simple, but they are not trivial; also they are based on concrete applications. For this reason we will shortly present these applications in Section 6.6.

Since the sets of strategies of both the Operator and the Inspectorate are the same as those for the games analysed in Chapters 3 to 5, we will refer to them. For this reason, we recommend to study these chapters with some care before entering into this one.

Note that only for the No-No and Se-No inspection games with $N \geq 2$ possible time points for $k = 1$ interim inspection in Sections 6.1 and 6.3 the optimal expected time point for the start of the illegal activity resp. the optimal expected interim inspection time point is considered.

6.1 No-No: Any number of inspection opportunities and one interim inspection; three inspection opportunities and two interim inspections

Let us begin with the case of $N = 3$ possible time points for $k = 1$ interim inspection, which may be performed at three possible time points $1, 2$ or 3. Thus, as explained in Section 3.1, the set of pure strategies of the Inspectorate is given by $J_{3,1}$ and that of the Operator by I_3; see (3.1) and (3.2). The matrix of the detection times, i.e., the payoff matrix of this inspection game, is given by Table 6.1.

Table 6.1 Payoff matrix of the No-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection and with errors of the second kind.

	1	2	3
0	$1+3\beta$	$2+2\beta$	$3+\beta$
1	3	$1+2\beta$	$2+\beta$
2	2	2	$1+\beta$
3	1	1	1

For two cases we explain the payoff and its computation. Let $i = 0$ and $j = 1$, i.e., the Operator starts the illegal activity at time point 0 while the Inspectorate performs its interim inspection at time point 1. Then the illegal activity is detected at 1 with probability $1 - \beta$ and not detected at 1 with β. If it is not detected at time point 1 then it will be detected at the final PIV with certainty; see the assumption (iii) of Chapter 2. Therefore, the detection time, i.e., the payoff to the Operator, is $(1 - \beta)\,1 + \beta\,4 = 1 + 3\,\beta$. If $i = j = 1$, then, according to assumption (x), the illegal activity will be detected only at the final PIV and is therefore 3.

In analogy to the case of $N = 3$ possible time points for $k = 1$ interim inspection on p. 23, let p_i, $i = 0, \ldots, 3$, be the probability that the illegal activity is started at time point i and q_j, $j = 1, 2, 3$, be the probability to perform the interim inspection at time point j. Thus, the sets of mixed strategies of both players are given by (3.4) and (3.5).

Using Table 6.1, the Operator's (expected) payoff, i.e., the expected detection time, is, for any

$\mathbf{p} \in P_3$ and any $\mathbf{q} \in Q_{3,1}$, in analogy to (3.6) given by

$$\begin{aligned} Op_{3,1}(\mathbf{p},\mathbf{q}) := p_0\,&[(1+3\,\beta)\,q_1 + (2+2\,\beta)\,q_2 + (3+\beta)\,q_3] \\ &+ p_1\,[3\,q_1 + (1+2\,\beta)\,q_2 + (2+\beta)\,q_3] \\ &+ p_2\,[2\,q_1 + 2\,q_2 + (1+\beta)\,q_3] + p_3\,. \end{aligned} \tag{6.1}$$

Like on p. 24 it can be shown that the pure strategy "starting the illegal activity at time point 3" is a strictly dominated strategy: Using the Operator's strategies $\mathbf{p}_1 := (0,0,0,1)^T$ and $\mathbf{p}_2 := (0,1/2,1/2,0)^T$, we get by (6.1)

$$\begin{aligned} Op_{3,1}(\mathbf{p}_2,\mathbf{q}) &= \frac{1}{2}\Big(5\,q_1 + (3+2\,\beta)\,(q_2+q_3)\Big) = \frac{1}{2}\Big(2\,(1-\beta)\,q_1 + 3 + 2\,\beta\Big) \\ &\geq \frac{3+2\,\beta}{2} \geq \frac{3}{2} > 1 = Op_{3,1}(\mathbf{p}_1,\mathbf{q}) \end{aligned}$$

for any $\mathbf{q} \in Q_{3,1}$, i.e., $\mathbf{p}_1$ is strictly dominated and thus not used in any optimal strategy.

The game theoretical solution of this inspection game, see Avenhaus et al. (2010), is presented in

Lemma 6.1. *Given the No-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection, and with errors of the second kind. The sets of mixed strategies are given by (3.4) and (3.5), and the payoff to the Operator by (6.1).*

Then optimal strategies and the optimal payoff $Op^*_{3,1} := Op_{3,1}(\mathbf{p}^*,\mathbf{q}^*)$ to the Operator are given by:

(i) For $0 \leq \beta < 1/6$ an optimal strategy of the Operator is given by

$$\mathbf{p}^* = \left(\frac{1}{3},\frac{1}{6},\frac{1}{2},0\right)^T,$$

and an optimal strategy of the Inspectorate by

$$\mathbf{q}^* = \frac{1}{1-\beta}\left(\frac{1}{3},\frac{1}{2},\frac{1}{6}-\beta\right)^T.$$

The optimal payoff to the Operator is

$$Op^*_{3,1} = \frac{11}{6} + \beta\,.$$

(ii) For $1/6 < \beta < 2/3$ an optimal strategy of the Operator is given by

$$\mathbf{p}^* = \left(\frac{2}{3},\frac{1}{3},0,0\right)^T,$$

and an optimal strategy of the Inspectorate by

$$\mathbf{q}^* = \frac{1}{1-\beta}\left(\frac{1}{3},\frac{2-3\,\beta}{3},0\right)^T.$$

The optimal payoff to the Operator is

$$Op^*_{3,1} = \frac{10}{6} + 2\,\beta\,.$$

(iii) For $2/3 < \beta \leq 1$ an optimal strategy of the Operator and an optimal strategy of the Inspectorate are given by

$$\mathbf{p}^* = (1,0,0,0)^T \qquad \textit{and} \qquad \mathbf{q}^* = (1,0,0)^T \,.$$

The optimal payoff to the Operator is

$$Op^*_{3,1} = \frac{6}{6} + 3\,\beta \,.$$

Proof. We have to show, see (19.11), that

$$Op_{3,1}(i,\mathbf{q}^*) \leq Op^*_{3,1} \leq Op_{3,1}(\mathbf{p}^*,j) \tag{6.2}$$

for all $i = 0,1,2,3$ and for all $j = 1,2,3$.

Ad (i): We have

$$Op_{3,1}(i,\mathbf{q}^*) = \begin{cases} Op^*_{3,1} & : i = 0,1,2 \\ 1 & : i = 3 \end{cases} \quad \text{and} \quad Op_{3,1}(\mathbf{p}^*,j) = Op^*_{3,1} \quad \text{for all} \quad j = 1,2,3\,,$$

i.e., the inequalities in (6.2) are fulfilled.

Ad (ii): We obtain

$$Op_{3,1}(i,\mathbf{q}^*) = \begin{cases} Op^*_{3,1} & : i = 0,1 \\ 3 - i + 1 & : i = 2,3 \end{cases} \quad \text{and} \quad Op_{3,1}(\mathbf{p}^*,j) = \begin{cases} Op^*_{3,1} & : j = 1,2 \\ \frac{8}{3} + \beta & : j = 3 \end{cases} \,.$$

Again, the inequalities in (6.2) are fulfilled.

Ad (iii): We get

$$Op_{3,1}(i,\mathbf{q}^*) = \begin{cases} 1 + 3\,\beta & : i = 0 \\ 3 - i + 1 & : i = 1,2,3 \end{cases} \quad \text{and} \quad \begin{array}{l} Op_{3,1}(\mathbf{p}^*,j) = j + (3 - j + 1)\,\beta \\ \text{for all } j = 1,2,3 \end{array} \,,$$

i.e., the inequalities in (6.2) are fulfilled again. □

Let us comment the results of Lemma 6.1: First, in contrast to Lemma 3.1 in which the cut-off value n^* can not be recognized yet (starting the illegal activity at time point 3 is a dominated strategy and is therefore not chosen), the existence of cutting-value n^* is strongly pronounced here for the cases (ii) and (iii). Looking at this inspection problem for $\beta = 0$ from the common sense point of view, the Inspectorate should perform its interim inspection in the middle of the reference time interval with the resulting detection time of 2. The game theoretical solution, however, leads to slightly shorter detection times in case (i); see also p. 25.

Second, solutions of most games are seldom intuitive. So it is also in this game. It is not trivial nor explainable that the Inspectorate performs, e.g., its interim inspection at time point 1 with the probability q^*_1 given in Lemma 6.1. It is rather a result. But what can be done is to explain the structure of the optimal strategies. From the common sense of view it is clear that when the non-detection probability β is high, e.g., $\beta > 2/3$ in case (iii), the Operator will start the illegal activity as early as possible, i.e., $p^*_0 = 1$, and so the Inspectorate will also perform its interim inspection as early as possible, i.e., $q^*_1 = 1$.

Third, it is interesting and surprising, that the Operator's optimal strategies are constant in the β-intervals $[0,1/6)$, $(1/6,2/3)$ and $(2/3,1]$, contrary to the Inspectorate's optimal strategies. This is no longer true in the No-No inspection game with $N=3$ possible time points for $k=2$ interim inspections; see Lemma 6.2.

Fourth, the case $\beta=1$ deserves special attention. In that case, the illegal activity will be detected only at the end of the reference time interval. Thus, the Operator will start it right at the beginning and the detection time will be 4, independently of what the Inspectorate does. Formally, Table 6.1 shows that the first strategy of the Operator dominates all other ones, and that the Inspectorate is indifferent with respect to its pure strategies. Since we do not want to put too much emphasis on this unrealistic case, we have given in Lemma 6.1 for $\beta=1$ just one optimal strategy of the Inspectorate instead of infinitely many ones. We will proceed in the same way in all subsequent Lemmata of this chapter.

Fifth, using the optimal strategies of Lemma 6.1, we obtain, using the definition on the left hand side of (3.36) and (3.38),

$$\mathbb{E}_{\mathbf{p}^*}(S)=\begin{cases}\dfrac{7}{6} & : \ 0\le\beta<\dfrac{1}{6}\\ \dfrac{1}{3} & : \ \dfrac{1}{6}<\beta<\dfrac{2}{3}\\ 0 & : \ \dfrac{2}{3}<\beta\le 1\end{cases} \quad \text{and} \quad \mathbb{E}_{\mathbf{q}^*}(T_1)=\begin{cases}3-\dfrac{7}{6\,(1-\beta)} & : \ 0\le\beta<\dfrac{1}{6}\\ 2-\dfrac{1}{3\,(1-\beta)} & : \ \dfrac{1}{6}<\beta<\dfrac{2}{3}\\ 1 & : \ \dfrac{2}{3}<\beta\le 1\end{cases}\ .$$

The relation of $\mathbb{E}_{\mathbf{p}^*}(S)$ and $\mathbb{E}_{\mathbf{q}^*}(T_1)$ to n^* and $Op^*_{3,1}$ is illustrated in (6.13) and (6.14).

Finally, the optimal strategies and the optimal payoff to the Operator are not given for the limiting cases $\beta=1/6$ and $\beta=2/3$ because in reality exact values of β will rarely occur. Nevertheless we will consider them here in some detail; in the remainder of the monograph, however, we will exclude the limiting cases from our discussion. Note that the optimal strategies of the Inspectorate and the optimal payoff to the Operator as a function of β are continuous at $\beta=1/6$ and $\beta=2/3$, but not the optimal strategies of the Operator. It can be seen with the help of (6.2), that for $\beta=1/6$ the mixed strategies $\mathbf{p}^*(\lambda)$ and $\mathbf{q}^*$ given by

$$\mathbf{p}^*(\lambda):=\lambda\begin{pmatrix}1/3\\1/6\\1/2\\0\end{pmatrix}+(1-\lambda)\begin{pmatrix}2/3\\1/3\\0\\0\end{pmatrix},\lambda\in[0,1]\,, \qquad \text{and} \qquad \mathbf{q}^*=\begin{pmatrix}2/5\\3/5\\0\end{pmatrix}$$

are optimal strategies with the optimal payoff $Op^*_{3,1}=2$ to the Operator. Note that $\mathbf{p}^*(\lambda)$ is a convex combination of two mixed strategies and thus, a mixed strategy itself. For $\beta=2/3$ the mixed strategies

$$\mathbf{p}^*(\lambda):=\lambda\begin{pmatrix}2/3\\1/3\\0\\0\end{pmatrix}+(1-\lambda)\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\lambda\in[0,1]\,, \qquad \text{and} \qquad \mathbf{q}^*=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

are optimal strategies for both players leading to the optimal payoff $Op^*_{3,1}(2/3)=3$ to the Operator, which can again be seen with the help of (6.2).

We now generalize Lemma 6.1 to any number N of possible time points for $k=1$ interim inspection. Because the detection time is for $i<j$ given by $(1-\beta)(j-i)+\beta\,(N+1-i)=$

$j - i + \beta\,(N+1-j)$, we get in analogy to (3.11)

$$Op_{N,1}(i,j) := \begin{cases} j - i + \beta\,(N+1-j) & \text{for} \quad 0 \le i < j < N+1 \\ N+1-i & \text{for} \quad 1 \le j \le i < N+1 \end{cases} . \qquad (6.3)$$

The game theoretical solution of this inspection game, which is a generalization of Theorem 3.1 and which is published in this monograph for the first time, is presented in

Theorem 6.1. *Given the No-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection, and with errors of the second kind. The sets of mixed strategies are given by (3.13) and (3.14), and the payoff to the Operator by (3.15) using (6.3). Define the cut-off value n^* by*

$$n^* := \min\left\{ n : n \in \{1, \ldots, N\} \quad \textit{with} \quad \sum_{j=1}^{n} \frac{1}{N+1-j} \ge 1 - \beta \right\} . \qquad (6.4)$$

Then an optimal strategy for the Operator is given by

$$p_i^* = \begin{cases} \dfrac{1}{N}\,(N+1-n^*) & \textit{for} \quad i = 0 \\ \dfrac{(N+1-n^*)}{(N+1-i)\,(N-i)} & \textit{for} \quad i = 1, \ldots, n^*-1 \\ 0 & \textit{for} \quad i = n^*, \ldots, N \end{cases} , \qquad (6.5)$$

and an optimal strategy of the Inspectorate by

$$q_j^* = \begin{cases} \dfrac{1}{1-\beta}\,\dfrac{1}{N+1-j} & \textit{for} \quad j = 1, \ldots, n^*-1 \\ 1 - \dfrac{1}{1-\beta} \displaystyle\sum_{j=1}^{n^*-1} \dfrac{1}{N+1-j} & \textit{for} \quad j = n^* \\ 0 & \textit{for} \quad j = n^*+1, \ldots, N \end{cases} . \qquad (6.6)$$

The optimal payoff to the Operator is

$$Op_{N,1}^* := Op_{N,1}(\mathbf{p}^*, \mathbf{q}^*) = \sum_{j=1}^{n^*} \frac{N+1-n^*}{N+1-j} + \beta\,(N+1-n^*) . \qquad (6.7)$$

Proof. The proof goes along the same lines as that of Theorem 3.1. According to (6.5), we have $p_{n^*-1}^* > 0$, thus, the indifference principle in Theorem 19.1 implies, using (6.3) and (6.6), that

$$\begin{aligned} Op_{N,1}(\mathbf{p}^*, \mathbf{q}^*) &= Op_{N,1}(n^*-1, \mathbf{q}^*) = (N+2-n^*) \sum_{j=1}^{n^*-1} q_j^* + (1 + \beta\,(N+1-n^*))\, q_{n^*}^* \\ &= (1-\beta)\,(N+1-n^*) \sum_{j=1}^{n^*-1} q_j^* + 1 + \beta\,(N+1-n^*) \end{aligned}$$

$$= \sum_{j=1}^{n^*-1} \frac{N+1-n^*}{N+1-j} + 1 + \beta\,(N+1-n^*)\,, \tag{6.8}$$

i.e., (6.7). □

Note that the first comment made on p. 34 on interesting properties of the optimal strategies $\mathbf{p}^*$ and $\mathbf{q}^*$ holds here as well. In Lemma 6.1 we have seen that 3 regions of β-values need to be distinguished. For arbitrary N the regions of β-values are, using (6.4), given by

$$\left[0, 1 - \sum_{j=1}^{n^*-1} \frac{1}{N-j+1}\right), \ldots, \left(1 - \frac{1}{N} - \frac{1}{N-1}, 1 - \frac{1}{N}\right), \left(1 - \frac{1}{N}, 1\right]\,,$$

which for $N = 5$ possible time points gives

$$\left[0, \frac{13}{60}\right), \left(\frac{13}{60}, \frac{11}{20}\right), \left(\frac{11}{20}, \frac{4}{5}\right), \left(\frac{4}{5}, 1\right]\,, \tag{6.9}$$

see also p. 111. Note that in Lemma 9.2 the continuous time version of this inspection game is treated.

Interesting enough, there is a close analogy to Lemma 3.2:

Corollary 6.1. *Given the No-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection, and with errors of the second kind analysed in Theorem 6.1.*

Then the following bounds hold for the cut-off value $n^*(N)$ and for the optimal expected detection time $Op^*_{N,1}$:

$$\left(1 - \frac{1}{e^{1-\beta}}\right) N < n^*(N) < \left(1 - \frac{1}{e^{1-\beta}}\right) (N+1) + 1 \tag{6.10}$$

and

$$N + 1 - n^* < Op^*_{N,1} < N + 2 - n^*\,. \tag{6.11}$$

Proof. According to the proof of Lemma 3.2 we get from (3.31)

$$\frac{1}{1-\beta} \ln\left[\frac{N+1}{N+1-n}\right] \le \frac{1}{1-\beta} \sum_{j=1}^{n} \frac{1}{N+1-j} \le \frac{1}{1-\beta} \ln\left[\frac{N}{N-n}\right] \tag{6.12}$$

for any $n \in \{1, \ldots, N-1\}$. Thus, by (6.4) we get[1]

$$\sum_{j=1}^{n^*} \frac{1}{N+1-j} \ge 1 - \beta \qquad \text{and} \qquad \sum_{j=1}^{n^*-1} \frac{1}{N+1-j} < 1 - \beta\,.$$

Making use of (6.12), this leads to

$$1 - \beta \le \ln\left[\frac{N}{N-n^*}\right] \qquad \text{and} \qquad \ln\left[\frac{N+1}{N+2-n^*}\right] < 1 - \beta\,.$$

[1]Note that contrary to the case of $\beta = 0$, the $\ge$ sign in (6.4) cannot be replaced by the $>$ sign; see the footnote on p. 27.

Combining both inequalities leads to (6.10). Because (6.4) yields

$$1-\beta \leq \sum_{j=1}^{n^*} \frac{1}{N+1-j} < 1-\beta + \frac{1}{N+1-n^*}\,,$$

multiplying these two inequalities with $N+1-n^*$ and using (6.7) we get

$$(1-\beta)\,(N+1-n^*) \leq Op^*_{N,1} - \beta\,(N+1-n^*) < (1-\beta)\,(N+1-n^*)+1\,,$$

which is equivalent to (6.11). □

In analogy to the derivations in (3.36), we obtain for the optimal expected time point for the start of the illegal activity $\mathbb{E}_{\mathbf{p}^*}(S)$, using (6.5),

$$\mathbb{E}_{\mathbf{p}^*}(S) = \sum_{i=0}^{N} i\,p_i^* = n^* - \sum_{i=1}^{n^*} \frac{N+1-n^*}{N+1-i} = n^* - Op^*_{N,1} + \beta\,(N+1-n^*)\,, \tag{6.13}$$

which can be explicitly confirmed for the case of $N=3$ possible time points on p. 103. In case of $\beta = 0$, (6.13) reduces to (3.37). Using (6.6) and (6.8), we get

$$\begin{aligned}
(1-\beta)\sum_{j=1}^{N}(N+1-j)\,q_j^* &= n^*-1+(1-\beta)\,(N+1-n^*) - \sum_{j=1}^{n^*-1}\frac{N+1-n^*}{N+1-j} \\
&= n^*-1+(1-\beta)\,(N+1-n^*) - (Op^*_{N,1} - 1 - \beta\,(N+1-n^*)) \\
&= N+1-Op^*_{N,1}\,,
\end{aligned}$$

which leads, using (6.3) and $p_0^* > 0$, to

$$Op^*_{N,1} = \mathbb{E}_{\mathbf{q}^*}(T_1) + \beta\sum_{j=1}^{N}(N+1-j)\,q_j^* = \mathbb{E}_{\mathbf{q}^*}(T_1) + \frac{\beta}{1-\beta}\,(N+1-Op^*_{N,1})\,.$$

Thus, the optimal expected interim inspection time point $\mathbb{E}_{\mathbf{q}^*}(T_1)$ is given by

$$Op^*_{N,1} = (1-\beta)\,\mathbb{E}_{\mathbf{q}^*}(T_1) + \beta\,(N+1)\,, \tag{6.14}$$

which leads, using (6.13), to

$$\mathbb{E}_{\mathbf{p}^*}(S) + (1-\beta)\,\mathbb{E}_{\mathbf{q}^*}(T_1) = (1-\beta)\,n^*\,, \tag{6.15}$$

which for $\beta = 0$ coincides with (3.38) and (3.39). The case of $N=3$ possible time points on p. 103 confirms the relations (6.14) and (6.15). Note also, that in the continuous time version of this inspection game which is analysed in Section 9.3 we find corresponding results.

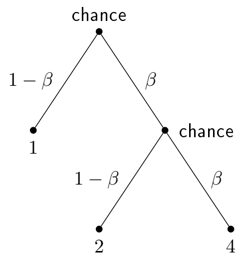
Let us now consider the inspection game with $N=3$ possible time points for $k=2$ interim inspections. Because the Operator's time point for starting the illegal activity does not depend on the number of interim inspections, his set of pure strategies is again given by I_3 with the set of mixed strategies P_3; see (3.2) and (3.4). In this game the Inspectorate's set of pure strategy is $\{(1,2),(1,3),(2,3)\}$, i.e., $J_{3,2}$ as given by (3.50) and therefore its set of mixed strategies is $Q_{3,2}$; see (3.52). The payoff matrix of this inspection game is given in Table 6.2.

Table 6.2 Payoff matrix of the No-No inspection game with $N = 3$ possible time points for $k = 2$ interim inspections and with errors of the second kind.

	$(1,2)$	$(1,3)$	$(2,3)$
0	$1+\beta+2\,\beta^2$	$1+2\,\beta+\beta^2$	$2+\beta+\beta^2$
1	$1+2\,\beta$	$2+\beta$	$1+\beta+\beta^2$
2	2	$1+\beta$	$1+\beta$
3	1	1	1

Let us explain the entry $i = 0$ and $(j_2, j_1) = (1, 2)$ of the payoff matrix. If the pure strategy combination $(0, (1, 2))$ is played, then the Operator starts the illegal activity at time point 0 and the Inspectorate performs its interim inspection at time points 1 and 2. Then the illegal activity is detected at 1 with probability $1 - \beta$ and not detected with probability β. In the latter case the illegal activity is detected at time point 2 again with probability $1 - \beta$ and not detected with probability β. In the latter case the Inspectorate detects the illegal activity at the final PIV with certainty. The decision tree in Figure 6.1 illustrates this situation.

Figure 6.1 Illustration of the computation of entry $(0, (1, 2))$ of the payoff matrix in Table 6.2.



Therefore, the Operator's payoff in case the pure strategy combination $(0, (1, 2))$ is played is $(1-\beta)\,1+\beta\,((1-\beta)\,2+\beta\,4) = 1+\beta+2\,\beta^2$. The remaining entries can be derived in a similar way.

With $\mathbf{p} := (p_0, p_1, p_2, p_3)^T \in P_3$ and $\mathbf{q} := (q_{(1,2)}, q_{(1,3)}, q_{(2,3)})^T \in Q_{3,2}$, where p_i is again the probability to start the illegal activity at time point i and $q_{(j_2,j_1)}$ the probability to perform the first interim inspection at time point j_2 and the second interim inspection at time point j_1, the (expected) payoff to the Operator, i.e., the expected detection time, is, using the payoff matrix in Table 6.2 and (19.3), given by

$$\begin{aligned} Op_{3,2}(\mathbf{p}, \mathbf{q}) := {} & p_0\,[(1+\beta+2\,\beta^2)\,q_{(1,2)} + (1+2\,\beta+\beta^2)\,q_{(1,3)} + (2+\beta+\beta^2)\,q_{(2,3)}] \\ & + p_1\,[(1+2\,\beta)\,q_{(1,2)} + (2+\beta)\,q_{(1,3)} + (1+\beta+\beta^2)\,q_{(2,3)}] \\ & + p_2\,[2\,q_{(1,2)} + (1+\beta)\,q_{(1,3)} + (1+\beta)\,q_{(2,3)}] + p_3\,. \end{aligned} \tag{6.16}$$

Starting the illegal activity at time point 3 is again a strictly dominated strategy, which can be seen as follows: Using $\mathbf{p}_1 := (0,0,0,1)^T$ and $\mathbf{p}_2 := (1/3,1/3,1/3,0)^T$, (6.16) implies

$$Op_{3,2}(\mathbf{p}_2,\mathbf{q}) = \frac{1}{3}\Big((4+3\,\beta+2\,\beta^2)\,q_{(1,2)} + (4+4\,\beta+\beta^2)\,q_{(1,3)} + (4+3\,\beta+2\,\beta^2)\,q_{(2,3)}\Big)$$

$$\geq \frac{1}{3}\Big(4+3\,\beta+2\,\beta^2\Big) \geq \frac{4}{3} > 1 = Op_{3,2}(\mathbf{p}_1,\mathbf{q})$$

for any $\mathbf{q} \in Q_{3,2}$, i.e., $\mathbf{p}_1$ is a strictly dominated strategy.

The game theoretical solution of this inspection game, see Avenhaus et al. (2010) and Avenhaus and Krieger (2013a), is presented in

Lemma 6.2. *Given the No-No inspection game with $N = 3$ possible time points for $k = 2$ interim inspections, and with errors of the second kind. The sets of mixed strategies are given by (3.4) and (3.52) with $N = 3$, and the payoff to the Operator by (6.16).*

Then optimal strategies and the optimal payoff $Op^*_{3,2} := Op_{3,2}(\mathbf{p}^*,\mathbf{q}^*)$ to the Operator are given by:

(i) For $0 \leq \beta < 1/2$ an optimal strategy of the Operator is given by

$$\mathbf{p}^* = \frac{1}{3+2\,\beta+\beta^2}\,\left(1+\beta,1,1+\beta+\beta^2,0\right)^T , \tag{6.17}$$

and an optimal strategy of the Inspectorate by

$$\mathbf{q}^* = \frac{1}{1-\beta}\,\frac{1-2\,\beta}{3+2\,\beta+\beta^2}\,\left(\frac{1+\beta+2\,\beta^2+\beta^3}{1-2\,\beta},1+\beta+\beta^2,1+\beta\right)^T .$$

The optimal payoff to the Operator is

$$Op^*_{3,2} = \frac{4+6\,\beta+5\,\beta^2+2\,\beta^3}{3+2\,\beta+\beta^2} .$$

(ii) For $1/2 < \beta \leq 1$ an optimal strategy of the Operator and an optimal strategy of the Inspectorate is given by

$$\mathbf{p}^* = (1,0,0,0)^T \qquad \textit{and} \qquad \mathbf{q}^* = (1,0,0)^T . \tag{6.18}$$

The optimal payoff to the Operator is

$$Op^*_{3,2} = 1+\beta+2\,\beta^2 .$$

Proof. We have to show, see (19.11), that

$$Op_{3,2}(i,\mathbf{q}^*) \leq Op^*_{3,2} \leq Op_{3,2}(\mathbf{p}^*,(j_2,j_1)) \tag{6.19}$$

for all $i = 0,1,2,3$ and for all $(j_2,j_1) \in \{(1,2),(1,3),(2,3)\} = J_{3,2}$; see (3.50).

Ad (i): We get

$$Op_{3,2}(i,\mathbf{q}^*) = \begin{cases} Op^*_{3,2} & : \quad i = 0,1,2 \\ 1 & : \quad i = 3 \end{cases} \quad \text{and} \quad \begin{array}{l} Op_{3,2}(\mathbf{p}^*,(j_2,j_1)) = Op^*_{3,2} \\ \text{for all } (j_2,j_1) \in J_{3,2} \end{array} ,$$

i.e., the inequalities in (6.19) are fulfilled.

Ad (ii): We obtain

$$Op_{3,2}(i,\mathbf{q}^*) = \begin{cases} Op^*_{3,2} & : \quad i=0 \\ 1+2\,\beta & : \quad i=1 \\ 2 & : \quad i=2 \\ 1 & : \quad i=3 \end{cases}$$

and

$$Op_{3,2}(\mathbf{p}^*,(j_2,j_1)) = \begin{cases} Op^*_{3,2} & : \quad (j_2,j_1)=(1,2) \\ 1+2\,\beta+\beta^2 & : \quad (j_2,j_1)=(1,3) \\ 2+\beta+\beta^2 & : \quad (j_2,j_1)=(2,3) \end{cases}$$

Again, the inequalities in (6.19) are fulfilled. □

Let us comment the results of Lemma 6.2: Like for the results in Lemma 6.1, it can be seen that the cut-off value n^* is strongly pronounced here for the case (ii). Interesting enough, and contrary to the case $k=1$ interim inspection, the Operator's optimal strategy is a function of β and not constant in the respective β-regions, and that there are only two regions of β-values with different solutions. We will come back to the last point on p. 113.

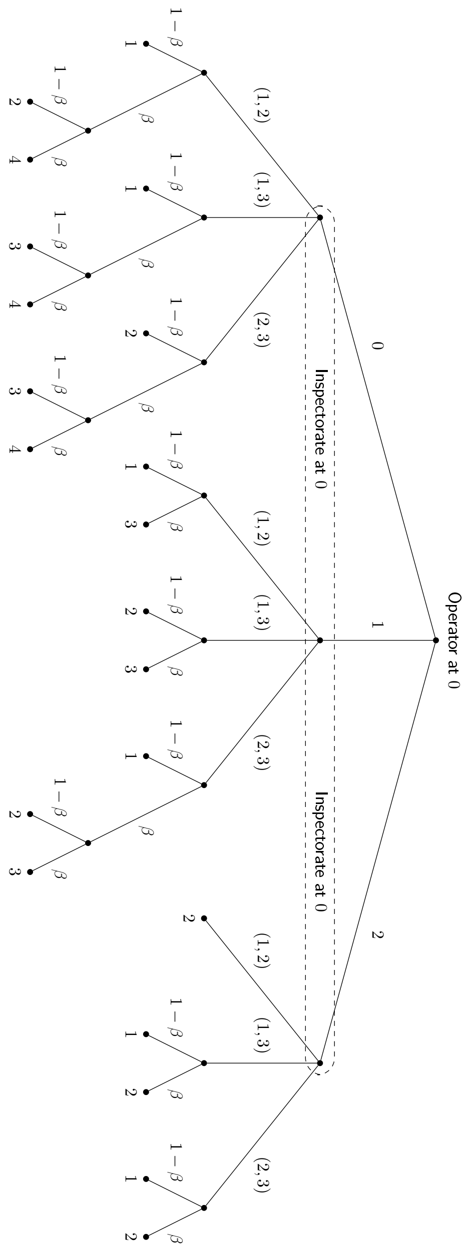
Before proceeding let us present the No-No inspection game with $N=3$ possible time points for $k=2$ interim inspections as an extensive form game, even though the presentation in normal form in Table 6.2 is the natural way to describe this No-No inspection game. In Figure 6.2 the game is represented in extensive form which is the appropriate way to describe games over time, if decision are taken sequentially. Due to the comment on p. 50, the game starts with the Operator and the chance moves can be identified via the probabilities $1-\beta$ and β.

At the beginning of the reference time interval, the Operator decides at which time point he will start the illegal activity. The Inspectorate, not knowing the Operator's decision, decides also at the beginning when to perform its two interim inspections. Its lack of knowledge is described by its information set, the encircled area, in Figure 6.2. After both players' decisions, chance moves decide whether the illegal activity will be detected during the next interim inspection.

Let us repeat that we do not need to describe the No-No inspection game in extensive form. In all of the following No-Se, Se-No and Se-Se inspection games, however, the extensive form is the appropriate form to describe these games. Therefore, it is helpful to see in which way the graphical representations of these variants differ.

Finally, we consider the case of $k=3$ interim inspections. The treatment of this case is simple, since the Inspectorate has no real choice because it has only the pure strategy $(1,2,3)$, i.e., it has to perform its interim inspections at any possible time point. The game is depicted in Table 6.3. The entries in the payoff matrix can again be determined with help of a kind of decision tree like in Figure 6.1. If $\beta>0$, then the Operator will always choose time point $i=0$ for the start of the illegal activity, since the detection time is then as large as possible. In case of $\beta=0$ the Operator is, of course, indifferent between all of his pure strategies $0,1,2$ and 3, because the detection time is always 1.

Figure 6.2 Extensive form of the No-No inspection game with $N = 3$ possible time points for $k = 2$ interim inspections and with errors of the second kind.



Instead of formulating a Lemma, we just mention that the optimal strategies of both players for $\beta > 0$ are given by $\mathbf{p}^* = (1,0,0,0)^T$ and $\mathbf{q}^* = q^*_{(1,2,3)} = 1$, and the optimal payoff to the Operator by $Op^*_{3,3} = 1 + \beta + \beta^2 + \beta^3$.

Table 6.3 Payoff matrix of the No-No inspection game with $N = 3$ possible time points for $k = 3$ interim inspections and with errors of the second kind.

	$(1,2,3)$
0	$1+\beta+\beta^2+\beta^3$
1	$1+\beta+\beta^2$
2	$1+\beta$
3	1

Let us conclude this section with the No-No inspection game with $N = 5$ possible time points for $k = 1$ and $k = 2$ interim inspection(s), because it has been of practical interest; see Avenhaus et al. (2010) and the comments on p. 123. For $\beta > 0$ and $k = 1$ interim inspection, the 6×5 payoff matrix of this inspection game, see Avenhaus et al. (2010), is presented in Table 6.4. The entries are of first order in β.

Table 6.4 Payoff matrix of the No-No inspection game with $N = 5$ possible time points for $k = 1$ interim inspection and with errors of the second kind.

	1	2	3	4	5
0	$1+5\beta$	$2+4\beta$	$3+3\beta$	$4+2\beta$	$5+\beta$
1	5	$1+4\beta$	$2+3\beta$	$3+2\beta$	$4+\beta$
2	4	4	$1+3\beta$	$2+2\beta$	$3+\beta$
3	3	3	3	$1+2\beta$	$2+\beta$
4	2	2	2	2	$1+\beta$
5	1	1	1	1	1

Because a manual solution would already be somewhat cumbersome, Canty's Mathematica® programs, see Canty (2003), have been used for the determination of the optimal strategies and the optimal expected detection time. Just to illustrate the results, the optimal expected detection time $Op^*_{5,1}$ is given by

$$Op^*_{5,1} = \begin{cases} \frac{77}{30} + 2\beta & \text{for} \quad 0 \le \beta < \frac{13}{60} \\ \frac{47}{20} + 3\beta & \text{for} \quad \frac{13}{60} < \beta < \frac{11}{20} \\ \frac{9}{5} + 4\beta & \text{for} \quad \frac{11}{20} < \beta < \frac{4}{5} \\ 1 + 5\beta & \text{for} \quad \frac{4}{5} < \beta \le 1 \end{cases}, \tag{6.20}$$

where the four regions of β-values are given by (6.9). We see that with increasing β the optimal expected detection time increases continuously from 2.57 to 6, which is intuitive.

For $\beta > 0$ and $k = 2$ interim inspections, the 6×10 payoff matrix of this inspection game, see Avenhaus et al. (2010), is given in Table 6.5. This time the entries are of second order in β.

Table 6.5 Payoff matrix of the No-No inspection game with $N = 5$ possible time points for $k = 2$ interim inspections and with errors of the second kind.

	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$
0	$1+\beta+4\beta^2$	$1+2\beta+3\beta^2$	$1+3\beta+2\beta^2$	$1+4\beta+\beta^2$	$2+\beta+3\beta^2$
1	$1+4\beta$	$2+3\beta$	$3+2\beta$	$4+\beta$	$1+\beta+3\beta^2$
2	4	$1+3\beta$	$2+2\beta$	$3+\beta$	$1+3\beta$
3	3	3	$1+2\beta$	$2+\beta$	3
4	2	2	2	$1+\beta$	2
5	1	1	1	1	1

	$(2,4)$	$(2,5)$	$(3,4)$	$(3,5)$	$(4,5)$
0	$2+2\beta+2\beta^2$	$2+3\beta+\beta^2$	$3+\beta+2\beta^2$	$3+2\beta+\beta^2$	$4+\beta+\beta^2$
1	$1+2\beta+2\beta^2$	$1+3\beta+\beta^2$	$2+\beta+2\beta^2$	$2+2\beta+\beta^2$	$3+\beta+\beta^2$
2	$2+2\beta$	$3+\beta$	$1+\beta+\beta^2$	$1+2\beta+\beta^2$	$2+\beta+\beta^2$
3	$1+2\beta$	$2+\beta$	$1+2\beta$	$2+\beta$	$1+\beta+\beta^2$
4	2	$1+\beta$	2	$1+\beta$	$1+\beta$
5	1	1	1	1	1

Consequently, the solution is complicated; the optimal expected detection time $Op^*_{5,2}$ is a ratio of polynomials in β up to the fifth order:

$$Op^*_{5,2} = \begin{cases} \dfrac{59+133\beta+128\beta^2+62\beta^3+12\beta^4}{34+48\beta+30\beta^2+8\beta^3} & \text{for} \quad 0 \le \beta \le 0.15 \\ \dfrac{34+76\beta+53\beta^2-2\beta^3-26\beta^4-12\beta^5}{20+24\beta+10\beta^2-6\beta^3-8\beta^4} & \text{for} \quad 0.16 \le \beta \le 0.17 \\ \dfrac{26+48\beta+23\beta^2+2\beta^3-14\beta^4-12\beta^5}{16+10\beta-5\beta^2-4\beta^3-2\beta^4} & \text{for} \quad 0.18 \le \beta \le 0.23 \\ \dfrac{19+44\beta+50\beta^2+24\beta^3}{6(2+2\beta+\beta^2)} & \text{for} \quad 0.24 \le \beta \le 0.29 \\ \dfrac{23+77\beta+102\beta^2+72\beta^3}{15+27\beta+18\beta^2} & \text{for} \quad 0.30 \le \beta \le 0.52 \\ \dfrac{10+28\beta+33\beta^2+36\beta^3}{7+10\beta+3\beta^2} & \text{for} \quad 0.53 \le \beta \le 0.74 \\ 1+\beta+4\beta^2 & \text{for} \quad 0.75 \le \beta \le 1 \end{cases} . \tag{6.21}$$

Table 6.6 illustrates the pure strategies which are mixed in the Inspectorate's optimal strategy in case of $N = 5$ possible time points for $k = 2$ interim inspections. Again, like in case of $\beta = 0$ in Tables 3.5 and 3.6, a kind of step structure can be conjectured.

Table 6.6 Pure strategies which are mixed in the Inspectorate's optimal strategy.

	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$	$(2,4)$	$(2,5)$	$(3,4)$	$(3,5)$	$(4,5)$
$0 \leq \beta \leq 0.15$	$=0$	>0	>0	$=0$	$=0$	>0	>0	$=0$	>0	$=0$
$0.16 \leq \beta \leq 0.17$	$=0$	>0	>0	$=0$	$=0$	>0	$=0$	>0	>0	$=0$
$0.18 \leq \beta \leq 0.23$	$=0$	>0	>0	$=0$	$=0$	>0	$=0$	>0	$=0$	$=0$
$0.24 \leq \beta \leq 0.29$	>0	>0	>0	$=0$	$=0$	>0	$=0$	$=0$	$=0$	$=0$
$0.30 \leq \beta \leq 0.52$	>0	>0	$=0$	$=0$	>0	>0	$=0$	$=0$	$=0$	$=0$
$0.53 \leq \beta \leq 0.74$	>0	>0	$=0$	$=0$	>0	$=0$	$=0$	$=0$	$=0$	$=0$
$0.75 \leq \beta \leq 1$	$=1$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$

Looking at all cases of N and k we have considered in this section together with the case $N = k = 5$ and counting the number of regions of β-values we get

- $(N, k) = (3, 1)$: 3 regions; Lemma 6.1;
- $(N, k) = (3, 2)$: 2 regions; Lemma 6.2;
- $(N, k) = (3, 3)$: 1 region;
- $(N, k) = (5, 1)$: 4 regions; see (6.9);
- $(N, k) = (5, 2)$: 7 regions; see (6.21);
- $(N, k) = (5, 5)$: 1 region.

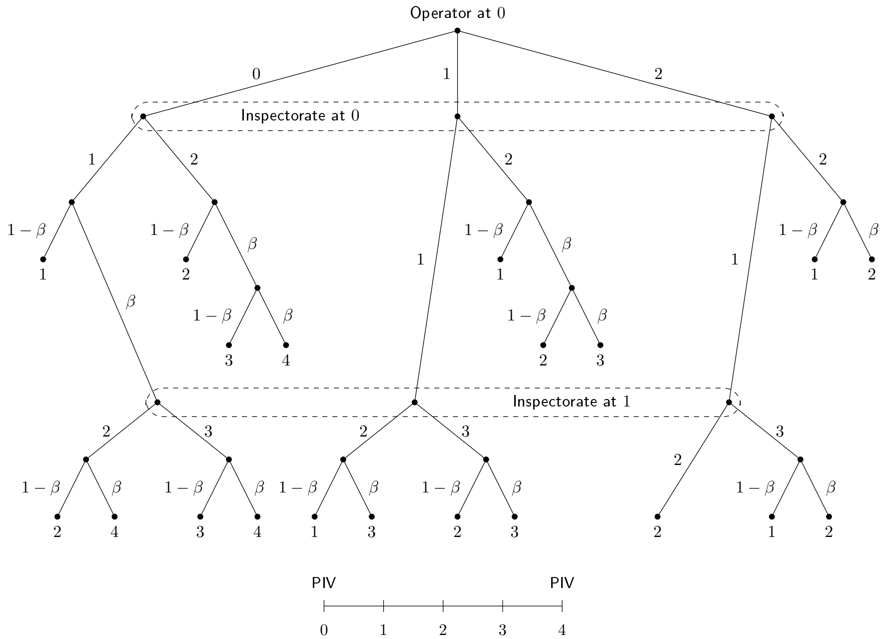
Except for the cases $k = N$, it is difficult to imagine a kind of rule for the number of regions of β-values; see also p. 123.

6.2 No-Se: Three inspection opportunities and two interim inspections

In this variant of the inspection game, the Operator decides at the beginning of the reference time interval when to start the illegal activity, whereas the Inspectorate decides at the beginning of the reference time interval only when the first interim inspection is performed. This means that in case there is only $k = 1$ interim inspection, the No-Se inspection game is identical to the No-No inspection game. Thus, in case of any number N of possible time points for $k = 1$ interim inspection and for $\beta \geq 0$, Theorem 6.1 and Corollary 6.1 hold here as well.

As mentioned on p. 49, the No-Se inspection game with any number N of possible time points for $k > 1$ interim inspection have not been analysed so far. However, for $\beta > 0$, and as already announced at the beginning of Section 4.1, the special case of $N = 3$ possible time points for $k = 2$ interim inspections has been analysed in Avenhaus and Krieger (2013a) and the results will be presented now. The extensive form of this No-Se inspection game is represented in Figure 6.3. Due to the sequential behaviour of the Inspectorate this extensive form has a more complicated structure than that of the No-No inspection game in Figure 6.2. According to the comment on p. 50, the game starts as in Figure 6.2 with the Operator and the chance moves can be identified via the probabilities $1-\beta$ and β.

Figure 6.3 Extensive form of the No-Se inspection game with $N = 3$ possible time points for $k = 2$ interim inspections and with errors of the second kind.



In this game the Operator decides at time point 0 when to start the illegal activity, namely at 0, 1, 2 or at 3; see also the next paragraph. The Inspectorate, however, decides at time point 0 only if to perform the first interim inspection at time point 1 or 2. In case that the first interim inspection is carried through at time point 1, it decides if to perform the second interim inspection at time point 2 or 3. In case the first interim inspection takes place at time point 2, the second one has to be performed at time point 3.

Before explaining the information structure in this game, let us shortly discuss time point 3 as start of the illegal activity. We have seen for the No-No inspection game in Lemma 6.2 that time point 3 is never chosen in an optimal strategy, see (6.17) and (6.18), which is also intuitive because starting at time point 3 is a strictly dominated strategy; see p. 107. Does this mean that we can a priori exclude this time point in the No-Se inspection game? We think that two stages have to be distinguished: First, building the model which reflects the reality as close as possible and – only thereafter – finding solutions of the game, e.g., by identifying dominated strategies. Thus, we need to model time point 3 as a potential starting point for the illegal activity. But even modellers have to make compromises: In the extensive form of the game described in Figure 6.3 we have excluded time point 3 for the sake of transparency and only for this reason! In the subsequent description of the strategy sets and (expected) payoffs, however, this strategy and its corresponding probability is included again; see P_3 and (6.24).

Let us look in more detail at the case, that the Inspectorate decides at time point 0 to perform the first interim inspection at time point 1 and the Operator starts the illegal activity at time point 0, since here it can be observed the first time in this monograph in which way second kind errors affect the information structure of the game. The illegal activity will be detected at time point 1 with probability $1-\beta$ and the detection time is 1. With probability $\beta\,(1-\beta)$ it will be detected at the second interim inspection and the detection time is 2 or 3 if it takes place at time points 2 or 3. With probability β^2 it will be detected only at the final PIV and the detection time is 4.

Also, it is important to realize that the Inspectorate, having performed the first interim inspection at time point 1, and having not detected the illegal activity which started at time point 0, i.e., the left node in the information set "Inspectorate at 1", does not know at time point 1 if the Operator acted this way, or else will start the illegal activity only at time point 1 or 2.

As in Section 6.1, let p_i be the Operator's probabilities to choose the time points $0, 1, 2$ and 3 for the start of the illegal activity, i.e., his set of mixed strategies is given by P_3; see (3.4) and the comment on p. 114. As introduced in Section 5.7, let $h_3(j_2), j_2 = 1, 2$, be the probability that the first interim inspection is performed at time point j_2, and let $h_2(j_1|1), j_1 = 2, 3$, be the probability that the second interim inspection is performed at time point j_1 when the first one has been performed at time point 1. For the sake of simplicity we define here

$$h_3 := h_3(1)\,, \quad 1-h_3 = h_3(2) \quad \text{and} \quad h_2 := h_2(2|1)\,, \quad 1-h_2 = h_2(3|1)\,. \tag{6.22}$$

Because we have $h_2(3|2) = 1$ by definition and thus, it does not need to be considered here. Thus, the Inspectorate's set of behavioural strategies is instead of $H_{3,2}$, see (5.7), given by

$$\tilde{H}_{3,2} := \{\mathbf{h} := (h_3, h_2) \in [0,1]^2 : h_3, h_2 \in [0,1]\}\,. \tag{6.23}$$

The (expected) payoff to the Operator, i.e., the expected detection time, is, for any $\mathbf{p} \in P_3$ and any $\mathbf{h} \in \tilde{H}_{3,2}$, given by

$$\begin{aligned}
&Op_{3,2}(\mathbf{p}, \mathbf{h}) \\
&:= p_0\,[h_3\,(1-\beta+\beta\,(h_2\,(2+2\,\beta)+(1-h_2)\,(3+\beta)))+(1-h_3)\,(2+\beta+\beta^2)] \\
&+ p_1\,[h_3\,(h_2\,(1+2\,\beta)+(1-h_2)\,(2+\beta))+(1-h_3)\,(1+\beta+\beta^2)] \\
&+ p_2\,[h_3\,(h_2\,2+(1-h_2)\,(1+\beta))+(1-h_3)\,(1+\beta)]+p_3\,.
\end{aligned} \tag{6.24}$$

The fact that starting the illegal activity at time point 3 is indeed a strictly dominated strategy can be seen as follows: Using the strategies $\mathbf{p}_1 := (0,0,0,1)^T$ and $\mathbf{p}_2 := (1/3, 1/3, 1/3, 0)^T$, (6.24) implies

$$\begin{aligned}
Op_{3,2}(\mathbf{p}_2, \mathbf{h}) &= \frac{1}{3}\left(4+3\,\beta+2\,\beta^2+\beta\,(1-\beta)\,h_3\,(1-h_2)\right) \\
&\geq \frac{1}{3}\left(4+3\,\beta+2\,\beta^2\right) \geq \frac{4}{3} > 1 = Op_{4,2}(\mathbf{p}_1, \mathbf{h})
\end{aligned}$$

for any $\mathbf{h} \in \tilde{H}_{3,2}$, i.e., $\mathbf{p}_1$ is a strictly dominated strategy.

The game theoretical solution of this inspection game, see Avenhaus and Krieger (2013a), is presented in

Lemma 6.3. *Given the No-Se inspection game with $N = 3$ possible time points for $k = 2$ interim inspections, and with errors of the second kind. The sets of mixed resp. behavioural strategies are given by (3.4) and (6.23), and the payoff to the Operator by (6.24).*

Then optimal strategies and the optimal payoff $Op^*_{3,2} := Op_{3,2}(\mathbf{p}^*, \mathbf{h}^*)$ to the Operator are given by:

(i) For $0 \le \beta < 1/2$ an optimal strategy of the Operator is given by

$$\mathbf{p}^* = \frac{1}{3 + 2\,\beta + \beta^2}\,\left(1 + \beta, 1, 1 + \beta + \beta^2, 0\right)^T ,$$

and an optimal strategy of the Inspectorate by

$$h_3^* = 1 - \frac{1}{1-\beta}\,\frac{(1 - 2\,\beta)\,(1 + \beta)}{3 + 2\,\beta + \beta^2} \qquad \textit{and} \qquad h_2^* = 1 - \frac{(1 - 2\,\beta)\,(1 + \beta + \beta^2)}{2 + \beta^2 - \beta^3} .$$

The optimal payoff to the Operator is

$$Op^*_{3,2} = \frac{4 + 6\,\beta + 5\,\beta^2 + 2\,\beta^3}{3 + 2\,\beta + \beta^2} .$$

(ii) For $1/2 < \beta \le 1$ an optimal strategy of the Operator and an optimal strategy of the Inspectorate is given by

$$\mathbf{p}^* = (1, 0, 0, 0)^T \qquad \textit{and} \qquad h_3^* = h_2^* = 1 .$$

The optimal payoff to the Operator is

$$Op^*_{3,2} = 1 + \beta + 2\,\beta^2 .$$

Proof. Using (19.10), we have to show that

$$Op_{3,2}(\mathbf{p}, \mathbf{h}^*) \le Op^*_{3,2} \le Op_{3,2}(\mathbf{p}^*, \mathbf{h}) \tag{6.25}$$

for any $\mathbf{p} \in P_3$ and any $\mathbf{h} \in \tilde{H}_{3,2}$.

Ad (i): We have $Op_{3,2}(\mathbf{p}, \mathbf{h}^*) = Op^*_{3,2} = Op_{3,2}(\mathbf{p}^*, \mathbf{h})$ for any $\mathbf{p} \in P_3$ and any $\mathbf{h} \in \tilde{H}_{3,2}$, i.e., the inequalities in (6.25) are fulfilled as equalities.

Ad (ii): (6.24) yields for any $\mathbf{p} \in P_3$

$$Op_{3,2}(\mathbf{p}, (1,1)) = p_0\,(1 + \beta + 2\,\beta^2) + p_1\,(1 + 2\,\beta) + p_2\,2 + p_3 ,$$

which is, because of $1 + \beta + 2\,\beta^2 > 1 + 2\,\beta > 2$, maximized for $p_0^* = 1$, i.e., the left hand inequality in (6.25) is fulfilled. For any $\mathbf{h} \in \tilde{H}_{3,2}$, (6.24) implies

$$Op_{3,2}(\mathbf{p}^*, \mathbf{h}) = 2 + \beta + \beta^2 - h_3\,(1 - \beta)\,(1 + h_2\,\beta) ,$$

which is minimized in case of $\beta \in [0,1)$ for

$$h_2^*(h_3) = \begin{cases} 1 & \text{for} \quad h_3 \in (0,1] \\ [0,1] & \text{for} \quad h_3 = 0 \end{cases} ,$$

which leads to

$$Op_{3,2}(\mathbf{p}^*, (h_3, h_2^*(h_3))) = \begin{cases} 2 + \beta + \beta^2 - h_3\,(1-\beta)\,(1+\beta) & \text{for} \quad h_3 \in (0,1] \\ 2 + \beta + \beta^2 & \text{for} \quad h_3 = 0 \end{cases} ,$$

and which is minimized for $h_3^* = 1$. If $\beta = 1$, then $Op_{3,2}(\mathbf{p}^*, \mathbf{h}) = 4 = Op_{3,2}^*$ for any $\mathbf{h} \in \tilde{H}_{3,2}$, i.e., especially for $h_3^* = h_2^* = 1$. □

Comparing the results of Lemmata 6.2 and 6.3, we see that the Operator's optimal strategies and the optimal payoff to the Operator coincide, even though the structure of the extensive form of both inspection games is so different; see Figures 6.2 and 6.3. That the Operator's optimal strategies and the optimal payoff are the same is actually not so surprising, because the payoffs (6.16) and (6.24) can be transformed into each other: If one replaces in (6.16) the probabilities $q_{(1,2)}$, $q_{(1,3)}$ and $q_{(2,3)}$ by (recall the definitions in (6.22))

$$q_{(1,2)} = h_3\,h_2\,, \quad q_{(1,3)} = h_3\,(1-h_2) \quad \text{and} \quad q_{(2,3)} = 1 - h_3\,, \tag{6.26}$$

one obtains (6.24), and if one replaces in (6.24) the probabilities h_3 and h_2 by

$$h_3 = 1 - q_{(2,3)} \quad \text{and} \quad h_2 = q_{(1,2)}/(1 - q_{(2,3)})\,, \tag{6.27}$$

provided the appropriate ratio exists, one obtains (6.16). Thus, it is no longer surprising that the two inspection games lead to the same optimal payoffs. Note that the equivalence of the payoffs (6.16) and (6.24) could have also been used to proof Lemma 6.3.

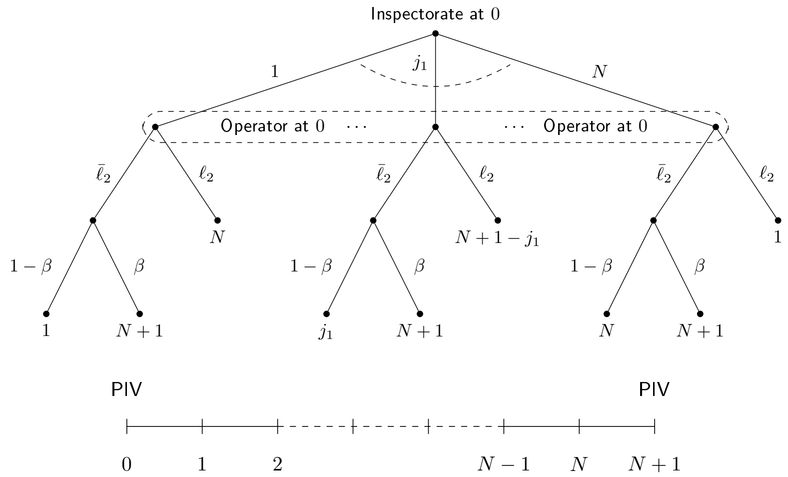
6.3 Se-No: Any number of inspection opportunities and one interim inspection; three inspection opportunities and two interim inspections

Since the case of any number N of possible time points for $k = 1$ interim inspection can be analysed as easily as that of any special number N, we do this here. Only thereafter we turn to the case of $N = 3$ possible time points for $k = 2$ interim inspections.

In Figure 6.4 the extensive form of the Se-No inspection game is represented. In line with the comment on p. 50, the games in Figures 6.4 and 6.5 start with the Inspectorate's decision at time point 0. Again, the chance moves can be identified via the probabilities $1 - \beta$ and β.

As described in Section 4.2, the Inspectorate decides at time point 0 at which time point j_1, $j_1 = 1, \ldots, N$, of the reference time interval the interim inspection is performed. The Operator also decides at time point 0 – not knowing the Inspectorate's interim inspection time point – whether to start the illegal activity immediately ($\bar{\ell}_2$) or to postpone its start (ℓ_2). If he starts it at time point 0 ($\bar{\ell}_2$), then it is detected with probability $1 - \beta$ at time point j_1 and with probability β at time point $N+1$, the final PIV. If the Operator postpones the start to time point j_1 (ℓ_2), then the detection time is $N+1-j_1$. The lack of information on the Operator's

Figure 6.4 Extensive form of the Se-No inspection game with $N > k$ possible time points for $k = 1$ interim inspection and with errors of the second kind.



side is indicated by the information set in Figure 6.4: He does not know when the Inspectorate will perform its interim inspection. The detection times are given at the end nodes again. For $\beta = 0$, Figure 6.4 reduces to Figure 4.2. The normal form of this extensive form game is given in Table 6.7.

Table 6.7 Payoff matrix for the Se-No inspection game represented in Figure 6.4.

	1	$\cdots$	j_1	$\cdots$	N
$\bar{\ell}_2$	$(1-\beta)\,1+\beta\,(N+1)$	$\cdots$	$(1-\beta)\,j_1+\beta\,(N+1)$	$\cdots$	$(1-\beta)\,N+\beta\,(N+1)$
ℓ_2	N	$\cdots$	$N+1-j_1$	$\cdots$	1

Because the Inspectorate chooses at time point 0 one of the time points $1, \ldots, N$ for its only interim inspection, its set of pure strategies is $J_{N,1}$ as given by (3.10). If q_{j_1} denotes the probability to choose time point j_1 for the interim inspection, then $\mathbf{q} := (q_1, \ldots, q_N)^T$ with $q_{j_1} \geq 0$ for all $j_1 = 1, \ldots, N$ and $\sum_{j_1=1}^{N} q_{j_1} = 1$ is a mixed strategy of the Inspectorate, and its set of mixed strategy is given by $Q_{N,1}$; see (3.14). Like in Section 4.2, let $1 - g_2$ be the Operator's probability to start the illegal activity at time point 0 and g_2 the probability to postpone it. In the latter case he starts the illegal activity after the interim inspection at time point j_1. Formally, the set of behavioural strategies G_1 of the Operator is given by (4.7).

Using Table 6.7, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $g_2 \in G_1$ and any $\mathbf{q} \in Q_{N,1}$, given by

$$Op_{N,1}(g_2, \mathbf{q}) := \sum_{j_1=1}^{N} q_{j_1} \Big((1-g_2)\,((1-\beta)\,j_1 + \beta\,(N+1)) + g_2\,(N+1-j_1) \Big) . \tag{6.28}$$

The game theoretical solution of this inspection game, see Avenhaus et al. (2010), is presented in

Lemma 6.4. *Given the Se-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection, and with errors of the second kind. The sets of behavioural resp. mixed strategies are given by (4.7) and (3.14), and the payoff to the Operator by (6.28).*

Then an optimal strategy of the Operator is given by

$$g_2^* = \frac{1-\beta}{2-\beta}\,, \tag{6.29}$$

and an optimal strategy $\mathbf{q}^* := (q_1^*, \ldots, q_N^*)^T$ of the Inspectorate fulfils the conditions

$$\sum_{j_1=1}^{N} j_1\, q_{j_1}^* = \frac{1-\beta}{2-\beta}\,(N+1) \qquad \textit{with} \qquad \sum_{j_1=1}^{N} q_{j_1}^* = 1\,. \tag{6.30}$$

The optimal payoff to the Operator is

$$Op_{N,1}^* := Op_{N,1}(g_2^*, \mathbf{q}^*) = \frac{N+1}{2-\beta}\,. \tag{6.31}$$

Proof. We have to show that

$$Op_{N,1}(i, \mathbf{q}^*) \leq Op_{N,1}^* \leq Op_{N,1}(g_2^*, j_1) \tag{6.32}$$

for all $i = \bar{\ell}_2, \ell_2$ and for all $j_1 = 1, \ldots, N$; see (19.11). The left hand inequality of (6.32) is equivalent to

$$\sum_{j_1=1}^{N} q_{j_1}^*\,((1-\beta)\,j_1 + \beta\,(N+1)) \leq \frac{N+1}{2-\beta} \qquad \text{and} \qquad \sum_{j_1=1}^{N} q_{j_1}^*\,(N+1-j_1) \leq \frac{N+1}{2-\beta}\,,$$

which holds as equalities because of (6.30). The right hand side of (6.32) is by (6.29) equivalent to

$$\frac{N+1}{2-\beta} \leq \frac{1}{2-\beta}\,((1-\beta)\,j_1 + \beta\,(N+1)) + \frac{1-\beta}{2-\beta}\,(N+1-j_1)$$

for all $j_1 = 1, \ldots, N$, which is also fulfilled as equality for all j_1. This completes the proof. □

Let us comment the results of Lemma 6.4: First, they are – as expected – generalizations of the results of Lemma 4.2.

Second, from a *theoretical* point of view it is interesting to note that in case the pure strategy $(0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 at position j_1^*, of the Inspectorate fulfils the condition, see Table 6.7,

$$(1-\beta)\,j_1^* + \beta\,(N+1) = N + 1 - j_1^*\,,$$

which means that

$$j_1^* = \frac{1-\beta}{2-\beta}\,(N+1) \tag{6.33}$$

is an integer, then according to (6.30) the Inspectorate can use this *pure* strategy, i.e., it can announce the time point j_1^* in advance. The larger the error of the second kind probability β is, the smaller is this time point j_1^*, which is intuitive. For the sake of illustration, consider the case of $N = 3$ possible time points. Then (6.33) yields $j_1^* = 1$ for $\beta = 2/3$ and $j_1^* = 2$ for $\beta = 0$. In case of $N = 5$ possible time points, (6.33) gives $j_1^* = 1$ for $\beta = 5/6$, $j_1^* = 2$ for $\beta = 3/5$ and $j_1^* = 3$ for $\beta = 1/4$. Let us mention, however, that for *practical* applications these results are not so interesting, since these special β-values will be hardly realized.

Third, using (6.30) and (6.31), we obtain for the optimal expected interim inspection time point

$$\mathbb{E}_{\mathbf{q}^*}(T_1) := \sum_{j_1=1}^{N} j_1\, q_{j_1}^* = \frac{1-\beta}{2-\beta}\,(N+1) = (1-\beta)\, Op_{N,1}^*\,, \tag{6.34}$$

and for *all* optimal strategies $\mathbf{q}^*$ of the Inspectorate and *all* $\beta \geq 0$

$$\mathbb{E}_{\mathbf{q}^*}(T_1) + Op_{N,1}^* = N + 1\,.$$

The optimal expected time point for the start of the illegal activity is, using (6.34), given by

$$\mathbb{E}_{(g_2^*,\mathbf{q}^*)}(S) = (1-g_2^*)\,0 + g_2^*\,\mathbb{E}_{\mathbf{q}^*}(T_1) = \frac{(1-\beta)^2}{2-\beta}\, Op_{N,1}^*\,.$$

A comparison of the system quantities of the discrete time No-No and Se-No inspection games is presented in Table 4.2 on p. 74.

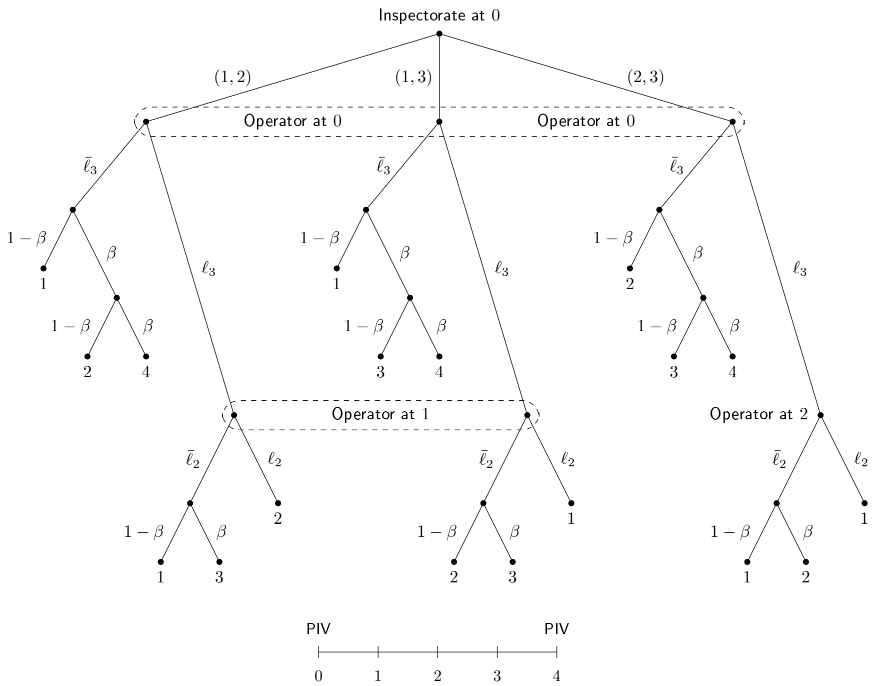
We now turn to the case of $N = 3$ possible time points for $k = 2$ interim inspections and errors of the second kind. The extensive form of this inspection game is represented in Figure 6.5.

The Inspectorate decides at the beginning of the reference time interval, i.e., at time point 0, where to place its two interim inspections: It has the three possibilities $(1,2)$, $(1,3)$ and $(2,3)$. The Operator also decides at time point 0 whether to start the illegal activity immediately ($\bar{\ell}_3$) or not (ℓ_3). In latter case, he decides after the first interim inspection whether to start his illegal activity now ($\bar{\ell}_2$) or postpone its start again (ℓ_2). In the latter he must start it after the second interim inspection with certainty due to assumption (iv) of Chapter 2.

We see that the information structure of that inspection game is more complicated than in the case of $k = 1$ interim inspection, see Figure 6.4: The Operator has three information sets: The information set named "Operator at 0" illustrates the fact that both player choose their strategies at time point 0 independently of each other; see assumption (viii) of Chapter 2. The information set named "Operator at 1" is due to the fact that after the first interim inspection at time point 1 he does not know when the second interim inspection will be performed. The information set named "Operator at 2" consists of only one node because after the first interim inspection at time point 2 he knows that the second one will be performed at time point 3.

Let $q_{(j_2,j_1)}$ be the Inspectorate's probabilities to choose the pair (j_2, j_1) of time points for the two interim inspections. Because $J_{3,2}$ as given by (3.50) is the Inspectorate's pure strategy set, its set of mixed strategies is $Q_{3,2}$ and given by (3.52) for $N = 3$. Let $1 - g_3$ and g_3 be the Operator's probabilities to start the illegal activity at time point 0 ($\bar{\ell}_3$) or to postpone its start (ℓ_3). In the latter case let $1 - g_2$ and g_2 be the probabilities to start the illegal activity right after the first interim inspection at time point 1 ($\bar{\ell}_2$) or to postpone its start again (ℓ_2). Thus, the set of behavioural strategies of the Operator is G_2 as given by (4.15), where here – in contrast to (4.15) – we write g_2 instead of $g_2(1)$, and thus, $g_2(2) = 1 - g_2$.

Figure 6.5 Extensive form of the Se-No inspection game with $N = 3$ possible time points for $k = 2$ interim inspections and with errors of the second kind.



Suppose the Inspectorate plays the pure strategy $(2, 3)$ and the Operator decides for ℓ_3. Then at time point 2, the Operator will start the illegal activity immediately at 2 ($\bar{\ell}_2$) because then he assures himself the detection time $1 + \beta$ which is, if $\beta > 0$, larger than the detection time 1 in case of postponing the start until time point 3. Thus, in case $(2, 3)$ and ℓ_3 is played, the detection time is $1 + \beta$.

Using Figure 6.5, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $\mathbf{g} \in G_2$ and any $\mathbf{q} \in Q_{3,2}$, given by

$$\begin{aligned} Op_{3,2}(\mathbf{g}, \mathbf{q}) := {} & q_{(1,2)} \left[(1 - g_3)\,(1 + \beta + 2\,\beta^2) + g_3\,((1 - g_2)\,(1 + 2\,\beta) + g_2\,2)\right] \\ & + q_{(1,3)} \left[(1 - g_3)\,(1 + 2\,\beta + \beta^2) + g_3\,((1 - g_2)\,(2 + \beta) + g_2)\right] \\ & + q_{(2,3)} \left[(1 - g_3)\,(2 + \beta + \beta^2) + g_3\,(1 + \beta)\right]. \end{aligned} \tag{6.35}$$

The game theoretical solution of this inspection game, see Avenhaus et al. (2010), however, with an incorrect probability $q^*_{(2,3)}$, and Avenhaus and Krieger (2013a), is presented in

Lemma 6.5. *Given the Se-No inspection game with $N = 3$ possible time points for $k = 2$ interim inspections, and with errors of the second kind. The sets of behavioural resp. mixed strategies are given by (4.15) and (3.52) for $N = 3$, and the payoff to the Operator by (6.35).*

Then optimal strategies and the optimal payoff $Op^*_{3,2} := Op_{3,2}(\mathbf{g}^*, \mathbf{q}^*)$ *to the Operator are given by:*

(i) For $0 \leq \beta < 1/2$ *an optimal strategy of the Operator is given by*

$$g_3^* = \frac{2 - 2\,\beta + \beta^2 - \beta^3}{3 - 3\,\beta + 2\,\beta^2 - \beta^3} \qquad \text{and} \qquad g_2^* = \frac{1}{2 + \beta^2}\,,$$

and an optimal strategy of the Inspectorate by

$$\mathbf{q}^* = \frac{1}{3 - 3\,\beta + 2\,\beta^2 - \beta^3}\,\left(1 + \beta + \beta^2 + \beta^3, 1 - 2\,\beta + \beta^2 - 2\,\beta^3, 1 - 2\,\beta\right)^T\,.$$

The optimal payoff to the Operator is

$$Op^*_{3,2} = \frac{4 - \beta + \beta^2}{3 - 3\,\beta + 2\,\beta^2 - \beta^3}\,.$$

(ii) For $1/2 < \beta \leq 1$ *an optimal strategy of the Operator and an optimal strategy of the Inspectorate is given by*

$$g_3^* = 0\,, \quad g_2^* \in [0,1] \qquad \text{and} \qquad \mathbf{q}^* = (1,0,0)^T\,.$$

The optimal payoff to the Operator is

$$Op^*_{3,2} = 1 + \beta + 2\,\beta^2\,.$$

Proof. We have to show that, see (19.10), the saddle point criterion

$$Op_{3,2}(\mathbf{g}, \mathbf{q}^*) \leq Op^*_{3,2} \leq Op_{3,2}(\mathbf{g}^*, \mathbf{q}) \tag{6.36}$$

is fulfilled for any $\mathbf{g} \in G_2$ and any $\mathbf{q} \in Q_{3,2}$.

Ad (i): We have $Op_{3,2}(\mathbf{g}, \mathbf{q}^*) = Op^*_{3,2} = Op_{3,2}(\mathbf{g}^*, \mathbf{q})$ for any $\mathbf{g} \in G_2$ and any $\mathbf{q} \in Q_{3,2}$, i.e., the inequalities in (6.36) are fulfilled as equalities.

Ad (ii): (6.35) implies for any $\mathbf{g} \in G_2$

$$Op_{3,2}(\mathbf{g}, \mathbf{q}^*) = (1 - g_3)\,(1 + \beta + 2\,\beta^2) + g_3\,((1 - g_2)\,(1 + 2\,\beta) + g_2\,2)\,,$$

which is maximized for

$$g_2^*(g_3) = \begin{cases} 0 & \text{for} \quad g_3 \in (0,1] \\ [0,1] & \text{for} \quad g_3 = 0 \end{cases}\,,$$

i.e., we get

$$Op_{3,2}((g_3, g_2^*(g_3)), \mathbf{q}^*) = \begin{cases} 1 + \beta + 2\,\beta^2 + g_3\,\beta\,(1 - 2\,\beta) & \text{for} \quad g_3 \in (0,1] \\ 1 + \beta + 2\,\beta^2 & \text{for} \quad g_3 = 0 \end{cases}\,,$$

which is maximized for $g_3^* = 0$, i.e., the left hand inequality of (6.36) is fulfilled. For any $\mathbf{q} \in Q_{3,2}$ we further get, using (6.35) and the normalization of $\mathbf{q}$,

$$Op_{3,2}(\mathbf{g}^*, \mathbf{q}) = q_{(1,2)}\,(1 + \beta + 2\,\beta^2) + q_{(1,3)}\,(1 + 2\,\beta + \beta^2) + q_{(2,3)}\,(2 + \beta + \beta^2)$$

$$= 2 + \beta + \beta^2 - (1-\beta)\left((1+\beta)\, q_{(1,2)} + q_{(1,3)}\right),$$

which is in case of $\beta \in [0,1)$ minimized for $q^*_{(1,2)} = 1$, i.e., the right hand inequality of (6.36) is also fulfilled. If $\beta = 1$, then $Op_{3,2}(\mathbf{g}^*, \mathbf{q}) = 4 = Op^*_{3,2}$ for any $\mathbf{q} \in Q_{3,2}$, i.e., especially for $\mathbf{q}^* = (1,0,0)^T$. □

Comparing the optimal strategies and payoffs of the No-No and No-Se inspection games in Lemmata 6.2 and 6.3, we see that they are as complicated as those of the Se-No inspection game for case (i). All these results show that explicit optimal strategies and payoffs may be feasible for some larger numbers N of possible time points and k interim inspections, but cannot be expected for the general case of any N and $k < N$.

The results of Lemma 6.5 (i) simplify for $\beta = 0$ as expected to (4.18), (4.26) and (4.21).

Like in Section 6.1, let us conclude this section with the case of $N = 5$ possible time points since it has been of practical interest; see Avenhaus et al. (2010). For $k = 1$ interim inspection we have according to (6.31) that $Op^*_{5,1} = 6/(2-\beta)$ for all $\beta \geq 0$, contrary to the situation in the corresponding No-No inspection game; see (6.20). In case of $k = 2$ interim inspections the Operator's set of pure strategies is explicitly given by

$$\{\bar{\ell}_3, \ell_3\} \times \{\bar{\ell}_2(1), \ell_2(1)\} \times \{\bar{\ell}_2(2), \ell_2(2)\} \times \{\bar{\ell}_2(3), \ell_2(3)\}\,, \tag{6.37}$$

i.e., there are $2^4 = 16$ pure strategies: Again, $\bar{\ell}_3$ and ℓ_3 are the Operator's decisions at time point 0. If he behaves legally at time point 0 (ℓ_3), he decides at the first interim inspection time point at j_2, $j_2 = 1, \ldots, 3$, to start the illegal activity immediately ($\bar{\ell}_2(j_2)$) or not ($\ell_2(j_2)$). If the Operator postpones the start again, i.e., in case of $\ell_2(j_2)$, he must start the illegal activity at the time point j_1 of the second interim inspection.

Because $\ell_2(4)$ is, in case of $\beta > 0$, strictly dominated by $\bar{\ell}_2(4)$, in the former case the Operator's payoff is 1 while in the latter one it is $1\,(1-\beta) + 2\,\beta = 1 + \beta$, the decisions $\bar{\ell}_2(4)$ and $\ell_2(4)$ are excluded from (6.37). The 9×10 payoff matrix of this inspection game is represented in Table 6.8; see Avenhaus et al. (2010) with a correction in column $(2,5)$, where the payoffs $3 + \beta$ instead of $3 + 2\,\beta$ have to be utilized. Because the 8 pure strategies $\bar{\ell}_3 * **$ lead for any pure strategy of the Inspectorate to the same payoff to the Operator, they are abbreviated by $\bar{\ell}_3$.

For $\beta = 0$ we get, using (4.36), $Op^*_{5,2} = 2$. For $\beta > 0$ optimal strategies and the optimal payoff have been obtained again with the help of M. Canty's Mathematica® programs; see Canty (2003). We present only the optimal expected detection time which is much simpler than that of the corresponding No-No inspection game; see (6.21):

$$Op^*_{5,2} = \begin{cases} \dfrac{6}{3-2\,\beta} & \text{for} \quad 0 \leq \beta < 2/3 \\[2ex] \dfrac{8 - 5\,\beta + 3\,\beta^2}{5 - 9\,\beta + 8\,\beta^2 - 3\,\beta^3} & \text{for} \quad 2/3 < \beta < 3/4 \\[2ex] 1 + \beta + 4\,\beta^2 & \text{for} \quad 3/4 < \beta \leq 1 \end{cases} . \tag{6.38}$$

We see that like in the corresponding case of the No-No inspection game, the optimal expected detection time increases continuously from 2 to 6, which again is intuitive.

Looking at all cases of N and k we have considered in this section together with the case $N = k = 5$ and counting the number of regions of β-values we get

Table 6.8 Payoff matrix of the Se-No inspection game with $N = 5$ possible time points for $k = 2$ interim inspections and with errors of the second kind.

	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$
$\bar{\ell}_3$	$1+\beta+4\beta^2$	$1+2\beta+3\beta^2$	$1+3\beta+2\beta^2$	$1+4\beta+\beta^2$	$2+\beta+3\beta^2$
$\ell_3\,\bar{\ell}_2(1)\,\bar{\ell}_2(2)\,\bar{\ell}_2(3)$	$1+4\beta$	$2+3\beta$	$3+2\beta$	$4+\beta$	$1+3\beta$
$\ell_3\,\bar{\ell}_2(1)\,\bar{\ell}_2(2)\,\ell_2(3)$	$1+4\beta$	$2+3\beta$	$3+2\beta$	$4+\beta$	$1+3\beta$
$\ell_3\,\bar{\ell}_2(1)\,\ell_2(2)\,\bar{\ell}_2(3)$	$1+4\beta$	$2+3\beta$	$3+2\beta$	$4+\beta$	3
$\ell_3\,\bar{\ell}_2(1)\,\ell_2(2)\,\ell_2(3)$	$1+4\beta$	$2+3\beta$	$3+2\beta$	$4+\beta$	3
$\ell_3\,\ell_2(1)\,\bar{\ell}_2(2)\,\bar{\ell}_2(3)$	4	3	2	1	$1+3\beta$
$\ell_3\,\ell_2(1)\,\bar{\ell}_2(2)\,\ell_2(3)$	4	3	2	1	$1+3\beta$
$\ell_3\,\ell_2(1)\,\ell_2(2)\,\bar{\ell}_2(3)$	4	3	2	1	3
$\ell_3\,\ell_2(1)\,\ell_2(2)\,\ell_2(3)$	4	3	2	1	3

	$(2,4)$	$(2,5)$	$(3,4)$	$(3,5)$	$(4,5)$
$\bar{\ell}_3$	$2+2\beta+2\beta^2$	$2+3\beta+\beta^2$	$3+\beta+2\beta^2$	$3+2\beta+\beta^2$	$4+\beta+\beta^2$
$\ell_3\,\bar{\ell}_2(1)\,\bar{\ell}_2(2)\,\bar{\ell}_2(3)$	$2+2\beta$	$3+\beta$	$1+2\beta$	$2+\beta$	$1+\beta$
$\ell_3\,\bar{\ell}_2(1)\,\bar{\ell}_2(2)\,\ell_2(3)$	$2+2\beta$	$3+\beta$	2	1	$1+\beta$
$\ell_3\,\bar{\ell}_2(1)\,\ell_2(2)\,\bar{\ell}_2(3)$	2	1	$1+2\beta$	$2+\beta$	$1+\beta$
$\ell_3\,\bar{\ell}_2(1)\,\ell_2(2)\,\ell_2(3)$	2	1	2	1	$1+\beta$
$\ell_3\,\ell_2(1)\,\bar{\ell}_2(2)\,\bar{\ell}_2(3)$	$2+2\beta$	$3+\beta$	$1+2\beta$	$2+\beta$	$1+\beta$
$\ell_3\,\ell_2(1)\,\bar{\ell}_2(2)\,\ell_2(3)$	$2+2\beta$	$3+\beta$	2	1	$1+\beta$
$\ell_3\,\ell_2(1)\,\ell_2(2)\,\bar{\ell}_2(3)$	2	1	$1+2\beta$	$2+\beta$	$1+\beta$
$\ell_3\,\ell_2(1)\,\ell_2(2)\,\ell_2(3)$	2	1	2	1	$1+\beta$

- Any N and $k = 1$: 1 region; Lemma 6.4;
- $(N, k) = (3, 2)$: 2 regions; Lemma 6.5;
- $(N, k) = (5, 2)$: 3 regions; see (6.38);
- $(N, k) = (5, 5)$: 1 region.

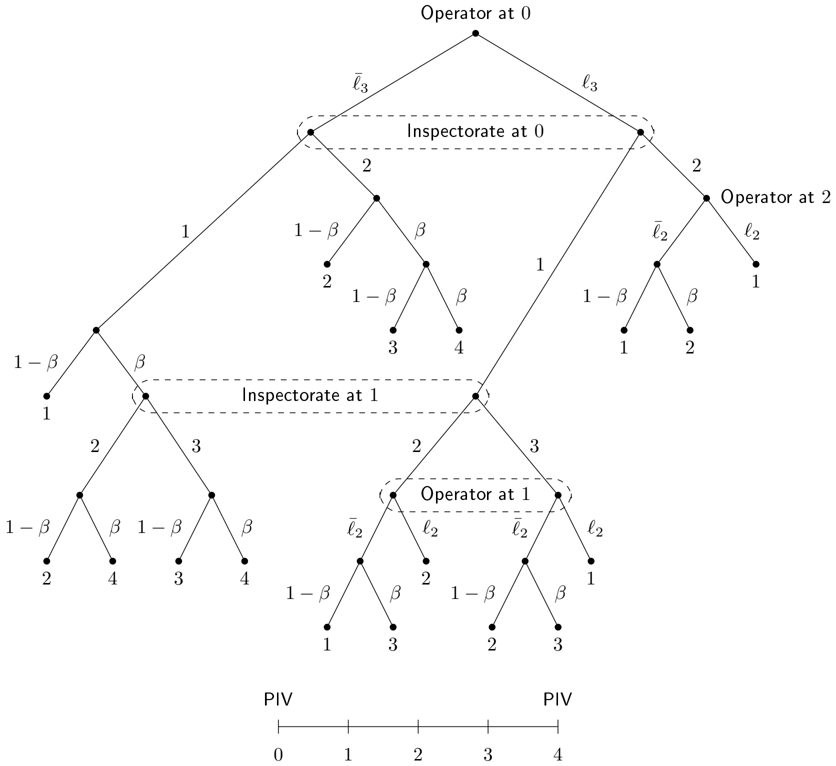
Again, except for the cases $k = N$ and like in the No-No inspection game, see p. 113, it seems to be difficult to imagine a kind of rule for the number of regions of β-values.

6.4 Se-Se: Three inspection opportunities and two interim inspections

The extensive form of the Se-Se inspection game is represented in Figure 6.6, where in line with the comment on p. 50 the game starts with the Operator's decision at time point 0, and again, the chance moves can be identified via the probabilities $1 - \beta$ and β.

The Operator behaves like in the Se-No inspection game in Section 6.3: He decides at time point 0 to start the illegal activity immediately or to postpone it. In the latter case he decides after the first interim inspection which is performed either at time point 1 or 2 to start the

Figure 6.6 Extensive form of the Se-Se inspection game with $N = 3$ possible time points for $k = 2$ interim inspections and with errors of the second kind.



illegal activity immediately or to postpone its start again. In the latter case he has to start the illegal activity after the second interim inspection at time point 2 or 3.

The Inspectorate behaves like in the No-Se inspection game in Section 6.2: It decides at time point 0 only if to perform the first interim inspection at time point 1 or 2. In case the first interim inspection is carried through at time point 1 it decides to perform the second interim inspection at time point 2 or 3. In case the first interim inspection takes place at time point 2, the second one has to be performed at time point 3.

Again, as already explained in case of the No-Se inspection game, errors of the second kind affect the information structure of the game decisively: Here, the information set of the Inspectorate at time point 1 is created by $\beta > 0$; it would degenerate to a one point set if β were zero.

A similar argumentation as on p. 121 shows that, if the Operator decides at time point 0 for postponing the illegal activity (ℓ_3) and the Inspectorate decides for the time point 2 as the first interim inspection time point, then the Operator will behave illegally immediately at time point 2 ($\bar{\ell}_2$) leading to the detection time $1 + \beta$.

In line with Section 6.2, let $1 - g_3$ resp. g_3 be the probabilities that the Operator decides at time point 0 to start the illegal activity immediately ($\bar{\ell}_3$) resp. to postpone its start (ℓ_3), and let $1-g_2$ resp. g_2 be the probabilities to start the illegal activity right after the first interim inspection at time point 1 ($\bar{\ell}_2$) or to postpone its start again (ℓ_2). Thus, the set of behavioural strategies of the Operator is again given by G_2, see (4.15), where here, and as in Section 6.2, we have abbreviated $g_2(1)$ with g_2. Thus, $g_2(2) = 1 - g_2$.

Like on p. 115, let h_3 resp. $1-h_3$ be the probability that the Inspectorate performs the first interim inspection at time points 1 resp. at time point 2, and let h_2 resp. $1-h_2$ be the probability to perform its second interim inspection at time point 2 resp. 3. We use here the same abbreviations as the one introduced before (6.23). Thus, the Inspectorate's set of behavioural strategies is again given by $\tilde{H}_{3,2}$; see (6.23).

According to Figure 6.6, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $\mathbf{g} \in G_2$ and any $\mathbf{h} \in \tilde{H}_{3,2}$, given by

$$\begin{aligned} Op_{3,2}(\mathbf{g},\mathbf{h}) := (1-g_3)\Big[&h_3\Big(1-\beta+\beta\,[h_2\,(2+2\,\beta)+\,(1-h_2)\,(3+\beta)]\Big) \\ &+(1-h_3)\,(2+\beta+\beta^2)\Big] \\ +\,g_3\Big[&h_3\Big(h_2\,[(1-g_2)\,(1+2\,\beta)+g_2\,2] \\ &\quad+(1-h_2)\,[(1-g_2)\,(2+\beta)+g_2]\Big) \\ &+(1-h_3)\,(1+\beta)\Big]\,. \end{aligned} \tag{6.39}$$

The game theoretical solution of this inspection game, see Avenhaus and Krieger (2013a), is presented in

Lemma 6.6. *Given the Se-Se inspection game with $N=3$ possible time points for $k=2$ interim inspections, and with errors of the second kind. The sets of behavioural strategies are given by (4.15) and (6.23), and the payoff to the Operator by (6.39).*

Then optimal strategies and the optimal payoff $Op^*_{3,2} := Op_{3,2}(\mathbf{g}^*,\mathbf{h}^*)$ to the Operator are given by:

(i) For $0 \le \beta < 1/2$ an optimal strategy of the Operator is given by

$$g_3^* = \frac{2-2\,\beta+\beta^2-\beta^3}{3-3\,\beta+2\,\beta^2-\beta^3} \qquad \text{and} \qquad g_2^* = \frac{1}{2+\beta^2}\,,$$

and an optimal strategy of the Inspectorate by

$$h_3^* = 1-\frac{1-2\,\beta}{3-3\,\beta+2\,\beta^2-\beta^3} \qquad \text{and} \qquad h_2^* = 1-\frac{1-2\,\beta}{2-\beta}\,.$$

The optimal payoff to the Operator is

$$Op^*_{3,2} = \frac{4-\beta+\beta^2}{3-3\,\beta+2\,\beta^2-\beta^3}\,.$$

(ii) For $1/2 < \beta \le 1$ an optimal strategy of the Operator and an optimal strategy of the Inspectorate is given by

$$g_3^* = 0\,, \quad g_2^* \in [0,1] \qquad \text{and} \qquad h_3^* = h_2^* = 1\,.$$

The optimal payoff to the Operator is

$$Op^*_{3,2} = 1 + \beta + 2\,\beta^2\,.$$

Proof. According to (19.10), we prove that

$$Op_{3,2}(\mathbf{g}, \mathbf{h}^*) \le Op^*_{3,2} \le Op_{3,2}(\mathbf{g}^*, \mathbf{h}) \tag{6.40}$$

for any $\mathbf{g} \in G_2$ and any $\mathbf{h} \in \tilde{H}_{3,2}$.

Ad (i): Using (6.39) we have $Op_{3,2}(\mathbf{g}, \mathbf{h}^*) = Op^*_{3,2} = Op_{3,2}(\mathbf{g}^*, \mathbf{h})$ for any $\mathbf{g} \in G_2$ and any $\mathbf{h} \in \tilde{H}_{3,2}$, i.e., the inequalities in (6.40) are fulfilled as equalities.

Ad (ii): (6.39) implies

$$Op_{3,2}(\mathbf{g}, \mathbf{h}^*) = 1 + \beta + 2\,\beta^2 - g_3\,(g_2 + \beta)\,(2\,\beta - 1)\,,$$

which, because of $2\,\beta - 1 \ge 0$, is maximized for $g_3^* = g_2^* = 0$, i.e., the left hand inequality in (6.40) is fulfilled. Furthermore, (6.39) leads to

$$\begin{aligned} &Op_{3,2}(\mathbf{g}^*, \mathbf{h}) \\ &\quad = h_3 \Big(1 - \beta + \beta\,[h_2\,(2 + 2\,\beta) + (1 - h_2)\,(3 + \beta)]\Big) + (1 - h_3)\,(2 + \beta + \beta^2)\,, \end{aligned} \tag{6.41}$$

which is minimized in case of $\beta \in (0, 1)$ for

$$h_2^*(h_3) = \begin{cases} [0,1] & \text{for} \quad h_3 = 0 \\ 1 & \text{for} \quad h_3 \in (0,1] \end{cases}\,,$$

i.e., we get

$$Op_{3,2}(\mathbf{g}^*, (h_3, h_2^*(h_3))) = \begin{cases} 2 + \beta + \beta^2 & \text{for} \quad h_3 = 0 \\ 2 + \beta + \beta^2 - h_3\,(1 - \beta)\,(1 + \beta) & \text{for} \quad h_3 \in (0,1] \end{cases}\,,$$

which is minimized for $h_3^* = 1$, i.e., the right hand inequality of (6.40) is fulfilled. If $\beta = 1$ then (6.41) simplifies to $Op_{3,2}(\mathbf{g}^*, \mathbf{h}) = 4$ for any $\mathbf{h} \in \tilde{H}_{3,2}$, i.e., especially for $h_3^* = h_2^* = 1$. □

We see that the optimal strategy $\mathbf{g}^*$ of the Operator as well as the optimal expected detection time are the same as in the Se-No inspection game; see Lemma 6.5.

Again, as in the case of Lemmata 6.2 and 6.3, we can identify the payoffs (6.35) and (6.39) of the Se-No and Se-Se inspection game: If we replace in (6.35) the probabilities $q_{(1,2)}, q_{(1,3)}$ and $q_{(2,3)}$ like in (6.26), we obtain (6.39), and if one replaces in (6.39) the probabilities h_3 and h_2 by (6.27) we get (6.35). And of course again, the optimal strategies $\mathbf{q}^*$ and $\mathbf{h}^*$ as given in Lemmata 6.5 and 6.6 reflect these relations.

Let us look again at Figure 6.6. If the Operator has started the illegal activity at time point 0, then from his point of view the game is finished. Only if he decides at this time point to postpone the start of the illegal activity he enters into a new game, knowing his and the Inspectorate's prior decisions. This motivates the attempt to cut the information set of the Inspectorate at time point 1, which leads to a subgame starting at time point 1 when the Inspectorate decides to perform an interim inspection at time points 2 or 3 and the Operator decides to behave illegally or to postpone the illegal activity. If we then proceed like in the recursive approach to the step by step game presented in Section 5.3, i.e., if we replace the subgame by its optimal payoff to the Operator and furthermore, use on the left hand side of the tree the same strategy of the Inspectorate as that obtained in the subgame, we arrive again at a game which can be represented as a 2×2 matrix game. Solving this game we obtain the same optimal payoff to the Operator as that of the original game and also the same optimal strategy of the Operator. The optimal strategy of the Inspectorate, however, is different. Thus, the situation is similar to that in case of the step by step game: To cut the relevant information set is intuitive, and the recursive treatment leads to the same optimal payoff to the Operator, but not to the same optimal strategies of both players as in the original game. We do not really understand why this is so.

6.5 Comparison of variants

Having analysed in major detail the four variants in Table 2.1 with $N = 3$ possible time points for $k = 1, 2$ interim inspection(s), it seems now more than appropriate to compare the solutions of these variants.

Let us summarize the most interesting findings in the form of three statements. First, for $\beta = 0$ all four variants lead in case of $N = 3$ possible time points for $k = 2$ interim inspections to the same optimal expected detection time $4/3$, i.e., one third of the length of the reference time interval. The same is true for all $1/2 < \beta \leq 1$ with the optimal expected detection time $1 + \beta + 2\,\beta^2$.

Second, the No-No and No-Se inspection game on one hand, and the Se-No and Se-Se inspection game on the other, lead for $0 \leq \beta < 1/2$ to the same optimal expected detection times. The same holds for the optimal strategies of the Operator, but not for that of the Inspectorate because he uses different strategy sets.

Third, the optimal expected detection times of the No-No and No-Se inspection game are smaller than those of the Se-No and Se-Se inspection game for $0 < \beta < 1/2$. Thus, if we change for the moment the notation, we have for all $0 < \beta < 1/2$:

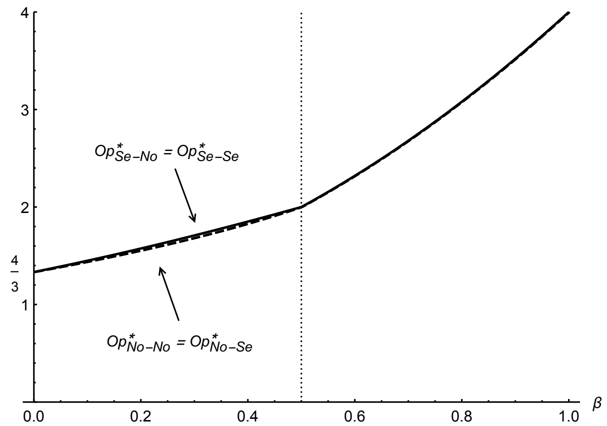
$$Op^*_{No-No} = Op^*_{No-Se} < Op^*_{Se-No} = Op^*_{Se-Se}\,. \tag{6.42}$$

At the end of Sections 6.1 an 6.3 we have shown for $N = 5$ possible time points and $k = 2$ interim inspections that for the No-No and for the Se-No inspection game even the regions of β-values of the optimal expected detection times are different. Thus, the properties which have been observed for $N = 3$ possible time points and $k = 2$ interim inspections do not hold any more for larger numbers N of possible time points and k interim inspections.

In Figure 6.7 the optimal expected detection times $Op^*_{\cdot-\cdot}$ are represented as functions of β.

In order to find a plausible interpretation of these results one observation is important: There is a distinct difference in information between the two players: The Operator knows when an interim inspection has taken place whereas the Inspectorate, after an interim inspection which did not prove illegal behaviour of the Operator, for $\beta > 0$ does not know if there was no illegal behaviour or else, if the Inspectorate did not detect it. This more comfortable role of the Operator may explain the results of the analysis for $0 < \beta < 1/2$; it does not explain, however, why for $1/2 < \beta < 1$ all variants lead to the same optimal expected detection times.

Figure 6.7 The optimal expected detection times as functions of β for all four variants with $N = 3$ possible time points for $k = 2$ interim inspections.



For $\beta = 0$, and in case the Inspectorate does not detect the illegal activity during an interim inspection, it knows that the illegal activity has not yet been started. Thus, the Inspectorate's information state corresponds in some way to that of the Operator. One may doubt, however, if this is a reasonable explanation for the fact that in case of $\beta = 0$ all four variants lead to the same optimal expected detection times.

Of course the results of the analysis of the special case of $N = 3$ possible time points for $k = 2$ interim inspections cannot simply be extrapolated to more complicated situations, i.e., more possible time points and more interim inspections, even though the information argument given before holds here as well. On the contrary: Optimal strategies of two person zero-sum games exhibit frequently unexpected and surprising properties. But anyhow, if interim inspections have to be planned for more complicated cases than analysed here, and if analytical results for more complicated cases are not available, then one may – with reference to the cases discussed in this chapter – assume that the Operator will behave sequentially, and the Inspectorate may do what is easier from an organisational and financial point of view.

6.6 Applications to Nuclear Safeguards

We mentioned in the beginning of this chapter that the game theoretical models with $N = 3$ possible time points for $k = 1, 2$ and 3 interim inspection(s) describe a real case therefore, we will shortly present it here. In the following we consider only the No-No inspection game which is – as we saw in Section 6.2 – equivalent to the No-Se inspection game in the sense that the Operator's optimal strategies coincide as well as the optimal expected detection times, and that the Inspectorate's (optimal) strategies can be transformed into each other. The description which follows together with the subsequent discussion has been taken from Avenhaus et al. (2010). Without going into too many technical details, this inspection problem looks as follows.

In Europe, adjacent to nuclear power reactors there are spent fuel storages which are safeguarded both by the European Atomic Energy Community (EURATOM), and by the International Atomic Energy Agency (IAEA). In these storages, there are spent fuel elements of the reactors which are contained in sealed casks. Every three months interim inspections may be performed, i.e., $N = 3$ in the terminology of this monograph, and at the end of a calendar year a Physical Inventory Verification (PIV) is performed which is assumed to provide exact knowledge about the amount of spent fuel in the storage.

The basis of these inspections is the so-called significant quantity, i.e., the approximate amount of nuclear material for which the possibility of manufacturing a nuclear device cannot be excluded, and the so-called conversion time, i.e., the time required to convert different forms of nuclear material to the metallic components of a nuclear explosive device; see IAEA (2002).

For the subsequent quantitative analysis we consider a representative situation where there are M casks (80 to 190) with light water reactor spent fuel elements in the storage facility, and where each cask contains 19 spent fuel elements. Without going into the details of the usability of the plutonium for weapons in the spent fuel elements we assume that the average amount of plutonium in each spent fuel element is about 8 kg, i.e., each spent fuel element contains about the significant quantity of plutonium; see IAEA (2002). Thus, in order to illegally acquire one significant quantity, the seal of one cask need to be broken. In other words, during an interim inspection one broken seal has to be detected with sufficient probability $1-\beta$.

Let us assume that the Inspectorate needs about five minutes net time in the storage to check one seal. There is, however, overhead work to be done by the Inspectorate, primarily the evaluation of the findings outside the storage, and administrative work before and after the whole seal checking procedure. Therefore, during a one day visit only two to three hours may be available for checking seals in the storage.

Quite generally, let the total number of casks/seals be M, the number of seals to be checked – the sample size – be n, and the number of broken seals be r. Then according to the hypergeometric distribution law, the probability to detect at least one broken seal in case of drawing without replacement is, see Avenhaus and Canty (1996), given by

$$\mathbb{P}(\{\text{at least one broken seal in the sample}\}) = 1 - \mathbb{P}(\{\text{no broken seal in the sample}\})$$

$$= 1 - \frac{\binom{r}{0}\binom{M-r}{n-0}}{\binom{M}{n}} = 1 - \prod_{j=0}^{r-1}\left(1 - \frac{n}{M-j}\right) . \tag{6.43}$$

Then the sample size is defined as the smallest number n fulfilling the inequality

$$\mathbb{P}(\{\text{at least one broken seal in the sample}\}) \geq 1-\beta \, .$$

Because the seal of one cask need to be broken, we have $r = 1$ and obtain, using (6.43), for the sample size n

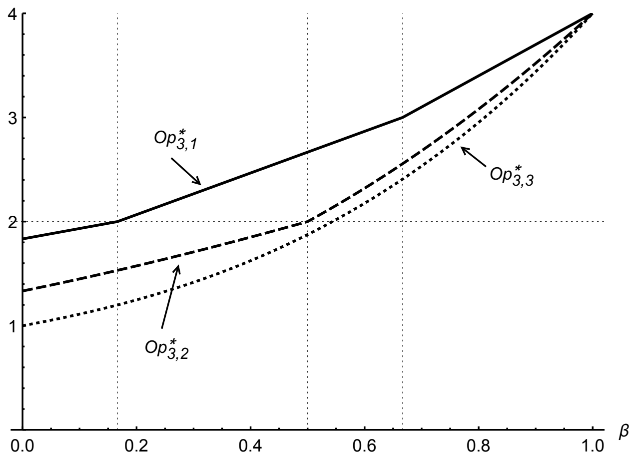
$$n = \lceil M\,(1-\beta) \rceil \, , \tag{6.44}$$

where the ceiling function $\lceil\,\rceil$ maps x to the least integer greater than or equal to x.

Note that there are situations, where inspectors tend to seal casks together. In those cases they can be treated as a "static" part reducing the number of casks to be checked. Therefore, the detection probability for the next interim inspection may be different from that for the former one. Note that in Section 11.1 inspections games with different non-detection probabilities are considered. However, they refer to different facilities.

Now let us come back to the game theoretical results and link them to the practice of interim inspections in spent fuel storages. In Figure 6.8 we have drawn the optimal expected detection times for the cases of $k = 1, 2$ and 3 interim inspection(s) from top to bottom; see Lemma 6.1 and 6.2 as well as Table 6.3.

Figure 6.8 Optimal expected detection times as functions of β for the No-No inspection game with $N = 3$ possible time points for $k = 1, 2$ and 3 interim inspection(s).



Let us comment on Figure 6.8: First, it can be seen that $Op^*_{3,3} < Op^*_{3,2} < Op^*_{3,1}$ for $\beta \in [0, 1)$ and $Op^*_{3,3} = Op^*_{3,2} = Op^*_{3,1}$ for $\beta = 1$. This result is clear due to the fact, that more possible interim inspection(s) lead(s) to the a shorter optimal expected detection time. In case of $\beta = 1$, i.e., the detection probability $1 - \beta$ is zero, any illegal activity is detected only at the end of the reference time interval and therefore the detection time is 4 quarters of a year.

Second, if the required optimal expected detection time is about 1.5 quarters of a year then we see that this expected detection time cannot be achieved with $k = 1$ interim inspection since $Op^*_{3,1} = 11/6 \approx 1.833$ for $\beta = 0$.

Finally, an important question from the practical point of view is whether the number of interim inspections can be reduced assuring a required optimal expected detection time. This question can be answered with the help of Figure 6.8. Suppose the required optimal expected detection time is 2 quarters of a year, i.e., 6 month. Then Figure 6.8 resp. the results of Lemmata 6.1 and 6.2, and Table 6.3 lead to the resulting non-detection probabilities β which are given in the second column of Table 6.9. Note that in IAEA practice the non-detection probabilities do not vary with the number of interim inspections but are fixed to some value. Using (6.44) with $M = 100$, the sample sizes n that achieve these detection probabilities are given in the third column, the total sample size in the fourth column, and the net time per interim inspection in the fifth column; recall that checking one seal takes five minutes net time.

Table 6.9 The non-detection probabilities (rounded to four digits), the sample size and the total sample size for $k = 1, 2$ and 3 interim inspection(s), and for a required optimal expected detection time of 2 quarters of a year.

	non-detection probability β	sample size n per interim inspection	total sample size	net time per interim inspection
$k = 1$	$1/6$	84	84	7h
$k = 2$	$1/2$	50	100	4h 10 min
$k = 3$	0.5437	46	138	3h 50 min

If the Inspectorate wants to take a decision between these three possibilities, it has to take into account additional aspects. If, for example, economic aspects are considered, the decision procedure can be formalized with the help of a cost model as follows: Let a be the overhead cost per inspection (travel and accommodation), and b the cost for checking one seal (Inspectorate manhour cost). Then, for a postulated optimal expected detection time of 2, the total cost of inspections are

$$\begin{aligned} a + b \cdot 84 \quad & \text{for} \quad k = 1 \\ 2 \cdot a + b \cdot 2 \cdot 50 \quad & \text{for} \quad k = 2 \\ 3 \cdot a + b \cdot 3 \cdot 46 \quad & \text{for} \quad k = 3\,. \end{aligned}$$

We see immediately, that from this cost model point of view $k = 1$ interim inspection is the best choice. Of course, more complicated cost models could lead to different results; see p. 199. If, for example, the checking of 84 seals can not be achieved in one day, see the comment on p. 130, contrary to the checking of 46 seals, overhead costs, e.g., staying overnight, may favour more than $k = 1$ interim inspection. Thus, a decision based on a cost model can only be made with the help of truly realistic cost data.

Note that a similar analysis can be performed for the case of $N = 5$ possible time points for $k = 1, 2$ and 5 interim inspection(s) at least for the No-No and the Se-No inspection games, the optimal expected detection times of which are presented at the end of Sections 6.1 and 6.3.

It should be mentioned, however, that considerations of this kind do not meet current IAEA practice: frequencies of interim inspections and inspection procedures are agreed upon by the international community, and the budget needed is provided accordingly. In Section 7.2 this idea is formalized.

Chapter 7

Legal behaviour, effectiveness and efficiency, extensions

So far and in line with the assumptions in Chapter 2 all inspection models of Part I were framed as follows: There is an Operator who plans an illegal activity, see assumption (iv), in such a way that an Inspectorate detects it as late as possible whereas the Inspectorate plans its interim inspections such that this illegal activity is detected as early as possible. This way, the inspection models were built out of purely technical quantities such as time points for the start of the illegal activity and interim inspections, detection probability and detection time, and number of interim inspections during the reference time interval.

But there is also the argument that inspections should be designed in such a way that the Operator is deterred from illegal behaviour or, to say it positively, is induced to legal behaviour; see p. 1. It turns out that in order to be able to analyse this issue, utilities have to be introduced which describe the gains and losses of both the Operator and the Inspectorate for all outcomes of the inspections. More than that, there are other problems which require the use of these utilities; examples will be given in the paragraph on "serious problems" in Section 7.1 and in Section 7.4. It should be mentioned already here that in many applications these utilities can be estimated only very roughly or not at all, which was another reason for assumption (iv): Because this way utilities do not have to be introduced and we were able to limit the analyses to the above mentioned technical parameters.

In order to understand the relations between these issues and problems, we first present in Section 7.1 a little the historical development of inspection games as understood by safeguards pioneers and the authors of this monograph; see Avenhaus and Shmelev (2011).

Since the concept of deterring the Operator from illegal behaviour will be considered also in Parts II and III, we will develop in Section 7.2 the analysis, which is taken from Avenhaus et al. (2010), in some detail and refer to it in the following chapters. Let us mention already now that it will turn out that for the inspection problems treated so far only the ratio of the Operator's gain in case of successful illegal behaviour to his loss in case of detected illegal behaviour has to be known. In Section 7.3 our understanding of the concepts of effectiveness and efficiency is introduced and illustrated with the help of the example considered in Section 7.2. Section 7.4 deals with variable sampling problems the use of which may lead to false alarms. It has already been mentioned on p. 8 that this possibility requires also, independently of the aspect of legal or illegal behaviour of the Operator, the use of payoff parameters. Also for this reason we consider in Section 7.4 utilities for false alarms, even though they will be used only in some chapters of Parts II and III.

7.1 Historical development

Two roots of present days' inspection games may be identified: The one is the scientific-technical area of Statistics, and of quality and process control in science and technology; as an example and highlight we take the work of Diamond (1982) which will be discussed in detail in Chapter 9. The other lies in the arms control and disarmament efforts in the beginning of the second half of the last century; here we just mention as prominent example the work of Dresher (1962) which will be discussed in Chapter 16. In the first area methods were developed for practical needs, whereas in the second one intended to design concepts for dealing with the new and frightening challenge posed by the invention of nuclear weapons. In both cases, utilities played no or no major role: In the first case because of the applied nature of the problems and their solutions, and in the second one because of their conceptual character. With the development of the IAEA safeguards system, which was sketched already in Section 1.2, this changed for several reasons. Initially, statisticians developed the system based on the concepts of material balance and data verification which were used already in the chemical and nuclear industry. Later, as already mentioned, the issue of legal behaviour became important, and we will deal with it Section 7.2.

Furthermore, and this is quite a different strand, it turned out that false alarms which cannot be avoided if variable sampling methods are used, see Sections 1.1 and 7.4, could no longer be handled as, e.g., in medical or biological research. There, the false alarm probability is just fixed conventionally when statistical tests are performed. Here, in international safeguards false alarms may have serious consequences – not zero-sum! – to both the Operator and the Inspectorate therefore, these consequences have to be taken into account explicitly.

In order to clarify these two aspects, Table 7.1 shows under which assumptions utilities in the form of payoff parameters have to be introduced. Let us mention in passing that variable sampling models will be considered in detail only in Parts II and III. Of course there are further reasons for the use of utilities which will be sketched in the final Chapter 17. Note that in case the Operator behaves illegally, e.g., by falsifying some data/items, false alarms can occur when the non-falsified data/items are verified.

Table 7.1 Overview on inspections and methods requiring the use of utility functions.

<table>
<tr><th></th><th>Operator behaves illegally</th><th>Operator behaves illegally or legally</th></tr>
<tr><td>Inspections based on Attribute Sampling</td><td>Utilities are not explicitly used</td><td>Utilities d, b and a for illegal and legal behaviour have to be used</td></tr>
<tr><td>Inspections based on Variable Sampling</td><td colspan="2">Utilities d, b and a for illegal and legal behaviour and utilities f and g for false alarms have to be used</td></tr>
</table>

In order to indicate how serious problems may become, not for the use in this monograph, we mention another important problem which was controversially discussed in the IAEA, see also Avenhaus et al. (2009b): If one fixes the inspection effort in *one* State, e.g., in terms of manpower and money, which shall induce this State to legal behaviour in the sense of the Non-Proliferation Treaty, then the use of utilities for the gains and losses of the State in case of legal and illegal behaviour is equivalent to using some value of the overall detection probability, i.e., of a technical quantity. If, however, *several* States with different incentives are considered, this is no longer possible. The attempts of IAEA administrators to do this anyhow, and they claimed to have good reasons, were commented by the American Statistician Carl A. Bennett (1922–2014) with the words: *They are looking for a technical solution of a political problem.* There are situations where even peaceful theoreticians have to fight, if necessary passionately, for their better cause.

7.2 Utilities for attribute sampling inspection schemes

In all inspection games of this Part I and according to assumption (iii) of Chapter 2 any illegal activity will be detected with certainty at the end of the reference time interval, the PIV. Therefore, if we normalize the gain (or loss) of the Operator in case of legal behaviour to zero, any illegal activity causes the loss $b > 0$ (gain $-b < 0$). In addition, according to the playing for time criterion, the Operator, starting an illegal activity, has a gain proportional to the detection time $\triangle t$, i.e., $d\,\triangle t$ for $d > 0$. If we denote the length of the reference time interval by T[1], the maximum gain of the Operator in case of illegal behaviour is $d\,T - b$ thus, we assume

$$d\,T - b > 0\,; \tag{7.1}$$

otherwise the Operator would never start an illegal activity. In sum, the payoff to the Operator is

$$\begin{cases} d\,\triangle t - b & \text{for} \quad \text{illegal behaviour and detection time } \triangle t \\ 0 & \text{for} \quad \text{legal behaviour} \end{cases} . \tag{7.2}$$

Let us assume now that the Operator decides before the beginning of the game whether to behave illegally or not at all. We will return to this point later on. With the terminology used before we get for the Operator's the optimal payoff

$$\begin{cases} d\,Op^* - b & \text{for} \quad \text{illegal behaviour and detection time } Op^* \\ 0 & \text{for} \quad \text{legal behaviour} \end{cases} ,$$

where Op^* stands for any of the optimal expected detection times determined in the previous chapters and also in Sections 9.1 – 9.3, 10.1 and 10.2, and in Chapter 11. Therefore, the Operator will behave legally if and only if

$$d\,Op^* - b < 0$$

or equivalently

$$Op^* < \frac{b}{d}\,. \tag{7.3}$$

[1] In this Part I we have $T = N + 1$ and in Part II we consider $T = t_0$. Part III does not fit into this scheme.

For the No-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection, we get according to Lemma 6.1 with (7.3)

$$\frac{b}{d} > \begin{cases} \frac{11}{6} + \beta & \text{for} \quad 0 \le \beta < \frac{1}{6} \\ \frac{10}{6} + 2\,\beta & \text{for} \quad \frac{1}{6} < \beta < \frac{2}{3} \\ \frac{6}{6} + 3\,\beta & \text{for} \quad \frac{2}{3} < \beta \le 1 \end{cases} . \tag{7.4}$$

We see that the Operator is induced to legal behaviour either if the non-detection probability β is small, or if the ratio b/d is large.

The question, however, remains what are the Inspectorate's equilibrium strategies in case of legal behaviour of the Operator. In order to answer this question we have to get back to the inspection game and we have to introduce the utilities of the Inspectorate which means, by the way, that we describe the inspection problems no longer as zero-sum games. We define these utilities as follows:

$$\begin{cases} -\,a\,\triangle t & \text{for} \quad \text{illegal behaviour and detection time } \triangle t \\ 0 & \text{for} \quad \text{legal behaviour} \end{cases} . \tag{7.5}$$

Here we assume $a > 0$ since detected illegal behaviour of the Operator is worse for the Inspectorate than legal behaviour. Note that in (7.5) there is no equivalent to the Operator's sanctions b in (7.2). Also note that if no illegal behaviour occurs, both players receive by definition payoff nil; this is also the best result for the Inspectorate and implies the idea that inspection costs are not part of the Inspectorate's payoff, but rather imposed by the external parameter k; see the end of Sections 6.6 and 10.2. In other words, the number of interim inspections k is not a strategic variable. This assumption describes the situation of an inspection organization, e.g., the IAEA, which works by international agreement with a fixed inspection budget for specific facilities and/or States.

For purpose of illustration let us consider again the No-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection. The bimatrix with the players' payoffs is given in Table 7.2. It is structural equivalent to the payoff matrix in Table 3.1, except for the two entries instead of one for each matrix element, for the payoff parameters, and for the legal strategy "le" of the Operator. Therefore, the Operator's set of pure strategies is here, in contrast to (3.2), given by

$$I := \{0, 1, 2, 3, \mathsf{le}\} \tag{7.6}$$

and his set of mixed strategies by

$$P := \left\{ \mathbf{p} := (p_0, p_1, p_2, p_3, p_{\mathsf{le}})^T \in [0,1]^5 : \sum_{i=0}^{3} p_i + p_{\mathsf{le}} = 1 \right\} . \tag{7.7}$$

Note that for the strategy sets (7.6) and (7.7) as well as for the below defined payoffs to both players the indices are omitted, because in this chapter we consider only this game.

Because the Inspectorate's behaviour is not influenced by the introduction of the legal behaviour strategy, its sets of pure and mixed strategies remain the same, i.e., they are given by (3.1) resp. (3.5).

Table 7.2 Normal form of the No-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection including legal behaviour of the Operator.

	1	2	3
0	$d-b$; $-a$	$2d-b$; $-2a$	$3d-b$; $-3a$
1	$3d-b$; $-3a$	$d-b$; $-a$	$2d-b$; $-2a$
2	$2d-b$; $-2a$	$2d-b$; $-2a$	$d-b$; $-a$
3	$d-b$; $-a$	$d-b$; $-a$	$d-b$; $-a$
le	0 ; 0	0 ; 0	0 ; 0

Because we deal here the first time with a non-zero-sum game, we present its analysis in some detail. Note that for this purpose we have to use the Nash equilibrium concept, which generalizes the saddle point concept and which is explained in Chapter 19.

With $\mathbf{p} := (p_0, p_1, p_2, p_3, p_{\mathsf{le}})^T \in P$ and $\mathbf{q} := (q_1, q_2, q_3)^T \in Q_{3,1}$, the (expected) payoffs – which can *no* longer be interpreted as detection times – to both players denoted by $Op(\mathbf{p},\mathbf{q})$ and $In(\mathbf{p},\mathbf{q})$ are, using (19.3) and (19.4), given by

$$Op(\mathbf{p},\mathbf{q}) = \mathbf{p}^T \begin{pmatrix} d-b & 2\,d-b & 3\,d-b \\ 3\,d-b & d-b & 2\,d-b \\ 2\,d-b & 2\,d-b & d-b \\ d-b & d-b & d-b \\ 0 & 0 & 0 \end{pmatrix} \mathbf{q} \tag{7.8}$$

and

$$In(\mathbf{p},\mathbf{q}) = -\,\mathbf{p}^T \begin{pmatrix} a & 2\,a & 3\,a \\ 3\,a & a & 2\,a \\ 2\,a & 2\,a & a \\ a & a & a \\ 0 & 0 & 0 \end{pmatrix} \mathbf{q}\,. \tag{7.9}$$

Using (19.7), we see that the game $(P, Q_{3,1}, Op, In)$ is strategically equivalent to the game $(P, Q_{3,1}, \widetilde{Op}, \widetilde{In})$ with

$$\widetilde{Op}(\mathbf{p},\mathbf{q}) := \mathbf{p}^T \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \\ b/d & b/d & b/d \end{pmatrix} \mathbf{q} \quad \text{and} \quad \widetilde{In}(\mathbf{p},\mathbf{q}) := -\mathbf{p}^T \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \\ b/d & b/d & b/d \end{pmatrix} \mathbf{q}\,, \tag{7.10}$$

i.e., strategically equivalent to the zero-sum game $(P, Q_{3,1}, \widetilde{Op}, -\widetilde{Op})$.

Using the results of Lemma 3.1, the game theoretical solution of this inspection game is presented in

Lemma 7.1. *Given the No-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection. The sets of mixed strategies are given by (7.7) and (3.5), and the payoffs to both players by (7.8) and (7.9).*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^* := Op(\mathbf{p}^*, \mathbf{q}^*)$ and $In^* := In(\mathbf{p}^*, \mathbf{q}^*)$:

(i) For

$$\frac{b}{d} < \frac{11}{6} \tag{7.11}$$

an equilibrium strategy of the Operator is given by

$$\mathbf{p}^* = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}, 0, 0\right)^T , \tag{7.12}$$

and an equilibrium strategy of the Inspectorate by

$$\mathbf{q}^* = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)^T . \tag{7.13}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^* = \frac{11}{6}\, d - b \qquad \text{and} \qquad In^* = -\,\frac{11}{6}\, a\,. \tag{7.14}$$

(ii) For

$$\frac{b}{d} > \frac{11}{6} \tag{7.15}$$

the Operator behaves legally, i.e., $\mathbf{p}^* = (0, 0, 0, 0, 1)^T$, and the Inspectorate's set of equilibrium strategies is given by

$$\frac{b}{d} \geq -2\, q_1^* - q_2^* + 3\,, \qquad \frac{b}{d} \geq q_1^* - q_2^* + 2\,, \qquad \frac{b}{d} \geq q_1^* + q_2^* + 1\,. \tag{7.16}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^* = In^* = 0\,. \tag{7.17}$$

Proof. The Nash conditions for the equilibria of this inspection game, are given by the two inequalities, see (19.5),

$$Op^* \geq Op(\mathbf{p}, \mathbf{q}^*) \qquad \text{and} \qquad In^* \geq In(\mathbf{p}^*, \mathbf{q}) \tag{7.18}$$

for any $\mathbf{p} \in P$ and any $\mathbf{q} \in Q_{3,1}$.

Ad (i): Inserting (7.12) and (7.13) into (7.8) and (7.9) respectively, we see immediately that both inequalities in (7.18) are fulfilled as equalities.

Ad (ii): The Nash condition (7.18) for the Inspectorate is fulfilled as equality. That of the Operator is equivalent to the following inequalities, see also (19.6),

$$0 \geq Op(0, \mathbf{q}^*)\,, \quad 0 \geq Op(1, \mathbf{q}^*)\,, \quad 0 \geq Op(2, \mathbf{q}^*)\,, \quad 0 \geq Op(3, \mathbf{q}^*)\,, \quad 0 \geq Op(\mathsf{le}, \mathbf{q}^*)\,.$$

The first three inequalities are explicitly given by

$$\begin{aligned} 0 &\geq (d-b)\, q_1^* + (2\,d-b)\, q_2^* + (3\,d-b)\, q_3^* \\ 0 &\geq (3\,d-b)\, q_1^* + (d-b)\, q_2^* + (2\,d-b)\, q_3^* \\ 0 &\geq (2\,d-b)\, q_1^* + (2\,d-b)\, q_2^* + (d-b)\, q_3^*\,. \end{aligned}$$

With $q_3^* = 1 - q_1^* - q_2^*$ these inequalities are seen immediately to be equivalent to (7.16). The inequality $0 \geq Op(3, \mathbf{q}^*) = d - b$ is fulfilled because (7.15) implies $b > d$, and the inequality $0 \geq Op(\mathsf{le}, \mathbf{q}^*)$ is fulfilled as equality because $Op(\mathsf{le}, \mathbf{q}^*) = 0$. □

Before commenting the results of this Lemma, let us turn to the case $11/6\,d - b = 0$. We justify the exclusion of this limiting case with the fact that in practice payoff parameters will be estimated, and are therefore always subject to uncertainty. Thus, the case $11/6\,d - b = 0$ is only of theoretically interest. For this reason we will not consider these limiting cases in following chapters. But let us discuss this case here: One can see immediately that the pair $(\mathbf{p}^*(\lambda), \mathbf{q}^*)$ with

$$\mathbf{p}^*(\lambda) := \lambda \begin{pmatrix} 1/3 \\ 1/6 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and $\mathbf{q}^*$ as given by (7.12) constitutes for any $\lambda \in [0,1]$ a Nash equilibrium and that the equilibrium payoffs to both players are zero. Thus, in this limiting case in equilibrium the Operator can randomize over illegal and legal behaviour, i.e., $\lambda \in (0,1)$, but his payoff is always the same as in case of legal behaviour. Note that except for limiting cases of this kind, randomizing over illegal and legal behaviour of the Operator has not been found in attribute sampling inspection games considered in this monograph. An intuitive explanation for this fact remains to be found; see also Avenhaus and Canty (1995). There are other inspection models where this is most certainly not the case; see Section 9.3.3 in Avenhaus and Canty (1996).

Returning to Lemma 7.1, we first observe that the equilibrium strategies of both players in case of illegal behaviour of the Operator, i.e., case (i), are the same as in the corresponding No-No inspection game; see Lemma 3.1. This is not surprising because in case of $p_{\mathsf{le}}^* = 0$ the payoff $\widetilde{Op}$ to the Operator according to (7.10) coincides with the payoff $Op_{3,1}$ to the Operator as given by (3.6), and it explains, in particular, why the equilibrium strategies (7.12) and (7.13) of both players do not depend on the payoff parameters a, b and d.

Second, for the purpose of illustration we assume $b/d = 2$ which fulfils with $T = N + 1 = 4$ and (7.1) the condition for legal behaviour of the Operator,

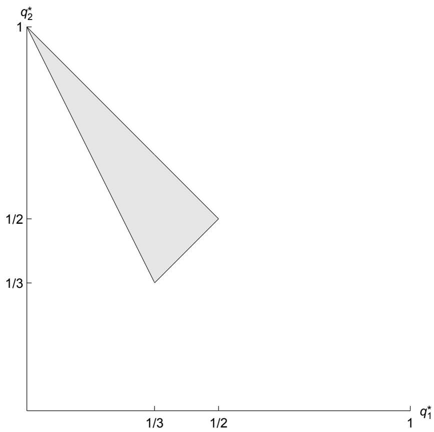
$$\frac{11}{6} < \frac{b}{d} < 4\,.$$

Then we receive by (7.16) the inequalities

$$2\,q_1^* + q_2^* \geq 1\,, \qquad q_1^* - q_2^* \leq 0 \qquad \text{and} \qquad q_1^* + q_2^* \leq 1\,,$$

which are illustrated in Figure 7.1. Here we see why Kilgour (1992) several years ago coined the term *cone of deterrence*.

Figure 7.1 Components q_1^* and q_2^* of an Inspectorate's equilibrium strategy (q_1^*, q_2^*, q_3^*) in case of legal behaviour of the Operator.



Third, because the Inspectorate's equilibrium strategy $\mathbf{q}^*$ given by (7.12) is an element of the cone of deterrence (7.16), the Inspectorate's equilibrium strategy in case of the Operator's illegal behaviour is also equilibrium strategy in case of the Operator's legal behaviour. Thus, $\mathbf{q}^*$ can be understood as a *robust* equilibrium strategy in the sense that given the payoff parameters d and b, the Inspectorate can just play $\mathbf{q}^*$ according to (7.12) and does not need to check whether (7.11) or (7.15) is fulfilled. In other words, $\mathbf{q}^*$ according to (7.12) is an equilibrium strategy – which as mentioned does not depend on payoff parameters – no matter whether the illegal or the legal game is played. This is very helpful for practical applications since, as mentioned, in many cases the payoff parameters can be estimated only roughly.

Since in this monograph many inspection games with a robust Inspectorate's equilibrium strategy are treated, we explain the logic behind the general procedure with the help of Lemma 7.1, however, using the following notation: The equilibrium strategy (7.13) is abbreviated by $\mathbf{q}^*_{\text{illegal}}$, and the optimal payoffs to the Operator (7.14) and (7.17) by Op^*_{illegal} and Op^*_{legal}, respectively. Then Lemma 7.1 – and other Lemmata and Theorems in which legal behaviour is an equilibrium strategy of the Operator – has the following structure:[2]

(i) For

$$Op^*_{\text{illegal}} > Op^*_{\text{legal}}\,, \tag{7.19}$$

[2]The equilibrium strategy of the Operator in case (i) as well as the equilibrium payoffs to the Inspectorate in cases (i) and (ii) are omitted, because they are not important for the subsequent explanations. Note that in Lemma 15.2 and Theorem 15.1 even three cases have to be distinguished.

the Operator behaves illegally and an equilibrium strategy of the Inspectorate is given by $\mathbf{q}^*_{\text{illegal}}$ with the equilibrium payoff Op^*_{illegal} to the Operator.

(ii) For

$$Op^*_{\text{illegal}} < Op^*_{\text{legal}}\,, \tag{7.20}$$

the Operator behaves legally, and the Inspectorate's set of equilibrium strategies is, using (7.8), given by

$$\begin{aligned} Q^*_{3,1} := \Big\{ \mathbf{q}^* \in Q_{3,1} : \quad & Op(0,\mathbf{q}^*) \le Op^*_{\text{legal}}\,, \quad Op(1,\mathbf{q}^*) \le Op^*_{\text{legal}} \\ & Op(2,\mathbf{q}^*) \le Op^*_{\text{legal}}\,, \quad Op(3,\mathbf{q}^*) \le Op^*_{\text{legal}} \Big\}\,. \end{aligned} \tag{7.21}$$

The equilibrium payoff to the Operator is Op^*_{legal}.

Now we come to the crucial point: It can be shown for many inspection games, that $\mathbf{q}^*_{\text{illegal}} \in Q^*_{3,1}$, i.e., $\mathbf{q}^*_{\text{illegal}}$ constitutes a robust equilibrium strategy: the Inspectorate can just play $\mathbf{q}^*_{\text{illegal}}$ and does not need to check whether (7.19) or (7.20) is fulfilled. At first sight this statement seems obvious, because the Operator's Nash condition yields in case (i)

$$Op(\mathbf{p}, \mathbf{q}^*_{\text{illegal}}) \le Op^*_{\text{illegal}}$$

for any $\mathbf{p} \in P$, and thus, by (7.20),

$$Op(\mathbf{p}, \mathbf{q}^*_{\text{illegal}}) \le Op^*_{\text{illegal}} < Op^*_{\text{legal}}\,,$$

i.e., the inequalities in (7.21) are fulfilled. Therefore, the only thing that needs to be checked on a case by case basis is whether $\mathbf{q}^*_{\text{illegal}}$ remains a meaningful expression under condition (7.20). While this is obvious for $\mathbf{q}^*_{\text{illegal}}$ according to (7.13), we will also treat inspection games where this is not true: If the Inspectorate's equilibrium strategy in case of illegal behaviour of the Operator depends on payoff parameters, then it might happen that, e.g.,

- a time point for an interim inspection that is well defined under condition (7.19) becomes smaller than the time point for the beginning of the game under condition (7.20); see Lemmata 10.3 and 12.2 and Theorem 12.1.
- a well-defined probability under condition (7.19) becomes larger than one under condition (7.20); see Lemma 15.2 and Theorem 15.1.

Note that the use of $>$ and $<$ in (7.19) and (7.20) reflects the fact that in many cases the payoff parameters can be estimated only roughly.

For easier reference we present in Table 7.3 the inspection games treated in this monograph in which an equilibrium strategy of the Operator is legal behaviour, and in which the Inspectorate possesses a robust equilibrium strategy.

Fourth, let us repeat that the analyses presented so far – introduction of payoff parameters, condition for legal behaviour and inspection strategies – can not only be applied to all cases considered in this Part I, but also grosso modo to the inspection problems in Part II.

Table 7.3 Overview of inspection games treated in this monograph in which an equilibrium strategy of the Operator is legal behaviour, and in which the Inspectorate possesses a robust equilibrium strategy.

Discrete time	
No-No inspection game $k = 1$ interim inspection	Lemma 7.1, p. 138
Continuous time	
No-No inspection game $k = 1$ interim inspection, $\alpha \geq 0, \beta \geq 0$	Lemma 9.3, p. 174
Se-No inspection game, one facility $k = 2$ interim inspections, $\alpha \geq 0, \beta \geq 0$	Lemma 10.3, p. 202 only for $\alpha = 0$
Se-Se inspection game $k = 1$ interim inspection, $\alpha \geq 0, \beta \geq 0$	Lemma 12.1, p. 237
Se-Se inspection game $k \geq 2$ interim inspections, $\alpha \geq 0, \beta \geq 0$	Lemma 12.2, p. 243 Theorem 12.1, p. 253 only for $\alpha = 0$
Critical time	
No-No inspection game with assumption (iv') on p. 282, $k = 1$ inspection, $\alpha \geq 0, \beta \geq 0$	Lemma 15.1, p. 284
No-No inspection game with assumption (iv') on p. 282, $k \geq 2$ inspections, $\alpha \geq 0, \beta \geq 0$	Lemma 15.2, p. 289 Theorem 15.1, p. 296 only for $\alpha = 0$ Corollary 15.1, p. 300
No-No inspection game with assumption (iv") on p. 307, $k \geq 1$ inspection(s), $\alpha \geq 0, \beta \geq 0$	Theorem 15.2, p. 310
Generalized Thomas-Nisgav inspection game $k \geq 1$ control, $\beta \geq 0$	Lemma 17.1, p. 358 Lemma 17.2, p. 363 Theorem 17.1, p. 366

Finally, let us mention that the set of equilibrium strategies of the Inspectorate in case of legal behaviour of the Operator is for the No-Se inspection game with $N = 3$ possible time points for $k = 1$ interim inspection the same as for the corresponding No-No inspection game, i.e., is given by (7.16), since – because of $k = 1$ – both inspection game are identical. For the Se-No and the Se-Se inspection game, however, the set of equilibrium strategies of the Inspectorate in case of legal behaviour of the Operator is different from the one given by (7.16). To demonstrate this statement we consider the Se-No inspection game with $N > 1$ possible time points for $k = 1$ interim inspection which is treated in Lemma 4.2. Using (7.3) and (4.11), we see that the Operator will behave legally if and only if

$$\frac{b}{d} > \frac{N+1}{2}\,. \tag{7.22}$$

If we take legal behaviour of the Operator into account, the payoff (4.8) to the Operator has to be modified, and is, for any $(g_2, g_1) \in [0,1]^2$ and any $\mathbf{q} \in Q_{N,1}$, given by

$$Op_{N,1}((g_2, g_1), \mathbf{q}) := \sum_{j_1=1}^{N} q_{j_1} \Big((1-g_2)\,(d\,j_1 - b) \\ + g_2 \Big[(1 - g_1(j_1))\,(d\,(N+1-j_1) - b) + g_1(j_1)\,0 \Big] \Big)\,, \tag{7.23}$$

where $g_1(j_1)$ is the probability that the illegal activity is not started at time point j_1. The Nash condition for the Operator is given by

$$Op^*_{N,1} := Op_{N,1}((g_2^*, g_1^*), \mathbf{q}^*) \geq Op_{N,1}((g_2, g_1), \mathbf{q}^*)$$

for any $(g_2, g_1) \in [0,1]^2$, which can be seen to be equivalent to the inequalities

$$Op^*_{N,1} \geq Op_{N,1}((0, g_1), \mathbf{q}^*) \qquad \text{and} \qquad Op^*_{N,1} \geq Op_{N,1}((1,0), \mathbf{q}^*)\,. \tag{7.24}$$

Because in case of legal behaviour of the Operator we have $Op^*_{N,1} = 0$, (7.24) is by (7.23) equivalent to

$$\frac{b}{d} \geq \sum_{j_1=1}^{N} q^*_{j_1}\, j_1 \geq N + 1 - \frac{b}{d}\,. \tag{7.25}$$

Condition (7.22) assures that the interval (7.25) is not empty. For $N = 3$ possible time points, (7.25) simplifies to

$$\frac{b}{d} \geq -2\,q_1^* - q_2^* + 3 \geq 4 - \frac{b}{d}\,.$$

Even though the left hand inequality coincides with the first inequality of (7.16), as a whole this set of equilibrium strategies is different from that given by (7.16). Thus, we have shown that the set of equilibrium strategies of the Inspectorate in case of legal behaviour of the Operator is different for the No-No and the Se-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection. We will come back to this observation in Part II, where, however, we show in Section 13 that the Nash conditions for the Inspectorate's equilibrium strategy in case of the No-No inspection game coincide with those for the Se-Se inspection game at least for a wide range of parameter combinations.

7.3 Effectiveness and efficiency

Let us consider again the important application which we have presented already several times: For good reasons, as already pointed out in the Introduction, we consider the nuclear material safeguards system of the IAEA in Vienna the most elaborate international verification system which served as a model for other ones, e.g., that of the Chemical Weapons Convention. Not only for these reason IAEA officials do not cease to claim that their system is effective and efficient, but what does this mean in quantitative terms?

According to online dictionaries and related Wikipedia entries, effectiveness cannot be understood as a unique construct. In the framework of inspections *we* formulate

Definition 7.1. *An inspection system is effective if it is able to bring about the results intended or doing the right thing. Furthermore, it is efficient if it is doing the right thing well, e.g., in the least expensive way.*

According to this Definition effectiveness has a dichotomic meaning: A single inspection or an inspection system is effective or not, but it may become more efficient if, e.g., a new cost reducing technique is used. Note that effectiveness can – due to other sources – also be understood as "The degree to which something is successful in producing a desired result". This definition is used in Canty and Listner (2020) and repeals the dichotomic meaning of Definition 7.1. In this monograph we use Definition 7.1.

As mentioned on p. 1, the objective of IAEA safeguards is essentially the "... timely detection of the diversion of nuclear material and the deterrence of such diversion by the risk of early detection"; IAEA (1972). In that sense *we* define

Definition 7.2. *IAEA safeguards is effective in the State under consideration, if this State, or any Operator acting on behalf of that State, is deterred from diverting nuclear material. Also, IAEA safeguards is efficient, if it deters the State or the Operator in the least expensive way.*

The key question is: When is a State or Operator deterred from such illegal behaviour? For the purpose of illustration let us return to the example considered in Section 7.2. For the No-No inspection game with $N = 3$ possible time points and $k = 1$ interim inspection the condition for legal behaviour is given by (7.4). Thus, if the parameters b, d and β of this inspection system fulfil this set of inequalities, then it is effective according to Definition 7.2.

Furthermore, let us assume that the non-detection probability β is a monotone decreasing function of the inspection effort; see for an example (6.43). Then, this inspection system is efficient, if for any given value of b/d the right hand sides of (7.4) are maximized, for the existence of a maximum replace the $>$ sign in (7.4) by $\geq$, i.e., if β is maximized such that the inequalities with $\geq$ instead of $>$ are still fulfilled.

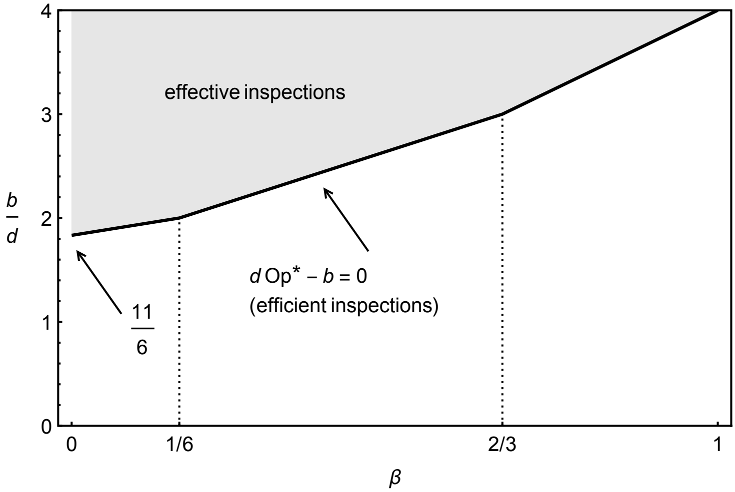
A representation of these results is given in Figure 7.2; see also Avenhaus and Krieger (2020). The shaded area characterizes effective inspections, and the solid line efficient ones. Extending what we said after (7.4) we see that the inspection system is effective either if the probability β of not detecting the illegal activity is small or the ratio b/d of sanctions to gains of the Operator in case of such behaviour is large enough.

Let us generalize these considerations to any inspection problem – not restricted to IAEA safeguards – analysed in this monograph and in the related literature. We formulate[3]

Definition 7.3. *An inspection system is effective if the equilibrium strategy of the inspected party (Operator) is legal behaviour in the sense of the purpose of the inspections. An equilibrium strategy of the inspecting party (Inspectorate) is efficient if the legal behaviour equilibrium is achieved at minimum cost.*

[3] Because the following statement is derived from Definitions 7.1 and 7.2 one might argue that it should be considered a Theorem. Since, however, the game theoretical context is new, we prefer the format of a Definition.

Figure 7.2 Effective and efficient inspections for the No-No inspection game with $N = 3$ possible time points for $k = 1$ interim inspection and with errors of the second kind.



These considerations shed new light on the diversion hypothesis which we discussed in connection with assumption (iv) on p. 17: If one wants to design an effective and efficient inspection system, then one has to determine the legal behaviour equilibrium of the inspection game that describes that system. This, however, requires the analysis of deviations from this equilibrium, namely illegal behaviour of the inspected party. In other words, the diversion hypothesis is needed for the determination of effective and efficient inspections.

7.4 Utilities for variable sampling inspection schemes and further extensions

Even though we have discussed already a large number of inspection models, many more possibilities exist as shown already in Figure 1.1. In the following we will present some extensions and future work and discuss why they have not yet been tackled and whether their analyses look promising. We will, however, not repeat in detail extensions which we mentioned already in the framework of the models of this Part I, e.g., the questions whether the No No and the No-Se inspection games on one hand, and the Se-No and the Se-Se inspection games on the other lead in general to the same optimal expected detection times.

First, let us consider *variable sampling problems* which arise when inspections are based on quantitative measurements and errors of the second kind, i.e., non-detection of illegal activities, as well as errors of the first kind, i.e., false alarms, cannot be avoided. Since the latter ones have negative consequences both to the Operator and to the Inspectorate, we can no longer describe problems of this kind with the help of zero-sum games. Instead and like in the case of the deterrence considerations in the previous section, we have to use non-zero-sum games.

For the sake of completeness, and for use in later chapters we present here the payoffs to both players in case false alarms cannot be avoided in playing for time inspection games: The payoffs to the two players (Operator, Inspectorate) are given by

$$\begin{array}{rl} (d\,\triangle t - b, -\,a\,\triangle t) & \text{for illegal behaviour and detection time } \triangle t \\ (-f, -g) & \text{for legal behaviour and a false alarm} \\ (0,0) & \text{for legal behaviour and no false alarm}\,, \end{array} \qquad (7.26)$$

where with T being the length of the reference time interval, $0 < f < b < d\,T$, $0 < g < a\,T$. Note that if $b \geq d\,T$ were true, the Operator would not have any incentive to behave illegally at all. The payoffs are normalized to zero for legal behaviour without false alarms, this is also the best result for the Inspectorate and implies the idea that inspection costs are not part of the Inspectorate's payoff, but rather imposed by the external parameter k, the number of interim inspections. The profit (loss) to the Operator (Inspectorate) grows proportionally with the time elapsed to detection of the illegal activity. A false alarm is resolved unambiguously with time independent costs $-f$ to the Operator and $-g$ to the Inspectorate, whereupon the game continuous. Note that because $-\,a\,\triangle t < 0$ and $-g < 0$, the most desirable outcome for the Inspectorate is legal behaviour of the Operator and no false alarm, i.e., its primary aim is to deter the Operator from behaving illegally; see Section 7.3.

Note that for the inspection games in Part III, the payoffs (7.26) are amended accordingly; see (14.1) and (14.2) in Chapter 14.

We note in passing that those inspection games, which do not take into account legal behaviour as well as false alarms, are still strategically equivalent to zero-sum games; see, e.g., the discussion before and after Lemma 7.1 and p. 398. In the example given in Section 7.2 we have shown this explicitly. For models which do take into account false alarms, this is no longer the case since false alarms cause, as mentioned, costs (negative gains) to both players.

In fact, practitioners have not yet raised false alarm problems neither in the context of discrete nor continuous time playing for time inspection games. Whereas theoreticians have already developed and analysed continuous time models with errors first and second kind, see Sections 9.4 and 10.3 and Chapter 12, discrete time models with errors first and second kind have not yet been considered. There is no doubt that interesting and useful models for discrete time could be developed once they would be asked for by practitioners.

Second, one can imagine situations where, e.g., in a State, there are several facilities of the same type, and that interim inspections have to be carried through such that the total number of interim inspections per reference time interval is fixed. This would mean, by the way, that then in general only the *expected* number of interim inspections per reference time interval and *per* facility would be fixed. The question then would be whether the Operator of the single facilities would act independently or if they would plan, if at all, only one illegal activity to be performed in one of the facilities. Contrary to the variable sampling case, this problem has already been raised by practitioners; see Avenhaus et al. (2010). Due to its complexity only a very special case has been studied so far: Two facilities, three possible time points for interim inspections, two interim inspections in total, no errors of the first and second kind and non-sequential planning of both players, State and Inspectorate. Any extension would be difficult, but not impossible, if demanded for. Let us mention already here, that this kind of problems will be considered for continuous time playing for time inspection games in Chapter 11. The case that for one facility the expected number of inspections is fixed, is analysed in Chapter 24.

Third, let us mention the so-called Inspectorate leadership principle. In general and according to Avenhaus et al. (2002), the leadership principle says that it can be advantageous in a competitive situation to be the first player to select and stay with a strategy. It was suggested first by von Stackelberg (1934) for pricing policies. Maschler (1966) applied this idea to sequential inspections. The notion of leadership consists of two elements: The ability of the player first to announce his strategy and make it known to the other player, and second to commit himself to playing it. This concept is particularly suitable for inspection games since an Inspectorate can credibly announce its strategy and stick to it, whereas the Operator cannot do so if he intends to act illegally. Therefore, it is reasonable to assume that the Inspectorate will take advantage of its leadership role.

Let us note first that each zero-sum game with a game theoretical solution, i.e., with optimal strategies and optimal payoffs, leadership has no effect. This is the essence of the minimax theorem: Each player can guarantee his optimal payoff even if he announces his (mixed) strategy to his opponent and commits himself to playing it.

Furthermore, it turns out that in inspection games where only errors of the second kind have to be taken into account, the resulting equilibria are up to special cases the same as those of the non-leadership version; see Avenhaus and von Stengel (1991).

In those cases in which false alarms cannot be avoided, however, the leadership version may result in equilibria which induce the Operator to legal behaviour whereas this is not so for the non-leadership version. In this context let us mention the work by Avenhaus and Okada (1992) in which sequential inspector-leadership games are considered, and which stands a bit out of the classification given in Chapter 2: The main objective of that investigation is to derive simple criteria for the determination of optimal inspection procedures from the equilibrium conditions for non-cooperative non-zero-sum two-person games. It is shown that, given the appropriate assumptions, one can arrive at "statistical" optimization criteria. In the simplest case one gets the global probabilities of the first and second kind errors, and in more complex cases the average run lengths for legal and illegal Operator behaviour.

Finally, because in this Part I false alarms are not considered, we will not explain the idea of inducing the Operator to legal behaviour here in more detail. Instead we will do this in Part II, where in Sections 9.4 and 10.3 and in Chapter 12 playing for time inspection games with errors of the first and second kind are considered.

Part II
Playing for Time: Continuous Time

The essential difference of the classes of inspection problems to be considered in Part II to those of Part I is the fact that the Inspectorate can perform its interim inspections at any point of time during the reference time interval, and that the Operator can start the illegal activity at any point of time during the reference time interval as well; in other words, time is now considered to be continuous.

Despite the fact that the difference between the assumptions of Parts II and I consists only in the nature of time, it will turn out that its consequences are considerable. Obviously, the analytical techniques are different: Now distributions of continuous random variables, such as the start of the illegal activity and the interim inspection time points, will have to be used, and in some cases methods of calculus will have to be applied.

There are, however, also differences in substance: Surprisingly enough in Chapters 10 – 12 the optimal interim inspection time points turn out to be pure strategies in the Se-No and the Se-Se inspection games which happens in the discrete time Se-No and the Se-Se inspection games only in rare cases; see p. 68. But also, some of the game theoretical solutions which will be presented in Part II, will turn out to be limiting cases of game theoretical solutions obtained in Part I.

In order that the three main parts of this monograph can be read independently in nearly arbitrary order, we repeat in Chapter 8 more or less verbally the general description of the inspection problems as well as the list of assumptions which has been presented in Chapter 2, and we will emphasize the differences between these lists.

Part II is structured in analogy to Part I: The inspection games in Chapters 9 – 12 differ by the planning of interim inspections and illegal activities and – which is new – several facilities are considered in Chapter 11. As in Part I, various relations to the discrete time inspection games treated in Part I and relations within the inspection games of Part II, such as optimal/equilibrium strategies, optimal/equilibrium payoffs and system quantities given in Table 1.1, are discussed throughout Part II, and especially in its final Chapter 13.

Chapter 8

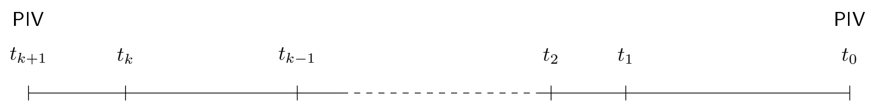
General assumptions

Even though the inspection problems considered in this Part II are very similar to those of Part I, and even though therefore many of the assumptions listed in Chapter 2 are the same as the ones needed here, we will formulate again a complete set of assumptions. For the sake of clarity we will mention explicitly which of the following assumptions are different from those of Chapter 2.

Again, we consider in this Part except for Chapter 11 one single inspected object, for example a production line, or a nuclear or chemical facility (for short: facility) which is subject to inspections in the framework of agreed rules, formal agreements or an international treaty, and a reference time interval of one time unit, e.g., a week, a month, or a calender year.

In order to separate the timeliness aspect of routine inspections from the overall goal of detecting failures or an illegal activity, it is assumed that a thorough and unambiguous inspection takes place at the beginning and end of the reference time interval with the help of which failures or an illegal activity will be detected with certainty once they have occurred. Such an inspection is called, according to some agreed wording, Physical Inventory Verification (PIV); see IAEA (2002) and Figure 8.1.

Figure 8.1 Time line for the continuous time inspection models including the time points $t_k, \ldots, t_1$ of the k interim inspections.



In addition, it is assumed that by agreement k less intrusive interim inspections are strategically placed during the reference time interval to reduce the time between start and detection of failures or the illegal activity below the length of the reference time interval. We assume, and this is the central difference to Part I, that the Inspectorate can perform its interim inspections at any points of time within the open reference time interval. Let $t_k, \ldots, t_1$ with $(0 =)t_{k+1} < t_k < \ldots < t_1 < t_0$ denote the time points of the interim inspection(s); see Figure 8.1. Note that t_0 is determined by the absolute length and scaling, e.g., days, weeks or quarters of a year, of the reference time interval. Thereafter, at the end of the reference time interval, i.e., at time point t_0, the above mentioned PIV takes place. For technical reasons we label again the k interim inspections in backward order. At an interim inspection a preceding failure or illegal activity will eventually be detected with some probability lower or equal than one. Also, associated with each interim inspection which is not preceded by a failure or illegal activity may be a false alarm which is assumed to be clarified with certainty.

Also, as a central difference to Part I, the Operator can start the illegal activity at any point of time within the reference time interval $[t_{k+1}, t_0)$.

Also, contrary to Chapters 3 to 6 of Part I, we will not assume in the corresponding chapters in this Part II that the Operator will start an illegal activity with certainty during the reference time interval. Instead, legal behaviour will also be a strategy of the Operator, and it will depend on the details of the problem and its model whether or not legal behaviour will be an equilibrium strategy. For this purpose, so-called utilities are introduced which describe the gains and losses of the Operator and of the Inspectorate in case of legal and illegal behaviour of the Operator; see (8.1) and (8.2).

Again, in Chapters 9 – 12 we assume, according to what has been said before, that the objective of the Operator is to place the start of the illegal activity such that the *detection time*, i.e., the time between the start of the illegal activity and its detection, is as long as possible, whereas the objective of the Inspectorate is to place its interim inspections such that the expected detection time is as short as possible. This means that we model this inspection game as a two person game where the payoffs to the Operator and to the Inspectorate are proportional to the expected detection time. Note that only in those cases where it is assumed that the Operator will behave illegally with certainty and where no false alarms are possible, the payoff to the Operator is the expected detection time itself.

Let us summarize the assumptions made so far:

(i) There are two players: the Operator of the facility or facilities under consideration and the Inspectorate.

(ii) The Inspectorate performs k interim inspections at the time points $(0 =)t_{k+1} < t_k < \ldots < t_1 < t_0$ in any facility if there are several ones.

(iii) The Inspectorate performs at the beginning and at the end of the reference time interval a regular inspection (Physical Inventory Verification, PIV) at which the illegal activity of the Operator – if he behaves illegally at all – is detected with certainty if it is not detected at a previous interim inspection.

(iv) The Operator may start at most once an illegal activity during the reference time interval $[t_{k+1}, t_0]$ in any facility if there are several ones.

(v) During an interim inspection the Inspectorate may commit an error of the first and second kind with probabilities α and β. These error probabilities are the same for all k interim inspections and all facilities if there are several ones. Only in Section 11.1 an exception is made. The "game" continues after an error of the first kind.

(vi) The number k of interim inspections is known to the Operator.[1]

[1] The possibility that the *expected* number of inspections is fixed and known to the Operator is addressed in Chapter 24; see also the comment on p. 18.

(vii) The Operator decides – if he behaves illegally at all – at the beginning of the reference time interval, i.e., at time point t_{k+1}, when to start the illegal activity, or he only decides whether to start the illegal activity immediately at time point t_{k+1} or to postpone its start; in the latter case he decides again after the first interim inspection; and so on.

The Inspectorate decides at the beginning of the reference time interval when to perform its interim inspections, or it decides only when to perform the first interim inspection, and after the first one when to perform the second interim inspection, and so on.

(viii) Both players decide independently of each other, i.e., no bindings agreements are made.

(ix) The payoffs to the two players (Operator, Inspectorate) are linear functions of the detection time $\triangle t$, i.e., the time between start and detection of the illegal activity, and are given as follows

$$\begin{array}{rl} (d\,\triangle t - b, -\,a\,\triangle t) & \text{for illegal behaviour and detection time } \triangle t \\ (-f, -g) & \text{for legal behaviour and a false alarm} \\ (0,0) & \text{for legal behaviour and no false alarm}\,, \end{array} \tag{8.1}$$

where

$$0 < f < b < d\,(t_0 - t_{k+1}) \qquad \text{and} \qquad 0 < g < a\,(t_0 - t_{k+1})\,. \tag{8.2}$$

(x) An (interim) inspection does not consume time. In case of the coincidence of the start of the illegal activity and the interim inspection, the illegal activity may be detected at the occasion of the next interim inspection or, with certainty, at the final PIV. In this sense the wording "... right after an interim inspection ..." is equivalent to "... at an interim inspection ...".

(xi) The game ends either at the interim inspection at which the illegal activity is detected or at the final PIV; see (iii).

We comment only on assumptions (v) and (ix) since the other comments given in Chapter 2 hold here as well. Regarding assumption (v), we mentioned in Chapter 2 that models, in which only errors of the second kind are taken into account, typically describe Attribute Sampling schemes. Models with errors of the first and second kind are typical for Variable Sampling problems; see Thyregod (1988) and Section 7.4. They describe, e.g., inspections which are based on quantitative measurements. Further assumptions on the relation between α and β will be made in the respective sections and chapters.

The payoffs (8.1) with (8.2) in assumption (ix) are the payoffs introduced in (7.26), and which are explained on p. 146. Let us note that if $d \geq (t_0 - t_{k+1})$ were true, then the Operator would not have any incentive to behave illegally at all, and that d as a proportionality factor changes appropriately if the time is measured differently: If we measure, for example, $t_0 - t_{k+1}$ in months resp. days instead of years, then d has to be divided by 12 resp. 365. Thus, it would be always better to write, e.g., $d\,(t_0 - t_{k+1})$, but this would lead to more cumbersome equations. Also note that if false alarms can be excluded and the Operator behaves illegally with certainty, then (8.1) implies that the game under consideration is strategically equivalent to a zero-sum game with the detection time as payoff to the Operator; see p. 398. Because $-\,a\,\triangle t < 0$ and $-g < 0$, the preferred Inspectorate's outcome is legal behaviour of the Operator and no false alarms, i.e., its primary aim is to deter the Operator from behaving illegally; see Section 7.3.

Chapter 9

No-No inspection game: Diamond model and extensions

Like in the Part I, we start Part II with that inspection game where both players, Operator and Inspectorate, plan the start of the illegal activity respectively the timing of the k interim inspections at the beginning of the reference time interval, i.e., we consider the No-No inspection game; see Table 2.1. Again, the payoff to the Operator is the detection time, i.e., the time elapsed from the start of the illegal activity until its detection, latest at the final PIV, which he wants to maximize. In contrast to Chapter 3, however, we are able to present a model and optimal strategies for any number k of interim inspections, and not just one. This model has been described and analysed by Diamond (1982); it is an ingenious piece of scientific work, and this chapter may be considered an homage to the author and his achievement.

Applications of this inspection model have been analysed in the framework of reliability studies by Derman (1961) and Diamond (1982): An operating unit may fail which creates cost which increase with the time until the failure is detected. The overall time interval is the time between normal replacements of the unit. A minimax analysis leads to a zero-sum game with the operating unit as Operator which cannot observe any inspection. A more common approach in reliability theory, which is not our topic, is to assume some knowledge about the distribution of the failure time. Another application is the planning of interim inspections in the framework of nuclear material safeguards which has been described already in Section 6.6 and will be addressed again in Section 10.2.

Following the same lines as in Part I, we start in Section 9.1 with the most simple, though by no means trivial case of just $k = 1$ interim inspection and no statistical errors following Avenhaus et al. (2002). Thereafter, in the central Section 9.2, the work of Diamond for any number k of interim inspections and no statistical errors is presented in a slightly modified way. Diamond's model is extended in Section 9.3 taking errors of the second kind into account, and a conjecture about optimal strategies and the optimal payoff is formulated for any number k of interim inspections. In Section 9.4 errors of the first and second kind are considered for the case of just $k = 1$ interim inspection; the results presented here are due to Sohrweide (2002), Avenhaus et al. (2003) and Krieger (2011). The chapter ends with Section 9.5 on the appropriate choice of the error first kind probability α.

Note that only in Sections 9.4 and 9.5 payoff parameters are used. This has been done for two reasons: First, Diamond's work should be presented in its original form and second, his model contains only technical parameters. Therefore, remembering what has been said in Section 1.4, it meets practitioners' requirements much better than a model containing also payoff parameters, satisfying for theoreticians as it may be.

9.1 One interim inspection

The inspection game analysed in this section is based on the following specifications:

(iv') The Operator starts once an illegal activity during the reference time interval $[0, t_0]$ in the only facility under consideration.

(v') During an interim inspection the Inspectorate does not commit an error of the first and of the second kind, i.e., the illegal activity, see assumption (iv'), is detected with certainty during the next interim inspection or with certainty during the final PIV; see assumption (iii).

(vii') The Operator decides at the beginning of the reference time interval, i.e., at time point 0, when to start the illegal activity.

The Inspectorate decides at the beginning of the reference time interval when to perform its interim inspection.

(ix') The payoffs to the two players (Operator, Inspectorate) are linear functions of the detection time $\triangle t$, i.e., the time between start and detection of the illegal activity, and are given as follows

$$(\triangle t, -\triangle t) \qquad \text{for illegal behaviour and detection time} \, \triangle t \, .$$

The remaining assumptions of Chapter 8 hold throughout this section.

The Operator starts the illegal activity at some time point s from the reference time interval $[0, t_0)$ and the Inspectorate performs its interim inspection at time point $t_1 \in (0, t_0)$. Thus, the sets of pure strategies of both players are given by

$$\mathcal{S} := \{s \in \mathbb{R} : 0 \leq s < t_0\} \qquad \text{and} \qquad \mathcal{T}_1 := \{t_1 \in \mathbb{R} : 0 < t_1 < t_0\} \, . \tag{9.1}$$

Because the letters S resp. T_1 are – as in Part I – reserved for the start of the illegal activity resp. the time point of the interim inspection, see below, the strategy sets (9.1) are denoted using calligraphic letters.

Again, the illegal activity is detected at the earliest inspection following the start of the illegal activity at time point s, i.e., at time point t_1 or at the final PIV. Then, the payoff to the Operator, i.e., the detection time, is given by

$$Op_1^*(s, t_1) := \begin{cases} t_1 - s & \text{for} \quad 0 \leq s < t_1 < t_0 \\ t_0 - s & \text{for} \quad 0 < t_1 \leq s < t_0 \end{cases} \, , \tag{9.2}$$

which can be seen as a generalization of the payoff matrix in Table 3.2. $Op_1(s, t_1)$ is also called the *payoff kernel* of the game. The use of $s < t_1$ and $t_1 \leq s$ in (9.2) is due to assumption (x) of Chapter 8 which says here that in case the start of the illegal activity coincides with the interim inspection, i.e., $s = t_1$, the illegal activity is detected at the final PIV. Again, the Inspectorate's payoff is the negative of the payoff to the Operator; see assumption (ix').

The payoff kernel given by (9.2) describes a kind of *duel* with reversed time where both players have an incentive to act early but after the other. By (9.2), the Operator's payoff is too small if he starts the illegal activity too late, so that he will select the time point s with a certain probability distribution from an interval $[0, b]$ where $b < t_0$. Consequently, the Inspectorate will not inspect later than time point b.

In analogy to Section 3.1 we first want to answer the question, if there exists a saddle point in pure strategies, i.e., a pair $(s^*, t_1^*) \in \mathcal{S} \times \mathcal{T}_1$ from which no player has an incentive to deviate:

$$Op_1(s, t_1^*) \le Op_1(s^*, t_1^*) \le Op_1(s^*, t_1) \tag{9.3}$$

for any $s \in \mathcal{S}$ and any $t_1 \times \mathcal{T}_1$. The answer to this question is no, since t_1^* – according to (9.3) – would have to be a minimum of the function $Op_1(s^*, t_1)$. Because only the infimum of $Op_1(s^*, t_1)$ for $t_1 \in \mathcal{T}_1$ exists, and is equal to zero, but not the minimum, a saddle point in pure strategies does not exist.

Therefore we must look for mixed strategies, which raises the question: What are mixed strategies for players with infinitely many pure strategies? The answer is that they can be represented, just as in matrix games, as probability distributions over the set of pure strategies. It is, however, more convenient to work with probability distribution functions on $\mathbb{R}$; see Karlin (1959a). Let the random variables S resp. T_1 represent the start of the illegal activity resp. the time point of the interim inspection. Their distribution functions are denoted by

$$P(s) := \mathbb{P}(S \le s) \qquad \text{and} \qquad Q(t_1) := \mathbb{P}(T_1 \le t_1)\,.$$

The intuitive meaning is straightforward: $P(s)$ is the probability that the illegal activity is started at time point s or earlier, and $Q(t_1)$ denotes the probability of the interim inspection is taking place at time point t_1 or earlier.

Using Lebesque-Stieltjes integrals, see Hewitt and Stromberg (1965) or Carter and Brunt (2000), the Operator's (expected) payoff, i.e., the expected detection time, is, for any P and any Q, given by

$$Op_1(P, Q) := \int_{[0,t_0)} \int_{(0,t_0)} Op_1(s, t_1)\, dQ(t_1)\, dP(s) \tag{9.4}$$

and to the Inspectorate by $In_1(P, Q) := -\,Op_1(P, Q)$. It can be shown, see Carter and Brunt (2000), that the double integral exist and that the order of integration can be changed.

Let us repeat, see also Chapter 19 for the case of matrix games, that a pair of distribution functions (P^*, Q^*) constitutes a saddle point of the game if and only if

$$Op_1(P, Q^*) \le Op_1(P^*, Q^*) \le Op_1(P^*, Q) \tag{9.5}$$

for any P and any Q. Also, it can be shown, like in Chapter 19 for matrix games, that a pair of distribution functions (P^*, Q^*) constitutes a saddle point if and only if

$$Op_1(s, Q^*) \le Op_1(P^*, Q^*) \le Op_1(P^*, t_1) \tag{9.6}$$

for any $s \in \mathcal{S}$ and any $t_1 \times \mathcal{T}_1$, i.e., both inequalities have only to be proven for the players' pure strategies. Again, P^* and Q^* are called *optimal* strategies, and $Op_1(P^*, Q^*)$ is called *optimal* payoff to the Operator.

So we have to look for distribution functions P^* and Q^* fulfilling (9.5). Success, however, is by no means guaranteed, since the payoff kernel $Op_1(s, t_1)$ is discontinuous on the diagonal $s = t_1$, and the existence of optimal strategies for those kind of games cannot be guaranteed without further assumptions; see Owen (1988). Fortunately, the games discussed in this section possesses optimal mixed strategies. Finding these optimal strategies is in general a difficult task.

The game theoretical solution of this inspection game, see Avenhaus et al. (2002), is presented in

Lemma 9.1. *Given the No-No inspection game on the reference time interval $[0, t_0]$ with $k = 1$ interim inspection. The sets of mixed strategies are given by the set of distribution functions on $\mathbb{R}$, and the payoff to the Operator by (9.4) using (9.2). Define the cut-off time point t^* by*

$$t^* := t_0 \left(1 - \frac{1}{e}\right).$$

Then an optimal strategy of the Operator is given by the distribution function

$$P^*(s) = \begin{cases} 0 & \text{for} \quad s < 0 \\ \dfrac{t_0 - t^*}{t_0 - s} & \text{for} \quad s \in [0, t^*) \\ 1 & \text{for} \quad s \geq t^* \end{cases}, \tag{9.7}$$

and an optimal strategy for the Inspectorate by the distribution function

$$Q^*(t_1) = \begin{cases} 0 & \text{for} \quad t_1 < 0 \\ \ln\left[\dfrac{t_0}{t_0 - t_1}\right] & \text{for} \quad t_1 \in [0, t^*) \\ 1 & \text{for} \quad t_1 \geq t^* \end{cases}. \tag{9.8}$$

The optimal payoff to the Operator is

$$Op_1^* := Op_1(P^*, Q^*) = t_0 - t^* = \frac{t_0}{e}. \tag{9.9}$$

Proof. It can be easily seen that $P^*(s)$ given by (9.7) and $Q^*(t_1)$ given by (9.8) are distribution functions on $\mathbb{R}$.

Because the interim inspection may take place before or after the start of the illegal activity, the payoff $Op_1(s, Q^*)$ to the Operator is for any $s \in [0, t^*]$, using (9.2), given by

$$\begin{aligned} Op_1(s, Q^*) &= \int_0^s (t_0 - s)\, q^*(t_1)\, dt_1 + \int_s^{t^*} (t_1 - s)\, q^*(t_1)\, dt_1 \\ &= t_0 \int_0^s q^*(t_1)\, dt_1 + \int_s^{t^*} t_1\, q^*(t_1)\, dt_1 - s\,. \end{aligned} \tag{9.10}$$

[1]It's derivative with respect to $s \in (0, t^*)$ simplifies, using (9.8), to[2]

$$\frac{d}{ds}\, Op_1(s, Q^*) = t_0\, q^*(s) - s\, q^*(s) - 1 = q^*(s)\,(t_0 - s) - 1 = 0\,, \tag{9.11}$$

[1]If P^* and Q^* are optimal strategies, and s is in the support of P^*, then $Op_1(s, Q^*)$ is constant for any $s \in [0, t^*]$; see Karlin (1959a), Lemma 2.2.1.

[2]In Part I we mentioned that it is sometimes easier to verify a solution which has been guessed with some strange means than to find it. Here, the right hand equation $q^*(s)\,(t_0 - s) - 1 = 0$ of (9.11) could have also be used to find the solution.

which means, that $Op_1(s, Q^*)$ is constant for any $s \in (0, t^*)$. Moreover, because $Op_1(s, Q^*)$ is a continuous function in s, see (9.10), $Op_1(s, Q^*)$ is constant for any $s \in [0, t^*]$, and we get, using (9.10), $Op_1(t^*, Q^*) = t_0 - t^* = t_0/e$, i.e., (9.9). Using (9.4) and (9.7), we obtain[3]

$$Op_1(P^*, Q^*) = \int_{[0,t_0)} Op_1(s, Q^*)\, dP^*(s) = (t_0 - t^*)\,(P^*(t_0) - P^*(0^-)) = t_0 - t^* = \frac{t_0}{e}\,,$$

because $P^*(s)$ is a right-continuous function. For $s > t^*$, the Operator's payoff is

$$Op_1(s, Q^*) = \int_0^{t^*} (t_0 - s)\, q^*(t_1)\, dt_1 = t_0 - s < \frac{t_0}{e} = Op_1^*\,.$$

Therefore, we obtain, using (9.9), the inequality $Op_1(s, Q^*) \le Op_1^*$ for any $s \in [0, t_0)$, i.e., the left hand inequality of (9.6) is fulfilled.

We now prove the right hand inequality of (9.6). Because $P^*(s)$ according to (9.7) is a right-continuous function with an atom at $s = 0$, and because of the properties of Lebesgue-Stieltjes-integrals[4,5], we get, using (9.2),

$$\begin{aligned} Op_1(P^*, t_1) &= \int_{[0,t_0)} Op_1(s, t_1)\, dP^*(s) \\ &= \int_{[0,0]} Op_1(s, t_1)\, dP^*(s) + \int_{(0,t_1)} Op_1(s, t_1)\, dP^*(s) + \int_{[t_1,t_0)} Op_1(s, t_1)\, dP^*(s) \\ &= t_1\, P^*(0) + t_1\,(P^*(t_1) - P^*(0)) + t_0\,(P^*(t_0) - P^*(t_1)) - \int_{(0,t_0)} s\, dP^*(s) \\ &= -\,P^*(t_1)\,(t_0 - t_1) + t_0\, P^*(t_0) - \int_{(0,t_0)} s\, dP^*(s) \\ &= \begin{cases} t^* - \displaystyle\int_{(0,t_0)} s\, dP^*(s) & \text{for} \quad t_1 \in (0, t^*) \\ t_1 - \displaystyle\int_{(0,t_0)} s\, dP^*(s) & \text{for} \quad t_1 \ge t^* \end{cases} . \end{aligned} \tag{9.12}$$

Using partial integration[6] and (9.7), we obtain

$$\begin{aligned} \int_{(0,t_0)} s\, dP^*(s) &= t_0\, P^*(t_0^-) - 0\, P^*(0^+) - \int_{(0,t_0)} P^*(s)\, ds \\ &= t_0 - (t_0 - t^*) \int_0^{t^*} \frac{1}{t_0 - s}\, ds - (t_0 - t^*) = t^* - \frac{t_0}{e} = t^* - Op_1^*\,. \end{aligned} \tag{9.13}$$

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[3] For the interval $[0, t_0)$, we have $\int_{[0,t_0)} \mathbf{1}\, dP(t) = P(t_0^-) - P(0^-)$; see Theorem 6.1.4 and Section 4.1 in Carter and Brunt (2000).

[4] If the interval I is a union of a finite number of pairwise disjoint intervals $I = I_1 \cup I_2 \cup \ldots \cup I_n$, then $\int_I f(s)\, dP(s) = \sum_{j=1}^n \int_{I_j} f(s)\, dP(s)$; see Theorem 6.1.1 in Carter and Brunt (2000).

[5] For any interval I, and any function f defined at $a \in I$, we have $\int_{[a,a]} f(s)\, dP(s) = f(a)\,(P(a^+) - P(a^-))$; see Theorem 6.1.6 in Carter and Brunt (2000).

[6] Let $f, g : I \to \mathbb{R}$ be functions of bounded variation, and let the set of points at which f and g are both discontinuous be empty. Then, $\int_I f(s)\, dg(s) + \int_I g(s)\, df(s) = \mu_{fg}(I)$, where $\mu_{fg}(I)$ is in case of the open interval $I = (a, b)$, $a < b$, given by $\mu_{fg}((a, b)) = f(b^-)\, g(b^-) - f(a^+)\, g(a^+)$. This result is a special case of Theorem 6.2.2 in Carter and Brunt (2000).

Therefore, we get, using (9.4), that $Op_1(P^*, t_1) = Op_1(P^*, Q^*) = Op_1^*$ for any $t_1 \in (0, t^*)$. Finally, (9.12) and (9.13) imply for any $t_1 \geq t^*$

$$Op_1(P^*, t_1) = Op_1^* + t_1 - t^* \geq Op_1^* ,$$

which implies $Op_1^* \leq Op_1(P^*, t_1)$ for any $t_1 \in (0, t_0)$, i.e., the right hand inequality of (9.6) is fulfilled. □

Before we turn to the general case of any number k of interim inspections, we make some comments on the results of Lemma 9.1. A comprehensive discussion can be found after the proof of Theorem 9.1.

First, the surprising result – which can also be observed in the discrete time No-No inspections games, see Theorem 3.1 – is, that after time point t^* neither an illegal activity is started nor an interim inspection is performed. This result makes sense since detection is guaranteed to occur at the end of the reference time interval and the Operator will not wish to wait too long before starting the illegal activity. Furthermore, it is interesting to notice that the Operator will start the illegal activity with positive probability $P^*(0) = 1/e > 0$ at time point 0; see (9.7).

Second, it has been mentioned after (9.2) that this game can also be seen as a generalization of the discrete time inspection game with the payoff matrix given in Table 3.2. Comparing the results from Lemma 9.1 for $t_0 = 1$ with that from Theorem 3.2 we see that they coincide. This means that not only the payoff matrix $A/(N+1)$ merges for $N \to \infty$ with the payoff kernel (9.2), but also – and this is by no means obvious – the optimal distribution functions for $N \to \infty$ are the same. As a side remark note also that due to Lemma 3.3 the normalized optimal expected detection time $Op_{N,1}/(N+1)$, $N = 1, 2, \ldots$, in the discrete time No-No inspection game is a monotone decreasing function of N and therefore, always larger than that in the continuous time game; see also Figure 3.3.

Third, the optimal strategies of both players can also be formulated in another way – due to a brilliant idea of Diamond (1982). It can be directly seen that, using $h_1(x) = e^x$, the distribution function

$$\tilde{P}^*(s) = \begin{cases} 0 & \text{for} \quad s < 0 \\ \dfrac{1}{h_1(1)}\, h_1\left(1 - h_1^{-1}\left(\dfrac{h_1(1)}{t_0}\,(t_0 - s)\right)\right) & \text{for} \quad 0 \leq s < t_0\left(1 - \dfrac{1}{h_1(1)}\right) \\ 1 & \text{for} \quad s \geq t_0\left(1 - \dfrac{1}{h_1(1)}\right) \end{cases}$$

almost magically coincide with (9.7). The Inspectorate's optimal interim inspection time point can also be formulated with the help of the function $h_1(x)$: It is given by

$$t_1^*(U) = t_0\left(1 - \frac{h_1(1-U)}{h_1(1)}\right) , \tag{9.14}$$

where U is a uniformly distributed random variable on $[0, 1]$. To prove this statement we determine the distribution function of T_1 for any $t_1 \in (0, t^*)$, and get

$$\tilde{Q}^*(t_1) = \mathbb{P}(T_1 \leq t_1) = \mathbb{P}(t_1^*(U) \leq t_1) = \mathbb{P}\left(U \leq 1 - h_1^{-1}\left(\frac{h_1(1)}{t_0}\,(t_0 - t_1)\right)\right)$$

$$= 1 - h_1^{-1}\left(\frac{h_1(1)}{t_0}\,(t_0 - t_1)\right) = \ln\left[\frac{t_0}{t_0 - t_1}\right] ,$$

i.e., $Q^*(t_1)$ from (9.8). This representation of the Inspectorate's optimal strategy renders its application very easy: The Inspectorate uses a random number generator, realizes $U = u$ and inspects at time point $t_1^*(u)$. The optimal expected detection time can also be written as a function of $h_1(x)$: $Op_1^* = t_0/h_1(1)$. It is this representation of the optimal strategies with the help of the function $h_1(x)$ which allowed Diamond to find a game theoretical solution of his inspection model for any number k of interim inspections.

Fourth, like in the discrete time No-No inspection game we determine the optimal expected time point for the start of the illegal activity $\mathbb{E}_{P^*}(S)$, and the optimal expected interim inspection time point $\mathbb{E}_{Q^*}(T_1)$. Using the rules of integration in footnotes 3 – 6, and making use of (9.9), we get

$$\begin{aligned}\mathbb{E}_{P^*}(S) &:= \int_{[0,t_0)} s\, dP^*(s) = 0\,(P^*(0^+) - P^*(0^-)) + \int_{(0,t^*)} s\, dP^*(s) \\ &= t^*\,P^*(t^{*-}) - 0\,P^*(0^+) - \int_{(0,t^*)} P^*(s)\,ds = t^* - (t_0 - t^*)\int_0^{t^*} \frac{1}{t_0 - s}\,ds \\ &= t^* + Op_1^* \ln\left[1 - \frac{t^*}{t_0}\right] = t^* - Op_1^* = 2\,t^* - t_0\,, \end{aligned} \tag{9.15}$$

which is the same relation as given by (3.37) for the discrete time No-No inspection game. The optimal expected interim inspection time point is, using (9.14) with $h_1(x) = e^x$, given by

$$\mathbb{E}_{Q^*}(T_1) := \int_0^1 t_0\left(1 - \frac{h_1(1-u)}{h_1(1)}\right) du = t_0 \int_0^1 \left(1 - e^{-u}\right) du = Op_1^*\,, \tag{9.16}$$

which leads, using (9.15), to

$$\mathbb{E}_{P^*}(S) + \mathbb{E}_{Q^*}(T_1) = t^*\,, \tag{9.17}$$

and which is the same relation as given by (3.39) for the discrete time No-No inspection game.

9.2 Any number of interim inspections

Because the case of $k = 2$ interim inspections does not give any additional insights into the problem, we directly turn to the general case of k interim inspections.

The inspection game analysed in this section is based on the specifications made at the beginning of Section 9.1, where assumption (vii') holds for any number k of interim inspections, and on the remaining assumptions of Chapter 8.

Again, the Operator starts the illegal activity at some time point $[0, t_0)$, thus his set $\mathcal{S}$ of pure strategies is again given by (9.1). The Inspectorate has k interim inspections which it performs at time points $t_k, \ldots, t_1$, freely chosen from the reference time interval $(0, t_0)$, i.e., its set of pure strategies is

$$\mathcal{T}_k := \{(t_k, \ldots, t_1) \in \mathbb{R}^k : 0 < t_k < \ldots < t_1 < t_0\}\,. \tag{9.18}$$

Since, according to assumption (v') made at the beginning of Section 9.1 and assumption (x) of Chapter 8, the illegal activity is detected at the earliest inspection following the start of the illegal activity at time point s with certainty, the payoff to the Operator, i.e., the detection time, is given by

$$Op_k(s,(t_k,\ldots,t_1)) := t_n - s \qquad \text{for} \qquad t_{n+1} \le s < t_n\,, \quad n = 0,\ldots,k\,, \tag{9.19}$$

where $t_{k+1} := 0$. Again, the Inspectorate's payoff is the negative of the Operator's one.

A fundamental role in solving Diamond's inspection game play the functions $h_n(x)$, $n = 1,2,\ldots$, fulfilling the system

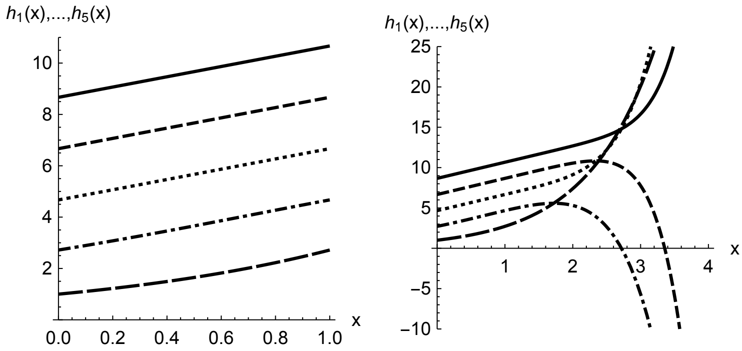
$$h_n'(x) = h_n(x) - h_{n-1}(x) \quad \text{for} \quad n = 1,2,\ldots \qquad \text{and} \qquad h_0(x) := 0\,, \tag{9.20}$$

of differential equations with the boundary conditions

$$h_1(0) := 1 \qquad \text{and} \qquad h_n(0) := h_{n-1}(1) \quad \text{for} \quad n = 2,3,\ldots\,. \tag{9.21}$$

This system has a unique solution, see Braun (1975), which, however, cannot be given explicitly for any number k of interim inspections. In Figure 9.1 the functions $h_1(x),\ldots,h_5(x)$ (from bottom to top) are illustrated for the range $[0,1]$ (left) and $[0,4]$ (right). It should be noted that the functions $h_n(x)$ are defined independently of k.

Figure 9.1 Graphical representation of the functions $h_1(x),\ldots,h_5(x)$.



The functions $h_n(x)$ are monotone increasing for any $x \in [0,1]$ and all $n = 1,2,\ldots$, which can be shown by induction: The function $h_1(x) = e^x$ is monotone increasing on $[0,1]$. even on $\mathbb{R}$. Let $h_n(x)$ be increasing on $[0,1]$ for a fixed $n \in \mathbb{N}$. Because $h_n(0) < h_n(1) = h_{n+1}(0)$, we get $h_{n+1}'(0) > 0$. Suppose there exist a $x_0 \in (0,1)$ with $h_{n+1}'(x_0) = 0$. Then (9.20) implies $h_{n+1}(x_0) = h_n(x_0)$. Let x_0 be the smallest number in $(0,1)$ with $h_{n+1}'(x_0) = 0$, i.e., $h_{n+1}'(x) > 0$ for any $x_0 \in [0,x_0)$. Because of the continuity of $h_{n+1}(x)$ we would get:

$$h_{n+1}(x_0) > h_{n+1}(0) = h_n(1) > h_n(x_0)\,,$$

which is a contradiction to $h_{n+1}(x_0) = h_n(x_0)$. Therefore, the functions $h_n(x)$ are monotone increasing for any $x \in [0,1]$ and all $n = 1,2,\ldots$.

The game theoretical solution of this inspection game is presented in Theorem 9.1, where we use a slightly different formulation of the optimal strategies and the optimal payoff compared to the original work of Diamond (1982).

Theorem 9.1. *Given the No-No inspection game on the reference time interval $[0, t_0]$ with k interim inspections. The sets of mixed strategies are given by the set of distribution functions on $\mathbb{R}$ resp. $\mathbb{R}^k$, and the payoff to the Operator by (9.4) using (9.19). Define the cut-off time point t^* by*[7]

$$t^* := t_0 \left(1 - \frac{h_1(0)}{h_k(1)}\right), \tag{9.22}$$

and the half-closed intervals $I_n, n = 1, \ldots, k$, by

$$I_n := \left[t_0 \left(1 - \frac{h_n(1)}{h_k(1)}\right), t_0 \left(1 - \frac{h_n(0)}{h_k(1)}\right)\right), \tag{9.23}$$

where the functions $h_n(x)$, $n = 1, 2, \ldots$ fulfil the system of differential equations (9.20) with the boundary conditions (9.21).

Then an optimal strategy of the Operator is given by the distribution function

$$P^*(s) = \begin{cases} 0 & \text{for} \quad s < 0 \\ \dfrac{1}{h_k(1)}\, h_{k-n+1}\left(1 - h_n^{-1}\left(\dfrac{h_k(1)}{t_0}(t_0 - s)\right)\right) & \text{for} \quad s \in I_n, n = 1, \ldots, k \\ 1 & \text{for} \quad s \geq t^* \end{cases} \tag{9.24}$$

and an optimal strategy for the Inspectorate by the interim inspection time points

$$t_n^*(U) = t_0 \left(1 - \frac{h_n(1-U)}{h_k(1)}\right), \qquad n = 1, \ldots, k, \tag{9.25}$$

where U is a random variable which is uniformly distributed on $[0, 1]$. Formally, the distribution function of the random vector $(T_k, \ldots, T_1)$ is, for any $(t_k, \ldots, t_1) \in \mathcal{T}_k$, given by

$$Q^*(t_k, \ldots, t_1) = \min\left\{1 - h_k^{-1}\left(h_k(1)\left(1 - \frac{t_k}{t_0}\right)\right), \ldots, 1 - h_1^{-1}\left(h_k(1)\left(1 - \frac{t_1}{t_0}\right)\right)\right\}.$$

The optimal payoff to the Operator is

$$Op_k^* := Op_k(P^*, Q^*) = t_0 - t^* = \frac{t_0}{h_k(1)}. \tag{9.26}$$

Proof. The proof is organized in three parts.

1. Because the functions $h_n(x)$, $n = 1, 2, \ldots$, are monotonely increasing, $h_n^{-1}(x)$ is monotonely increasing, and thus, $P^*(s)$ is easily seen to be monotonely increasing. To prove that $P^*(s)$ is a distribution function, the right-continuity of $P^*(s)$ has to be shown. Define for all $n = 1, \ldots, k$ – in contrast to the half-closed intervals I_n – the open intervals

$$\mathring{I}_n := \left(t_0 \left(1 - \frac{h_n(1)}{h_k(1)}\right), t_0 \left(1 - \frac{h_n(0)}{h_k(1)}\right)\right).$$

[7] Of course t^* depends on k. In order to avoid confusion with the interim inspection time points $t_k, \ldots, t_1$ which later on even occur with a star, we suppress the index k at t^* with the understanding that t^* is always used for a fixed k.

Because $h_n(x)$, $n = 1, 2, \ldots$, are differentiable function on $(0,1)$, it is clear that $P^*(s)$ is a differentiable function for all s from the set

$$A := \bigcup_{m=1}^{k} \mathring{I}_m \cup \left(t_0 \left(1 - \frac{1}{h_k(1)} \right), t_0 \right) ,$$

and hence a continuous functions for any $s \in A$. Furthermore, $P^*(s)$ is also continuous at all points of $[0, t_0)$ which are excluded from the set A. This can be seen as follows: Let s^* be given by

$$s^* = t_0 \left(1 - \frac{h_n(1)}{h_k(1)} \right)$$

for an n with $n = 1, \ldots, k-1$, i.e., $s^* \in I_n$. Then (9.24) implies

$$P^*(s^*) = \frac{1}{h_k(1)}\, h_{k-n+1}(0) \,.$$

We determine for a $s \in I_{n+1}$ the limit of $s \to s^*$. Because $s \in I_{n+1}$, it can be expressed as

$$s = t_0 \left(1 - \frac{h_{n+1}(1-u)}{h_k(1)} \right) ,$$

for $u \in [0,1)$. Then, $\lim_{u \to 1} s = s^*$, because of (9.21). Using (9.24), we get

$$P^*(s) = \frac{1}{h_k(1)}\, h_{k-n}(u) \to \frac{1}{h_k(1)}\, h_{k-n}(1) = \frac{1}{h_k(1)}\, h_{k-n+1}(0) = P^*(s^*) \,.$$

Thus, $P^*(s)$ is a continuous function on $(0, t_0)$, see also Figure 9.2, and with the monotonicity property shown above, a distribution function in $\mathbb{R}$.

2. Using the abbreviation $\mathbf{t} := (t_k, \ldots, t_1)$, we need to show that

$$Op_k(s, Q^*) \le Op_k(P^*, Q^*) \le Op_k(P^*, \mathbf{t}) \tag{9.27}$$

for any $s \in \mathcal{S}$ and any $\mathbf{t} \in \mathcal{T}_k$. To show the left hand inequality of (9.27), let $s \in \mathring{I}_n$ for an index $n = 1, \ldots, k$. Thus, s can be written as, see Figure 9.1,

$$s(y) = t_0 \left(1 - \frac{h_n(1-y)}{h_k(1)} \right) \qquad \text{for an } y \in (0,1) \,.$$

The only optimal interim inspection time point lying in the interval $\mathring{I}_n$ is $t_n^*(u)$. Therefore, the Operator's payoff is, using (9.19), $t_n^*(u) - s(y)$ if and only if $y < u < 1$ and it is – with $t_0^*(u) = 0$ – $t_{n-1}^*(u) - s(y)$ if and only if $0 < u \le y$, i.e., using $\mathbf{t}^*(u) = (t_k^*(u), \ldots, t_1^*(u))$,

$$Op_k(s(y), \mathbf{t}^*(u))$$

$$= \begin{cases} t_0 \left(1 - \dfrac{h_{n-1}(1-u)}{h_k(1)} \right) - t_0 \left(1 - \dfrac{h_n(1-y)}{h_k(1)} \right) & \text{for} \quad 0 < u \le y \\[2ex] t_0 \left(1 - \dfrac{h_n(1-u)}{h_k(1)} \right) - t_0 \left(1 - \dfrac{h_n(1-y)}{h_k(1)} \right) & \text{for} \quad y < u < 1 \end{cases} . \tag{9.28}$$

Therefore, we get for the payoff to the Operator

$$\frac{h_k(1)}{t_0}\, Op_k(s(y), \mathbf{t}^*(u))$$

$$= \int_0^y (h_n(1-y) - h_{n-1}(1-u))\, du + \int_y^1 (h_n(1-y) - h_n(1-u))\, du$$

$$= h_n(1-y) - \int_0^y h_{n-1}(1-u)\, du - \int_y^1 h_n(1-u)\, du\,. \tag{9.29}$$

Taking the first derivative with respect to y leads to

$$\frac{d}{dy}\left(\frac{h_k(1)}{t_0}\, Op_k(s(y), \mathbf{t}^*(u))\right) = -\, h_n'(1-y) - h_{n-1}(1-y) + h_n(1-y)\,. \tag{9.30}$$

Since the functions $h_n(x)$ fulfil the system of differential equations (9.20) for any $x \in (0,1)$, we obtain from (9.30)

$$\frac{d}{dy}\left(\frac{h_k(1)}{t_0}\, Op_k(s(y), \mathbf{t}^*(u))\right) = 0 \qquad \text{for any} \quad y \in (0,1)\,,$$

which implies that $Op_k(s(y), \mathbf{t}^*(u))$ is a constant function on the interval $\mathring{I}_n$. Using (9.29) one sees, that $Op_k(s(y), \mathbf{t}^*(u))$ is even a continuous function in y on the interval I_n, and thus, a constant function for any $s \in [0, t^*]$. The constant is for $n = 1$ and $y = 1$, using (9.28),

$$Op_k\,(t^*, \mathbf{t}^*(u)) = \frac{t_0}{h_k(1)}\,,$$

which leads finally, using (9.24), to

$$Op_k(P^*, Q^*) = \int_{[0,t_0)} Op_k(s, Q^*)\, dP^*(s) = \frac{t_0}{h_k(1)} = Op_k^*\,.$$

In case of $s \geq t^*$ the payoff to the Operator is

$$t_0 - s \leq \frac{t_0}{h_k(1)} = Op_k^*\,.$$

Therefore we have shown that $Op_k(s, Q^*) \leq Op_k^*$ for any $s \in [0, t_0)$, which is the left hand inequality of (9.27).

3. We now prove the right hand inequality of (9.27). Let $P(s)$ be any distribution function on $\mathbb{R}$ with an atom in $s = 0$ and $P(0^-) = 0$. Then we obtain with $t_{k+1} = 0$, (9.19) and the properties of the Lebesgue-Stieltjes-integrals, see the footnote 4 on p. 159,

$$Op_k(P, (t_k, \ldots, t_1)) = \int_{[0,t_0)} Op_k(s, (t_k, \ldots, t_1))\, d(s)$$

$$= \sum_{n=1}^{k} \int_{[t_{n+1}, t_n)} (t_n - s)\, dP(s) + \int_{[t_1, t_0)} (t_0 - s)\, dP(s)\,.$$

Because $P(s)$ is a distribution function, it is right-continuous in any point of $[0, t_0)$, we further obtain, using the result of footnote 5 on p. 159 and $P(0^-) = 0$,

$$\begin{aligned} Op_k(P, (t_k, \ldots, t_1)) &= t_k\, P(0^+) + \int_{(0,t_k)} (t_k - s)\, dP(s) + \sum_{n=0}^{k-1} \int_{[t_{n+1}, t_n)} (t_n - s)\, dP(s) \\ &= t_k\, P(0) + \sum_{n=0}^{k} t_n\, (P(t_n) - P(t_{n+1})) - \int_{(0,t_0)} s\, dP(s)\,. \end{aligned}$$

Then differentiation with respect to t_n leads for any $(t_k, \ldots, t_1) \in \mathcal{T}_k$ with $t_k, \ldots, t_1 \in A$ to

$$\begin{aligned} &\frac{\partial}{\partial t_n}\, Op_k(P^*, (t_k, \ldots, t_1)) \\ &= \begin{cases} P^*(t_k) - \dfrac{d}{dt_k}\, P^*(t_k)\, (t_{k-1} - t_k) & \text{for} \quad n = k \\ P^*(t_n) - P^*(t_{n+1}) - \dfrac{d}{dt_n}\, P^*(t_n)\, (t_{n-1} - t_n) & \text{for} \quad n = 1, \ldots, k-1 \end{cases} \,. \end{aligned} \tag{9.31}$$

The candidates $(t_k, \ldots, t_1) \in \mathcal{T}_k$ with $t_k, \ldots, t_1 \in A$ for being a minimum of the function $Op_k(P^*, (t_k, \ldots, t_1))$ have to fulfil the condition

$$\frac{\partial}{\partial t_n}\, Op_k(P^*, (t_k, \ldots, t_1)) = 0 \qquad \text{for all} \qquad n = 1, \ldots, k\,. \tag{9.32}$$

We now show that the vector $(t_k^*(u), \ldots, t_1^*(u))$, where $t_n^*(u)$ is defined by (9.25), is for any $u \in (0,1)$ a solution of (9.32), i.e., a candidate for a minimum. Let $n = 1, \ldots, k$ be an arbitrary but fixed number. Because $t_n \in \mathring{I}_n$ we get, using (9.24),

$$\frac{d}{dt_n}\, P^*(t_n) = \frac{1}{t_0}\, \frac{h'_{k-n+1}\left(1 - h_n^{-1}\left(\dfrac{h_k(1)}{t_0}\,(t_0 - t_n)\right)\right)}{h'_n\left(h_n^{-1}\left(\dfrac{h_k(1)}{t_0}\,(t_0 - t_n)\right)\right)}\,.$$

Therefore, (9.31) simplifies for $n = k$ and for any $u \in (0,1)$, using (9.20) and (9.25), to

$$\begin{aligned} &P^*(t_k^*(u)) - \frac{d}{dt_k}\, P^*(t_k)\Big|_{t_k = t_k^*(u)} (t_{k-1}^*(u) - t_k^*(u)) \\ &= \frac{h_1(u)}{h_k(1)} - \frac{1}{t_0}\, \frac{h_1'(u)}{h_k'(1-u)}\, \frac{t_0}{h_k(1)} (h_k(1-u) - h_{k-1}(1-u)) \\ &= 0\,, \end{aligned} \tag{9.33}$$

and the second equation of (9.31) to

$$\begin{aligned} &P^*(t_n^*(u)) - P^*(t_{n+1}^*(u)) - \frac{d}{dt_n}\, P^*(t_n)\Big|_{t_n = t_n^*(u)} (t_{n-1}^*(u) - t_n^*(u)) \\ &= \frac{h_{k-n+1}(u)}{h_k(1)} - \frac{h_{k-(n+1)+1}(u)}{h_k(1)} - \frac{1}{t_0}\, \frac{h'_{k-n+1}(u)}{h'_n(1-u)}\, \frac{t_0}{h_k(1)} (h_n(1-u) - h_{n-1}(1-u)) \\ &= 0\,. \end{aligned} \tag{9.34}$$

Thus, (9.33) and (9.34) prove that all vectors $(t_k^*(u), \ldots, t_1^*(u))|_{u\in(0,1)}$ are candidates for the minimum.

Now the proof proceeds in two steps. First, because for any $u \in (0,1)$ the time points $t_n^*(u)$ lie in the disjoint intervals $\mathring{I}_n$,$n = 1, \ldots, k$, it needs to be proven that any other vector $(t_k, \ldots, t_1) \in \mathcal{T}_k$ with $t_k, \ldots, t_1 \in A$, does not fulfil (9.32). Thus, cases such as no interim inspection is performed in some of the intervals $\mathring{I}_n$, e.g., $t_k, t_{k-1} \in \mathring{I}_{k-1}$, i.e., no interim inspection is performed in $\mathring{I}_k$), or at least two interim inspections are placed in the same interval, e.g., $t_k, t_{k-1} \in \mathring{I}_k$, have to be considered. Second, it is proven that even in case that at least one of the t_n is equal to a point at which $P^*(s)$ is not differentiable, (9.32) cannot be fulfilled. The proofs of these two steps are by no means trivial. Since, however, their technical details are far from the subject of this monograph, we refer the reader to the original work of Diamond (1982). □

Let us discuss the results of Theorem 9.1. First, the solution of Diamond's inspection game is for $k \geq 2$ considerably more complex compared to the one with $k = 1$ interim inspection, see Lemma 9.1, and has the following features; As before, the Operator will not start the illegal activity after time point t^* as given by (9.22), again due to the fact that detection is guaranteed to occur at the end of the interval and the Operator will not wish to wait too long before starting the illegal activity. The interim inspections take place in disjoint intervals. The interim inspection times are random, but it is a randomization over a one-parameter family of pure strategies where the interim inspection time points are fully correlated. One may understand this important property if one looks again at the No-No inspection game with $k = 2$ interim inspections analysed in Section 3.2: There also just one random experiment is performed because the time points (j_2, j_1) are realized according to $\mathbf{q} = (q_{(1,2)}, \ldots, q_{(N-1,N)})$. The Operator's optimal distribution function is piecewise defined on the k time intervals I_n, $n = 1, \ldots, k$, and has the atom $P^*(0) = 1/h_k(1)$ at the beginning of the reference time interval. For the optimal expected detection time we get from (9.26)

$$t_0 - Op_k^* = t^* \,,$$

which means that, like in Lemma 9.1, it is the time between the last possible interim inspection time point and the end of the reference time interval.

Second, in case of $k = 2$ or $k = 3$ interim inspections, the system of differential equations (9.20) and (9.21) has the solution

$$h_1(x) = e^x \,, \quad h_2(x) = e^x(e - x) \quad \text{and} \quad h_3(x) = \frac{1}{2}\, e^x(-2\,e + 2\,e^2 - 2\,e\,x + x^2) \,.$$

Therefore, we get for the atoms $P^*(0)$

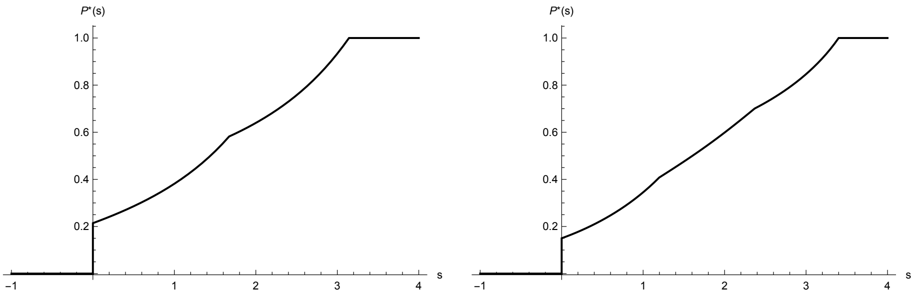
$$P^*(0) = \begin{cases} \dfrac{1}{e\,(e-1)} \approx 0.214 & \text{for} \quad k = 2 \\ \dfrac{2}{e\,(2\,e^2 - 4\,e + 1)} \approx 0.150 & \text{for} \quad k = 3 \end{cases}$$

as well as the optimal expected detection times

$$\begin{array}{ll} \dfrac{t_0}{h_2(1)} = \dfrac{t_0}{e\,(e-1)} \approx 0.214\,t_0 & \text{for} \quad k = 2 \\ \dfrac{t_0}{h_3(1)} = \dfrac{2\,t_0}{e\,(-4\,e + 2\,e^2 + 1)} \approx 0.150\,t_0 & \text{for} \quad k = 3 \end{array} .$$

The Operator's optimal distribution function is depicted in Figure 9.2 for $t_0 = 4$ and $k = 2, 3$ interim inspections. It can be seen that it is a piecewise differentiable function with an atom at $s = 0$.

Figure 9.2 Optimal strategy of the Operator for $t_0 = 4$ and $k = 2$ (left) and $k = 3$ (right) interim inspections.



Third, Diamond (1982) has shown that for large k the optimal interim inspection time points are uniformly distributed in their intervals and that the optimal expected detection time Op_k^* approaches $t_0/(2\,k + 2/3)$ rapidly with exponentially small error. It should be noted that Diamond also demonstrates computational solution methods for non-linear loss functions.

Fourth, the optimal expected interim inspection time points $\mathbb{E}_{\mathbf{q}^*}(T_n)$, $n = 1, \ldots, k$, can be determined as follows: Using (9.20), (9.25), and (9.26), we get for all $n = 1, \ldots, k$

$$\begin{aligned} \mathbb{E}_{Q^*}(T_n) &= \int_0^1 t_0 \left(1 - \frac{h_n(1-u)}{h_k(1)}\right) du = t_0 - Op_k^* \int_0^1 h_n(1-u)\, du \\ &= \mathbb{E}_{Q^*}(T_{n-1}) - Op_k^* \int_0^1 h_n'(1-u)\, du \\ &= \mathbb{E}_{Q^*}(T_{n-1}) - Op_k^* \left(h_n(1) - h_n(0)\right), \end{aligned} \tag{9.35}$$

where $T_0 := t_0$. Therefore, we obtain, using $h_1(0) = 1$ from (9.21), and (9.35),

$$\mathbb{E}_{Q^*}(T_n) = t_0 - Op_k^* \left(h_n(1) - 1\right), \tag{9.36}$$

which reduces for $k = 1$ interim inspection again to (9.16) and for $n = k$ to

$$\mathbb{E}_{Q^*}(T_k) = Op_k^* \,.$$

Note that due to the complicated structure of $P^*(s)$, $\mathbb{E}_{P^*}(S)$ cannot be given explicitly for $k \geq 2$ interim inspections. Thus, as a generalization of (9.15) we formulate, using (9.22) and (9.26), the conjecture:

$$\mathbb{E}_{P^*}(S) = t^* - Op_k^* = t_0 \left(1 - \frac{h_1(0)}{h_k(1)}\right) - \frac{t_0}{h_k(1)} = t_0 \left(1 - \frac{2}{h_k(1)}\right). \tag{9.37}$$

Fifth, it should be mentioned that in practice it may be difficult to plan and perform interim inspections within the continuous time model, since this may create too many problems both for the Operator and the Inspectorate. A practical solution could be to take the nearest possible time point to the optimal interim inspection time point(s).

Finally, if the Inspectorate deterministically inspects at time points

$$t_n = \frac{k-n+1}{k+1}\, t_0\,, \qquad n = 1, \ldots, k\,,$$

the illegal activity is detected at most after time $t_0/(k+1)$ has elapsed. This is indeed an optimal inspection strategy if the Operator can *observe* the interim inspections, i.e., in the Se-No and the Se-Se inspection game; see (10.26) for $\beta = 0$ and (12.50) for $\alpha = \beta = 0$ and using Lemma 22.1. In that case, the start of the illegal activity s is uniformly chosen from the set $\{0, t_k, \ldots, t_1\}$, conditional upon the Inspectorate's action. Then we get by (9.19)

$$\frac{1}{k+1}\,(t_k - 0) + \frac{1}{k+1}\,(t_{k-1} - t_k) + \ldots + \frac{1}{k+1}\,(t_0 - t_1) = \frac{t_0}{k+1}$$

for the expected detection time.

9.3 Any number of interim inspections; errors of the second kind

Let us now assume that we deal with an inspection problem where an illegal activity preceding an interim inspection is detected with probability $1-\beta$ as it is the case if this interim inspection uses an Attribute Sampling procedure; see Thyregod (1988).

The inspection game analysed in this section is based on the assumptions (iv'), (vii') for any number k of interim inspections, and (ix') made at the beginning of Section 9.1, and on the specification:

(v') During an interim inspection the Inspectorate may commit an error of the second kind with probability $\beta \geq 0$, i.e., the illegal activity, see assumption (iv'), is not detected during the next interim inspection with probability β. Note that if there is no interim inspection left, then it is detected with certainty at the final PIV; see assumption (iii). This non-detection probability is the same for all k interim inspections.

The remaining assumptions of Chapter 8 hold throughout this section. Note that the discrete time variant of this inspection game is treated in Section 6.1.

Let us start with $k = 1$ interim inspection. Again, the Operator starts the illegal activity at some time point s from the reference time interval $[0, t_0)$ and the Inspectorate performs its interim inspection at time point $t_1 \in (0, t_0)$. Thus, the sets of pure strategies of both players are again given by (9.1).

The payoff to the Operator, i.e., the detection time, is given by the payoff kernel

$$Op_1(s, t_1) = \begin{cases} (1-\beta)\,(t_1 - s) + \beta\,(t_0 - s) & \text{for} \quad 0 \leq s < t_1 < t_0 \\ t_0 - s & \text{for} \quad 0 < t_1 \leq s < t_0 \end{cases}\,, \tag{9.38}$$

because in case the illegal activity is started before the interim inspection takes place, i.e., $0 \leq s < t_1$, the illegal activity is detected at time point t_1 with probability $1-\beta$ or, if it is not detected at t_1 with probability β, it is detected at t_0, the final PIV, with certainty; see assumption (x) of Chapter 8. If the illegal activity is started after the interim inspection takes place, i.e., $t_1 \leq s < t_0$, then it is detected at time point t_0, again with certainty. Note that the payoff kernel (9.38) can be seen as a generalization of the payoff (6.3).

The game with payoff kernel (9.38) does not possess a saddle point in pure strategies; see the case $\beta = 0$ on p. 157. Therefore, mixed strategies and the Operator's payoff according to (9.4) have to be considered again.

The game theoretical solution of this inspection game, see Sohrweide (2002) and Avenhaus et al. (2003), is presented in Lemma 9.2, which is the continuous time version of the discrete time No-No inspection game treated in Theorem 6.1.

Lemma 9.2. *Given the No-No inspection game on the reference time interval* $[0, t_0]$ *with* $k = 1$ *interim inspection, and with errors of the second kind. The sets of mixed strategies are given by the set of distribution functions on* $\mathbb{R}$*, and the payoff to the Operator by (9.4) using (9.38). Define the cut-off time point* t^* *by*

$$t^* := t_0 \left(1 - \frac{1}{e^{1-\beta}}\right) . \tag{9.39}$$

Then an optimal strategy of the Operator is given by the distribution function

$$P^*(s) = \begin{cases} 0 & \textit{for} \quad s < 0 \\ \dfrac{t_0 - t^*}{t_0 - s} & \textit{for} \quad s \in [0, t^*) \\ 1 & \textit{for} \quad s \geq t^* \end{cases} , \tag{9.40}$$

and an optimal strategy for the Inspectorate by the distribution function

$$Q^*(t_1) = \begin{cases} 0 & \textit{for} \quad t_1 < 0 \\ \dfrac{1}{1-\beta} \ln \left[\dfrac{t_0}{t_0 - t_1}\right] & \textit{for} \quad t_1 \in [0, t^*) \\ 1 & \textit{for} \quad t_1 \geq t^* \end{cases} . \tag{9.41}$$

The optimal payoff to the Operator is

$$Op_1^* := Op_1(P^*, Q^*) = t_0 - t^* = \frac{t_0}{e^{1-\beta}} . \tag{9.42}$$

Proof. The proof is a straightforward generalization of the proof of Lemma 9.1 resp. a special case of the proof of Lemma 9.3, therefore, it is omitted here. □

Let us comment the results of Lemma 9.2: First, the first and the third comment – with $h_1(x) = e^{(1-\beta)\,x}$ – given after Lemma 9.1 hold here as well. The fourth comment has to be modified: From (9.15) we obtain for the optimal expected time point for the start of the illegal activity $\mathbb{E}_{P^*}(S)$, using (9.40) and (9.42),

$$\begin{aligned} \mathbb{E}_{P^*}(S) &= \int_{[0,t_0)} s \, dP^*(s) = t^* + Op_1^* \ln \left[1 - \frac{t^*}{t_0}\right] \\ &= t^* - (1-\beta)\, Op_1^* = t^* \,(2-\beta) - t_0 \,(1-\beta) , \end{aligned} \tag{9.43}$$

which for $\beta = 0$ coincides with (9.15). For the optimal expected interim inspection time point $\mathbb{E}_{Q^*}(T_1)$ we get, using (9.39), (9.41) and (9.42),

$$\mathbb{E}_{Q^*}(T_1) = \int_{(0,t_0)} t_1 \, dQ^*(t_1) = \frac{1}{1-\beta} \int_0^{t^*} \frac{t_1}{t_0 - t_1} \, dt_1$$

$$= t_0 - \frac{t^*}{1-\beta} = \frac{1}{1-\beta} \left(Op_1^* - t_0 \, \beta \right) , \tag{9.44}$$

which coincides for $\beta = 0$ with (9.16). Note that (9.44) can also be obtained in analogy to (9.16) using $h_1(x) = e^{(1-\beta)\,x}$. The optimal expected time point for the start of the illegal activity is, using (9.43) and (9.44), given by

$$\mathbb{E}_{P^*}(S) + (1-\beta) \, \mathbb{E}_{Q^*}(T_1) = t^* - t_0 \, \beta \, ,$$

which simplifies for $\beta = 0$ to (9.17). Also note that $\mathbb{E}_{Q^*}(T_1)$ as given by (9.44) and (6.14) coincide if we identify $N + 1$ with t_0.

Second, note that in this game the structure of the optimal strategies and the optimal payoff to the Operator, is not changed compared to the game with $\beta = 0$ in Lemma 9.1. If in addition errors of the first kind are considered, the equilibrium payoff to the Inspectorate will be structurally different from that of the Operator; see Lemma 9.3 (i).

Let us now turn to the case of $k > 1$ interim inspections. The Operator's set of pure strategies is again given by $\mathcal{S}$ in (9.1), and the Inspectorate's one by (9.18). The payoff to the Operator, i.e., the detection time, however, has to be amended accordingly. For $t_{n+1} \leq s < t_n$, $n = 0, \ldots, k$, it is given by ($\sum_{m=1}^{0} \ldots := 0$)

$$Op_k(s, (t_k, \ldots, t_1)) := (1-\beta) \, (t_n - s) + \beta \, (1-\beta) \, (t_{n-1} - s) + \ldots + \beta^n \, (t_0 - s)$$

$$= (1-\beta) \sum_{m=1}^{n} \beta^{n-m} \, (t_m - s) + \beta^n \, (t_0 - s) \, , \tag{9.45}$$

where $t_{k+1} = 0$. Again, the system of differential equations (9.20) has to be modified: Now, the functions $h_n(x)$, $n = 1, 2, \ldots$, have to satisfy the system

$$h_n'(x) = (1-\beta) \, h_n(x) - (1-\beta)^2 \sum_{m=1}^{n-1} \beta^{n-(m+1)} \, h_n(x) \tag{9.46}$$

of differential equations with $\sum_{m=1}^{0} \ldots := 0$, and with the boundary conditions

$$h_1(0) := 1 \qquad \text{and} \qquad h_n(0) := h_{n-1}(1) \quad \text{for} \quad n = 2, 3, \ldots \, . \tag{9.47}$$

This system has also a unique solution; see Braun (1975). It can be shown – analogously to the proof on p. 162 – that the functions $h_n(x)$ are monotone increasing for any $x \in [0, 1]$ and all $n = 1, 2, \ldots$.

Because we do not give a proof of the solution of this inspection game for any number k of interim inspections, the result is formulated as a conjecture for which we have a well-considered and strong evidence:

Conjecture 9.1. *Given the No-No inspection game on the reference time interval* $[0, t_0]$ *with* k *interim inspections, and with errors of the second kind. The sets of mixed strategies are given by the set of distribution functions on* $\mathbb{R}$ *resp.* $\mathbb{R}^k$*, and the payoff to the Operator by (9.4) using (9.45).*

Defining the cut-off time point t^* *and the half-closed intervals* $I_n, n = 1, \ldots, k$*, by (9.22) resp. (9.23), where the functions* $h_n(x)$*,* $n = 1, 2, \ldots$ *fulfil the system (9.46) of differential equations with the boundary conditions (9.47).*

Then an optimal strategy of the Operator is given by the distribution function (9.24), an optimal strategy for the Inspectorate by the interim inspection time points (9.25) with a uniformly distributed random variable U *on* $[0, 1]$*, and the optimal payoff to the Operator is given by (9.26).*

As we see the only difference to the case $\beta = 0$ treated in Theorem 9.1 is given by the modified differential equations (9.46) for the functions $h_n(x)$, $n = 1, 2, \ldots$. Whereas the necessary conditions for an optimum given by (9.32) can be shown to be fulfilled, the remaining part of the proof, especially the one for which we referred to Diamond's original work, has to be reproduced carefully.

Let us comment the conjecture: First, the first comment to Theorem 9.1 holds here as well.

Second, the first two differential equations of (9.46) are given by

$$h_1'(x) = (1 - \beta)\, h_1(x) \qquad \text{and} \qquad h_2'(x) = (1 - \beta)\, h_2(x) - (1 - \beta)^2\, h_1(x) \tag{9.48}$$

and thus we get, using (9.47) for $n = 2$,

$$h_1(x) = e^{(1-\beta)\,x} \qquad \text{and} \qquad h_2(x) = e^{(1-\beta)\,x} \left(e^{1-\beta} - x\,(1 - \beta)^2 \right) .$$

Therefore, we obtain for the optimal expected detection time in case of $k = 2$ interim inspections

$$Op_2^* = \frac{t_0}{h_2(1)} = \frac{t_0}{e^{1-\beta}} \frac{1}{e^{1-\beta} - (1 - \beta)^2} .$$

Note that $\lim_{\beta \to 1} Op_1^* = \lim_{\beta \to 1} Op_2^* = t_0$, but $Op_2^* < Op_1^*$ for $\beta < 1$, as one would have expected.

It can be conjectured, that (9.43) and (9.44) can be generalized accordingly to the case of $k > 1$ interim inspections, i.e., that

$$\mathbb{E}_{Q^*}(T_n) = t_0 - Op_k^*\,(h_n(1) - 1) \qquad \text{and} \qquad \mathbb{E}_{P^*}(S) = t^* - Op_k^* = t_0 \left(1 - \frac{2}{h_k(1)} \right)$$

holds; see (9.36) and (9.37).

9.4 One interim inspection; errors of the first and second kind

Sampling procedures with errors of the first and second kind have already been mentioned in Section 7.4. Since, however, we deal here the first time explicitly with these problems, we discuss them now and in Section 9.5 in more detail. The content of this section is based on the work by Sohrweide (2002), Avenhaus et al. (2003) and Krieger (2011).

Let us assume that in performing the only interim inspection a Variable Sampling procedure is used, see Thyregod (1988), this means that an error of the second kind may happen, i.e., that an illegal activity preceding the interim inspection is detected with probability $1 - \beta$ and furthermore, that an error of the first kind, i.e., a false accusation, may happen with probability α. According to standard statistical practice we assume that the value of the false alarm probability α is fixed a priori but we will come back to this point in Section 9.5. Also, we assume that the test procedure used for the interim inspection is *unbiased*, see Rohatgi (1976), which means that we assume

$$\alpha + \beta < 1\,. \tag{9.49}$$

In words: The false alarm probability α has to be smaller then the detection probability $1-\beta(\alpha)$, i.e., the probability of rejecting H_0, when false, has to be not less than the probability of rejecting H_0, when true. In Section 9.5 additional assumptions, such as that the non-detection probability β is a monotone decreasing function of the false alarm probability α, will be made.

The inspection game analysed in this section is based on the following specifications:

(iv') The Operator may start at most once an illegal activity during the reference time interval $[0, t_0]$ in the only facility under consideration.

(v') During an interim inspection the Inspectorate may commit an error of the first and second kind with probabilities α and β. If the interim inspection is performed before the start of the illegal activity, then only an error of the first kind (false alarm) may occur. If the interim inspection is performed after the start of the illegal activity, the only an error of the second kind (non-detection) may occur. The "game" continues after an error of the first kind.

(vii') The Operator decides at the beginning of the reference time interval, i.e., at time point 0, whether to behave legally, see assumption (iv'), or when to start the illegal activity.

The Inspectorate decides at the beginning of the reference time interval when to perform its interim inspection.

The remaining assumptions of Chapter 8 hold throughout this section.

Note that, as mentioned in Section 7.4, false alarms have negative consequences both for the Operator and the Inspectorate, we can no longer model the inspection problem as a zero-sum game, and payoff parameters have to be introduced. Thus, the payoffs to the two players (Operator, Inspectorate) are given by (8.1) and (8.2) for $k = 1$ interim inspection.

Again, the Operator starts the illegal activity at some time point s from the reference time interval $[0, t_0)$ and the Inspectorate performs its interim inspection at time point $t_1 \in (0, t_0)$. Thus, the sets of pure strategies of both players are again given by (9.1).

The Operator's payoff is, using (8.1), given by the payoff kernel

$$Op_1(s, t_1) := \begin{cases} d\,[(1-\beta)\,(t_1 - s) + \beta\,(t_0 - s)] - b & : \quad 0 \le s < t_1 < t_0 \\ d\,(t_0 - s) - f\,\alpha - b & : \quad 0 < t_1 \le s < t_0 \\ -f\,\alpha & : \quad \text{legal behaviour} \end{cases}, \tag{9.50}$$

because in case the illegal activity is started before the interim inspection takes place, i.e., $0 \le s < t_1$, the illegal activity is detected at time point t_1 with probability $1 - \beta$ or, if it is not detected at t_1 with probability β, it is detected at t_0 with certainty. In case the illegal activity is started after the interim inspection takes place, i.e., $t_1 \leq s < t_0$, it is detected at time point t_0. In this case, the interim inspection may cause a false alarm with probability α. Of course, this may also happen if the Operator behaves legally. Note that $Op_1(s, t_1)$ can no longer be interpreted as detection time.

The Inspectorate's payoff is accordingly given by

$$In_1(s,t_1) := \begin{cases} -a\,[(1-\beta)\,(t_1-s)+\beta\,(t_0-s)] & : \quad 0 \leq s < t_1 < t_0 \\ -a\,(t_0-s) - g\,\alpha & : \quad 0 < t_1 \leq s < t_0 \\ -g\,\alpha & : \quad \text{legal behaviour} \end{cases} . \tag{9.51}$$

Because the generalized Diamond inspection game with $\alpha = 0$ and $\beta > 0$ discussed in the last section does not possess an optimal strategy in pure strategies, it is not surprising that the game with payoff kernels (9.50) and (9.51) does not possess a Nash equilibrium in pure strategies. Therefore, mixed strategies and (expected) payoffs according to (9.4) have to be introduced again:

$$\begin{aligned} Op_1(P,Q) &:= \int_{[0,t_0)} \int_{(0,t_0)} Op_1(s,t_1)\,dQ(t_1)\,dP(s) \\ In_1(P,Q) &:= \int_{[0,t_0)} \int_{(0,t_0)} In_1(s,t_1)\,dQ(t_1)\,dP(s)\,. \end{aligned} \tag{9.52}$$

The game theoretical solution of this inspection game, see Sohrweide (2002) and Avenhaus et al. (2003), is presented in

Lemma 9.3. *Given the No-No inspection game on the reference time interval* $[0, t_0]$ *with* $k = 1$ *interim inspection, errors of the first and second kind, and an unbiased test procedure. The sets of mixed strategies are given by the set of distribution functions on* $\mathbb{R}$*, and the payoffs to both players by (9.52) using (9.50) and (9.51). Define the cut-off time point* t^* *by*

$$t^* := \left(t_0 - \frac{f}{d}\,\frac{\alpha}{1-\beta}\right)\left(1 - \frac{1}{e^{1-\beta}}\right). \tag{9.53}$$

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op_1^* := Op_1(P^*, Q^*)$ *and* $In_1^* := In_1(P^*, Q^*)$*:*

(i) For

$$t_0 - t^* > \frac{b}{d} \tag{9.54}$$

an equilibrium strategy of the Operator is given by the distribution function

$$P^*(s) = \begin{cases} 0 & \textit{for} \quad s < 0 \\ \dfrac{t_0 - t^* + \dfrac{g}{a}\,\dfrac{\alpha}{1-\beta}}{t_0 - s + \dfrac{g}{a}\,\dfrac{\alpha}{1-\beta}} & \textit{for} \quad s \in [0, t^*) \\ 1 & \textit{for} \quad s \geq t^* \end{cases} , \tag{9.55}$$

and an equilibrium strategy for the Inspectorate by the distribution function

$$Q^*(t_1) = \begin{cases} 0 & \textit{for} \quad t_1 < 0 \\ \dfrac{1}{1-\beta} \ln \left[\dfrac{t_0 - \dfrac{f}{d}\dfrac{\alpha}{1-\beta}}{t_0 - t_1 - \dfrac{f}{d}\dfrac{\alpha}{1-\beta}} \right] & \textit{for} \quad t_1 \in [0, t^*) \\ 1 & \textit{for} \quad t_1 \geq t^* \end{cases} . \tag{9.56}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_1^* = d\,(t_0 - t^*) - f\,\alpha - b \qquad \textit{and}$$

$$In_1^* = -a \left[\beta\,(t_0 - t^*) - \left(t_0 - t^* + \frac{g}{a}\frac{\alpha}{1-\beta} \right) \ln \left[1 - \frac{t^*}{t_0 + \dfrac{g}{a}\dfrac{\alpha}{1-\beta}} \right] \right] . \tag{9.57}$$

(ii) For

$$t_0 - t^* < \frac{b}{d} \tag{9.58}$$

the Operator behaves legally. The distribution function $Q^*(t_1)$ given by (9.56) is an equilibrium strategy of the Inspectorate.

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_1^* = -f\,\alpha \qquad \textit{and} \qquad In_1^* = -g\,\alpha\,. \tag{9.59}$$

Proof. Ad (i): It can be seen that $P^*(s)$ given by (9.55) and $Q^*(t_1)$ given by (9.57) are distribution functions on $\mathbb{R}$.

Since, as mentioned, we deal here with a non-zero-sum game, we have to show, generalizing (9.5) and (9.6), that the Nash conditions

$$Op_1(s, Q^*) \leq Op_1(P^*, Q^*) \qquad \text{and} \qquad In_1(P^*, t_1) \leq In_1(P^*, Q^*) \tag{9.60}$$

are fulfilled for any $s \in \mathcal{S}$ and any $t_1 \in \mathcal{T}_1$.

Explicitly, $Op_1(s, Q^*)$ is given by

$$Op_1(s, Q^*) = \int_0^s \left(d\,(t_0 - s) - f\,\alpha - b \right) q^*(t_1)\, dt_1$$

$$+ \int_s^{t^*} \left(d\,[(t_1 - s)\,(1-\beta) + (t_0 - s)\,\beta] - b \right) q^*(t_1)\, dt_1\,,$$

whereas $In_1(P^*, t_1)$ is given by

$$In_1(P^*, t_1) = \int_{[0,t_0)} In_1(s, t_1)\, dP^*(s)$$

$$= (-a) \left\{ \left[-(t_0 - t_1)\,(1-\beta) - \frac{g}{a}\,\alpha \right] P^*(t_1) + \left(t_0 + \frac{g}{a}\,\alpha \right) P^*(t_0) - \int_{(0,t_0)} s\, dP^*(s) \right\} .$$

With these two expressions for $Op_1(s, Q^*)$ and $In_1(P^*, t_1)$ the proof can be carried through analogously to that of Lemma 9.1. Especially we get for any $s \in [0, t^*]$

$$Op_1(s, Q^*) = d\,(t_0 - t^*) - f\,\alpha - b\,. \tag{9.61}$$

Inequality (9.54) assures that

$$Op_1^* > -f\,\alpha = Op_1(\text{legal behaviour}, t_1)$$

for any $t_1 \in (0, t_0)$.

Ad (ii): (9.58) implies for all $s > t^*$

$$d\,(t_0 - s) - f\,\alpha - b < d\,(t_0 - t^*) - f\,\alpha - b < -f\,\alpha\,.$$

Thus, we get, using (9.58), (9.59) and (9.61),

$$Op_1(s, Q^*) = \begin{cases} d\,(t_0 - t^*) - f\,\alpha - b & : \quad 0 \leq s \leq t^* \\ d\,(t_0 - s) - f\,\alpha - b & : \quad t^* < s < t_0 \end{cases} < -f\,\alpha = Op_1^*\,,$$

i.e., the left hand inequality of (9.60) is fulfilled for any $s \in \mathcal{S}$. The right hand inequality of (9.60) is fulfilled as equality. □

Let us comment the results of Lemma 9.3: First, the first and the third comment – with an appropriately defined function $h_1(x)$, see (9.68) – given after Lemma 9.1 hold here as well.

Second, in analogy to the derivations in (9.15), we get, using (9.55) and (9.57), for the expected time point for the start of the illegal activity in the equilibrium

$$\begin{aligned}
\mathbb{E}_{P^*}(S) &= t^* - \left(t_0 - t^* + \frac{g}{a}\,\frac{\alpha}{1-\beta}\right) \int_0^{t^*} \frac{1}{t_0 - s + \dfrac{g}{a}\,\dfrac{\alpha}{1-\beta}}\, ds \\
&= t^* + \left(t_0 - t^* + \frac{g}{a}\,\frac{\alpha}{1-\beta}\right) \ln\left[1 - \frac{t^*}{t_0 + \dfrac{g}{a}\,\dfrac{\alpha}{1-\beta}}\right] \\
&= t^* + \frac{In_1^*}{a} + \beta\,(t_0 - t^*)\,,
\end{aligned}$$

which coincides for $\alpha = 0$ with (9.43). The expected interim inspection time point in the equilibrium is, using (9.53), (9.56) and (9.57), given by

$$\begin{aligned}
\mathbb{E}_{Q^*}(T_1) &= \int_{(0,t_0)} t_1\, dQ^*(t_1) = \frac{1}{1-\beta} \int_0^{t^*} \frac{t_1}{t_0 - t_1 - \dfrac{f}{d}\,\dfrac{\alpha}{1-\beta}}\, dt_1 \\
&= t_0 - \frac{t^*}{1-\beta} - \frac{f}{d}\,\frac{\alpha}{1-\beta} = \frac{1}{1-\beta}\left(\frac{Op_1^* + b}{d} - t_0\,\beta\right)\,,
\end{aligned} \tag{9.62}$$

which for $d = 1$, $b = 0$ and $\alpha = 0$ coincides with (9.44). Because t^* according to (9.53) depends on f but not on g, $\mathbb{E}_{Q^*}(T_1)$ by (9.62) is a much simpler generalization of that for $\alpha = 0$, see (9.44), than $\mathbb{E}_{P^*}(S)$ by (9.62) compared to (9.43).

Third, taking into account errors of the first kind changes the structure of the solution fundamentally: The Operator's equilibrium strategy depends on the Operator's own payoff parameters. Also, the Inspectorate's equilibrium payoff is totally different from that of the Operator which is not the case if only errors of the second kind are taken into account: Using the payoff structure (8.1) we get for an Attribute Sampling procedures, i.e., $\alpha = 0$, according to Lemma 9.2

$$Op_1^* = d\,t_0\,e^{-(1-\beta)} - b \qquad \text{and} \qquad In_1^* = -\,e^{-(1-\beta)}\,a\,t_0\,.$$

Fourth, because $Q^*(t_1)$ is also an equilibrium strategy of the Inspectorate under condition (9.58), it is a robust equilibrium strategy in the sense that the Inspectorate can just play $Q^*(t_1)$ and does not need to check whether (9.54) or (9.58) is fulfilled; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

In the following Lemma it is shown under which conditions even a pure equilibrium strategy of the Inspectorate, in case of legal behaviour of the Operator, exists.

Corollary 9.1. *Given the No-No inspection game on the reference time interval $[0, t_0]$ with $k = 1$ interim inspection, errors of the first and second kind, and an unbiased test procedure analysed in Lemma 9.3.*

If

$$\frac{1}{2-\beta}\left(t_0 + \frac{f}{d}\,\alpha\right) < \frac{b}{d} \tag{9.63}$$

then the Operator behaves legally and the Inspectorate has a pure equilibrium strategy t_1^* with

$$t_0 - \frac{b}{d} \le t_1^* \le \frac{1}{1-\beta}\left(\frac{b}{d} - t_0\,\beta - \frac{f}{d}\,\alpha\right). \tag{9.64}$$

The equilibrium payoffs to the Operator and to the Inspectorate are given by (9.59).

Proof. We first prove that (9.63) yields

$$t_0 - t^* < \frac{b}{d}, \tag{9.65}$$

where t^* is given by (9.53), i.e., the Operator behaves legally because of (9.58). Suppose (9.65) does not hold. Then we get from (9.53) and (9.63)

$$\frac{1}{2-\beta}\left(t_0 + \frac{f}{d}\,\alpha\right) < t_0 - t^* = t_0 - \left(t_0 - \frac{f}{d}\,\frac{\alpha}{1-\beta}\right)\left(1 - \frac{1}{e^{1-\beta}}\right),$$

which is equivalent to

$$t_0\left(\frac{1}{2-\beta} - \frac{1}{e^{1-\beta}}\right) < \frac{f}{d}\,\alpha\left(\frac{1-e^{-(1-\beta)}}{1-\beta} - \frac{1}{2-\beta}\right). \tag{9.66}$$

Because the expression in brackets on the right hand side of (9.66) is larger than zero for $0 \le \beta < 1$, and because of $f < d\,t_0$, see (8.2) for $k = 1$, and $\alpha + \beta < 1$, we get from (9.66)

$$d\,t_0\left(\frac{1}{2-\beta} - \frac{1}{e^{1-\beta}}\right) < f\left(1 - \frac{1}{e^{1-\beta}} - \frac{1-\beta}{2-\beta}\right) < d\,t_0\left(\frac{1}{2-\beta} - \frac{1}{e^{1-\beta}}\right),$$

which is a contradiction. Thus, (9.65) is fulfilled.

Because of (9.65) and case (ii) of Lemma 9.3, the Operator behaves legally. Thus, using (9.50), the time point $t_1^* \in (0, t_0)$ is a pure Nash equilibrium strategy if and only if

$$-f\,\alpha \geq \begin{cases} d\,[(1-\beta)\,(t_1^* - s) + \beta\,(t_0 - s)] - b & : \quad 0 \leq s < t_1^* \\ d\,(t_0 - s) - f\,\alpha - b & : \quad t_1^* \leq s < t_0 \end{cases} . \tag{9.67}$$

Maximizing the right hand side of (9.67) with respect to s yields

$$-f\,\alpha \geq \begin{cases} d\,[(1-\beta)\,t_1^* + \beta\,t_0] - b \\ d\,(t_0 - t_1^*) - f\,\alpha - b \end{cases} ,$$

which is equivalent to (9.64). Inequality (9.63) assures the existence of t_1^* fulfilling (9.64). □

Note that even in case of $\alpha = \beta = 0$ it may happen that (9.65) is fulfilled but (9.63) is violated: For $t_0 = 6$ and $b/d = 2.5$ we have $d\,t_0 = 12/5\,b > b$, i.e., the left hand inequality in (8.2) is fulfilled for $k = 1$. Thus, we get by (9.53) and (9.63)

$$t_0 - t^* = \frac{t_0}{e} \approx 2.21 < 2.5 = \frac{b}{d} \qquad \text{and} \qquad \frac{t_0}{2} = 3 > 2.5 = \frac{b}{d} ,$$

i.e., (9.58) is fulfilled and the Operator behaves legally, but (9.63) is violated. Thus, $Q^*(t_1)$ as given by (9.56) is an equilibrium strategy of the Inspectorate and no pure equilibrium strategy t_1^* exists, because the set (9.64) is empty.

Corollary 9.1 is really surprising: First, the fact that the equilibrium strategy of the Inspectorate is deterministic – although not unique – is surprising in view of the sophistically randomized equilibrium strategy (9.56) in case of illegal behaviour of the Operator. Second, if (9.63) is fulfilled, then the interval given by (9.64) is the same as that of the Se-Se inspection game, to be considered in Section 12.1, given by (12.11) for $t_2 = 0$.

Let us conclude this section with a remark on the inspection game with $\alpha > 0$ and $\beta > 0$ and more than one interim inspection, see Krieger (2011): The system of differential equations to be solved in case of two interim inspections is similar to (9.48), and given by:

$$\begin{aligned} h_1'(x) &= (1-\beta)\,h_1(x) - \alpha\,f/d \\ h_2'(x) &= (1-\beta)\,h_2(x) - (1-\beta)^2\,h_1(x) - \alpha\,f/d\,, \end{aligned} \tag{9.68}$$

with $h_1(0) = A_2$ and $h_2(0) = h_1(1)$, where $A_2 > 0$ is determined by the condition $h_2(1) = t_0$. Recall that the construction of the optimal strategies in Theorem 9.1 is essentially based on the monotonicity property of the functions $h_n(x)$, $n = 1, 2, \ldots$. Finding conditions on α, β, f, d and t_0 leading to monotone increasing functions $h_1(x)$ and $h_2(x)$ fulfilling (9.68) for any $x \in [0, 1]$, however, has neither achieved for the case of $k = 2$ interim inspections nor does it seem feasible for any number $k > 2$ of interim inspections.

9.5 Choice of the false alarm probability

In Section 9.4 we have assumed that the false alarm probability α is a parameter of the model, and that the test procedure used for the interim inspection is unbiased; see (9.49). We now assume in addition that the non-detection probability $\beta(\alpha)$ is given as a function of α. Note that in signal detection theory the plot $(\alpha, 1-\beta(\alpha))$ for $\alpha \in [0,1]$ is called *Receiver Operating Characteristic* (ROC) and illustrates the performance of a binary classifier system as its discrimination threshold is varied; see Pepe (2004) or Krzanowski and Hand (2009). In Chapter 20 this concept together with some of its properties is explained in more detail with the help of some numerical examples. For a wide class of statistical tests it can be shown that $\beta(0)=1$ and $\beta(1)=0$ and that $\beta(\alpha)$ is a monotone decreasing function of α. Summing up, we assume in this section

$$\beta(0)=1\,, \quad \beta(1)=0\,, \quad \alpha+\beta(\alpha)<1 \quad \text{and} \quad \beta(\alpha) \text{ is monotone decreasing in } \alpha\,. \tag{9.69}$$

Since the choice of the value of α is up to the Inspectorate, one may ask what its appropriate value should be.

In practice, some conventional value is taken, e.g., $\alpha = 0.01$ or $\alpha = 0.05$. In the context of inspections, however, it is natural to use that value which is optimal in the sense of the Inspectorate's intentions. In the following we will present a procedure, keeping in mind that it has not yet applied in practice, since it depends crucially on the knowledge of the payoff parameters of both players. Also for these reasons we will limit the analysis to the case of just one interim inspection, i.e., we consider it primarily to be of theoretical interest. It should be mentioned here that we will come back to this issue in Sections 12.4, 15.5 and 16.4, therefore, we will go here into some major detail to which we will refer later on.

Using (9.57) and (9.59), the equilibrium payoff to the Operator is given by

$$Op_1^*(\alpha) := \begin{cases} Op_1^* & \text{for Operator's illegal behaviour} \\ -f\,\alpha & \text{for Operator's legal behaviour} \end{cases}$$

$$= \begin{cases} d\,t_0\, e^{-(1-\beta(\alpha))} - f\,\alpha \left(1 - \dfrac{1-e^{-(1-\beta(\alpha))}}{1-\beta(\alpha)}\right) - b & \text{for} \quad t_0 - t^* > b/d \\ -f\,\alpha & \text{for} \quad t_0 - t^* < b/d \end{cases}, \tag{9.70}$$

where $t^* = t^*(\alpha, \beta(\alpha))$ is given by (9.53). Define for any $\alpha \in [0,1]$

$$F(\alpha) := d\,t_0\, e^{-(1-\beta(\alpha))} - f\,\alpha \left(1 - \frac{1-e^{-(1-\beta(\alpha))}}{1-\beta(\alpha)}\right) - b\,. \tag{9.71}$$

Note that $F(\alpha)$ is equal to Op_1^*, see (9.70), if and only if $t_0 - t^* > b/d$, i.e., only for those $\alpha \in [0,1]$ for which we have $t_0 - t^*(\alpha, \beta(\alpha)) > b/d$.

Using (8.2) for $k=1$ interim inspection, (9.69), and (9.71) we get

$$F(0) = d\,t_0 - b > 0 \qquad \text{and} \qquad F(1) = \frac{d\,t_0}{e} - \frac{f}{e} - b < F(0)\,.$$

Furthermore, $F(\alpha)$ is a monotone decreasing function on $[0,1]$. To prove this statement, we define

$$\tilde{F}(\alpha, \beta) := d\,t_0\, e^{-(1-\beta)} - f\,\alpha \left(1 - \frac{1-e^{-(1-\beta)}}{1-\beta}\right) - b\,, \tag{9.72}$$

which implies $F(\alpha) = \tilde{F}(\alpha, \beta(\alpha))$. Let us assume – for an ease proof – that $\beta(\alpha)$ is a differentiable function on $(0,1)$. Thus, because of (9.69), we have $\beta'(\alpha) < 0$. Applying the chain rule from calculus, we get for any $\alpha \in (0,1)$

$$\frac{d}{d\alpha} F(\alpha) = \left(\frac{\partial}{\partial \alpha} \tilde{F}(\alpha, \beta), \frac{\partial}{\partial \beta} \tilde{F}(\alpha, \beta) \right) \Big|_{\alpha=\alpha, \beta=\beta(\alpha)} \begin{pmatrix} 1 \\ \beta'(\alpha) \end{pmatrix} . \tag{9.73}$$

Because $\beta - e^{-(1-\beta)}$ is a monotone increasing function in $[0,1]$ with value 0 for $\beta = 1$, we obtain, using (9.72), for any $\alpha \in (0,1)$, and thus, for any $\beta \in (0,1)$, see (9.69),

$$\frac{\partial}{\partial \alpha} \tilde{F}(\alpha, \beta) = f \left(\frac{1 - e^{-(1-\beta)}}{1 - \beta} - 1 \right) < 0 \, .$$

Again, using (9.72), leads for any $(\alpha, \beta) \in (0,1) \times (0,1)$, using (9.53), to

$$\frac{\partial}{\partial \beta} \tilde{F}(\alpha, \beta) = d \, e^{-(1-\beta)} \, t^* + f \, \alpha \, \frac{1 - e^{-(1-\beta)}}{(1-\beta)^2} > 0 \, .$$

Thus, (9.73) yields $F'(\alpha) < 0$ for any $\alpha \in (0,1)$.

For the purpose of illustration, we assume that the non-detection probability $\beta(\alpha)$ is, using (20.7), given by

$$\beta(\alpha) = \Phi \left(\Phi^{-1}(1 - \alpha) - \frac{\mu_1 - \mu_0}{\sigma} \right) , \tag{9.74}$$

which describes a statistical test problem where it has to be decided if the random variable X is Gaussian distributed with expectation μ_0 and variance σ^2, or else if X is Gaussian distributed with expectation $\mu_1 > \mu_0$, and the same variance σ^2. For the graphs we have chosen $(\mu_1 - \mu_0)/\sigma = 1.5$. Note that $\beta(\alpha)$ as given by (9.74) fulfils (9.69); see Chapter 20.

Figure 9.3 represents $F(\alpha)$ and $-f\,\alpha$ as well as the resulting $Op_1^*(\alpha)$. Depending on the regions of definition, see (9.70), $F(\alpha)$ and $-f\,\alpha$ are solid or dashed, and $Op_1^*(\alpha)$ is solid for any $\alpha \in [0,1]$. We have chosen $t_0 = 1, b = 6, f = 1.5$; the three graphs correspond to $d = 9$ (left top), $d = 12$ (right top) and $d = 14$ (bottom). Note that these parameters fulfil (8.2) for $k = 1$ interim inspection.

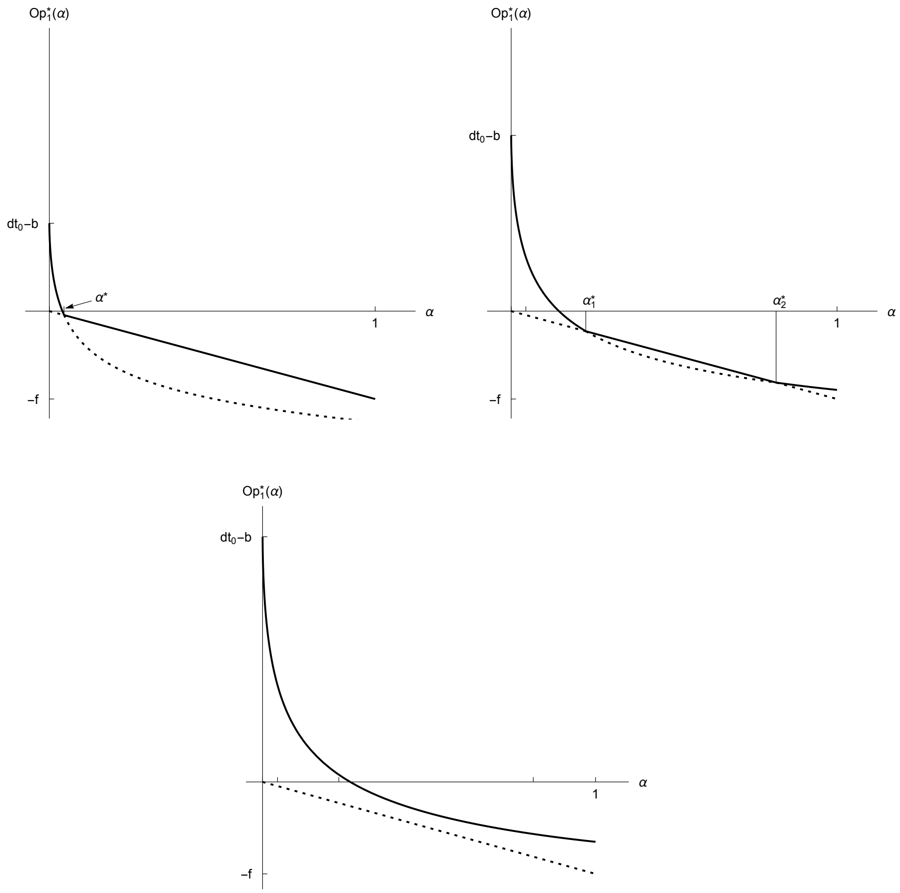
Because $F(\alpha)$ is a monotone decreasing function on $[0,1]$, we need to distinguish the cases:

$$\begin{aligned} &\text{(i):} \quad F(1) < -f \quad \text{and one intersection point} \\ &\text{(ii):} \quad F(1) > -f \quad \text{and} \quad \begin{cases} \text{two intersection points} \\ \text{no intersection point} \end{cases} . \end{aligned} \tag{9.75}$$

In case (i) the Operator will

$$\begin{matrix} \text{behave illegally} & & < & \\ \text{be indifferent} & \text{for } \alpha & = & \alpha^* \\ \text{behave legally} & & > & \end{matrix} \tag{9.76}$$

Figure 9.3 The equilibrium payoff (9.70) to the Operator for $t_0 = 1, b = 6, f = 1.5$ and $d = 9$ (top left), $d = 12$ (top right) and $d = 14$ (bottom).



where α^* is given by $F(\alpha^*) = -f\,\alpha^*$, and in case (ii) with two intersections the Operator will

$$\begin{array}{lcl} \text{behave illegally} & & \alpha < \alpha_1^* \ \text{ or } \ \alpha_2^* < \alpha \\ \text{be indifferent} & \text{for} & \alpha = \alpha_1^* \ \text{ or } \ \alpha = \alpha_2^* \\ \text{behave legally} & & \alpha_1^* < \alpha < \alpha_2^* \end{array} \tag{9.77}$$

where α_1^* and α_2^* with $\alpha_1^* < \alpha_2^*$ are given by $F(\alpha_1^*) = -f\,\alpha_1^*$ and $F(\alpha_2^*) = -f\,\alpha_2^*$. Because $F(\alpha) > -f\,\alpha$ for any $\alpha \in [0, 1]$ in case (ii) and no intersection point, the Operator will behave illegally for all values of α.

Now let us come back to the determination of the optimal value of α. One way would be to include α into the set of pure strategies of the Inspectorate and to solve this extended game.

This has been done for related problems, see Avenhaus and Canty (1996), Theorem 9.3, and leads to an equilibrium strategy of the Operator according to which he behaves legally or illegally with positive probabilities, i.e., the Operator can not be deterred from behaving illegally.

Another way is to apply the so-called *Inspectorate Leadership Principle*[8]: The Inspectorate chooses a value of α and announces it to the Operator in a *credible* way. Of course, the value of α is chosen in such a way that the Inspectorate's payoff is maximized.

Using (9.57) and (9.59), the equilibrium payoff to the Inspectorate is given by

$$In_1^*(\alpha) := \begin{cases} In_1^* & \text{for Operator's illegal behaviour} \\ -g\,\alpha & \text{for Operator's legal behaviour} \end{cases} ,$$

$$= \begin{cases} G(\alpha) & \text{for} \quad t_0 - t^* > b/d \\ -g\,\alpha & \text{for} \quad t_0 - t^* < b/d \end{cases} , \tag{9.78}$$

where $G(\alpha)$ is, using (9.53) with $t^* = t^*(\alpha, \beta(\alpha))$ and (9.57), for any $\alpha \in [0,1]$ defined by

$$\begin{aligned} G(\alpha) : = -a \Bigg[&\beta(\alpha)\,(t_0 - t^*(\alpha,\beta(\alpha))) \\ &- \left(t_0 - t^*(\alpha,\beta(\alpha)) + \frac{g}{a}\,\frac{\alpha}{1-\beta(\alpha)} \right) \ln \left[1 - \frac{t^*(\alpha,\beta(\alpha))}{t_0 + \dfrac{g}{a}\,\dfrac{\alpha}{1-\beta(\alpha)}} \right] \Bigg] . \end{aligned} \tag{9.79}$$

(9.78) implies that $G(\alpha)$ is equal to In_1^* if and only if $t_0 - t^* > b/d$. Because $t^*(0,1) = 0$, (9.69) together with (9.79) imply

$$G(0) = -\,a\,t_0 < 0 \quad \text{and} \quad G(1) = a\,\left(t_0 - t^*(1,0) + \frac{g}{a} \right) \ln \left[1 - \frac{t^*(1,0)}{t_0 + \dfrac{g}{a}} \right] .$$

In Figure 9.4, the solid curve represents $In_1^*(\alpha)$ for the sets of parameters used in Figure 9.3 and $a = 10$ and $g = 3$, which fulfil (8.2) for $k = 1$ interim inspection.

Figure 9.4 indicates that 1) $-g > G(1) > G(0)$, and 2) $G(\alpha)$ is a monotone increasing function of $\alpha \in [0,1]$, which, however, could not be proven yet for any differentiable function $\beta(\alpha)$ fulfilling (9.69) in contrast to the corresponding monotonicity property for $F(\alpha)$. Note that the difference in the plots of $G(\alpha)$ between the cases $d = 9, 12$ and $d = 14$ can only be recognized for values of α close to one.

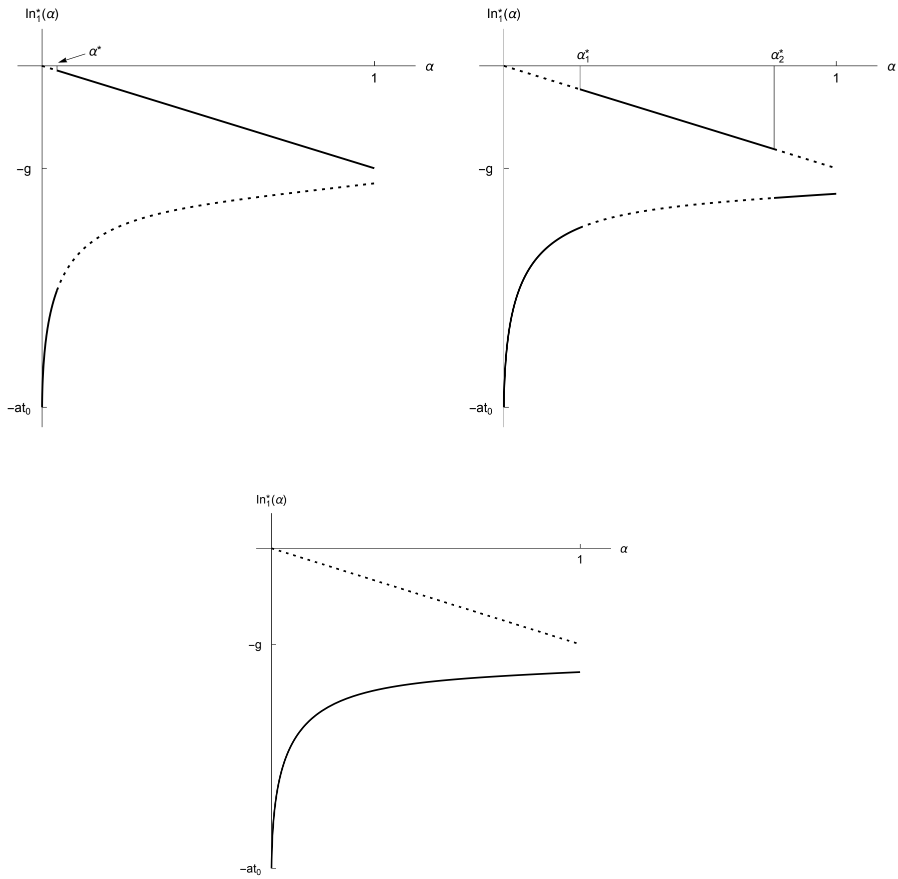
In case of $d = 9$ and $d = 12$ (top row), we see that for $\alpha = \alpha^*$ resp. $\alpha = \alpha_1^*$ and legal behaviour of the Operator the Inspectorate's payoff is maximized. Indeed one can show that these strategies constitute an equilibrium, in other words, the Operator is deterred from behaving illegall, or induced to legal behaviour, to say it in a positive way.

In case (ii) and no intersection point in (9.75), i.e., the bottom graph in Figure 9.4, the application of the Inspector Leadership Principle does not result in the deterrence of the Operator

[8] This principle has been introduced into Economic Theory by von Stackelberg (1934) and applied first to inspection games by Maschler (1966). For further details see Section 7.4 and Avenhaus et al. (2002).

for the chosen set of parameters and the specific function $\beta(\alpha)$ given by (9.74): Because the maximum of $In_1^*(\alpha)$ in $[0, 1]$ is attained at $\alpha = 1$, the optimal value of α is one. The Operator, however, is not deterred from behaving illegally, because $F(1) > -f$; see Figure 9.3. Note that at $\alpha = 1$ the Operator's payoff is even minimized.

Figure 9.4 The equilibrium payoff (9.78) to the Inspectorate for the sets of parameters used in Figure 9.3 and $a = 10$ and $g = 3$.



Chapter 10

Se-No inspection game for one facility: Avenhaus-Krieger model

Because there is no difference between the No-Se and the No-No inspection game with $k = 1$ interim inspection, the game theoretical solution presented in Lemma 9.2 also applies to the No-Se inspection game with $k = 1$ interim inspection and is not repeated here. What can be said in case of $k = 2$ interim inspections? Whereas in Section 4.1 the discrete time No-Se inspection game with $N = 4$ possible time point for $k = 2$ interim inspections is treated, there exist no game theoretical solution for this variant and continuous time.

Thus, we directly start with the analysis of the Se-No inspection game for any number k of interim inspections and non-vanishing errors of the second kind in Section 10.1, which is based on Avenhaus et al. (2009a), Avenhaus et al. (2010) and Avenhaus and Krieger (2013b). In Section 10.2 the practical application which has already been presented in Section 6.6 is discussed once more. The special case of $k = 2$ interim inspections and non-vanishing errors of the first and second kind, analysed in Avenhaus and Krieger (2010) and Avenhaus and Krieger (2011b), is presented in Section 10.3.

In this chapter, assumption (vii) of Chapter 8 is specified as follows:

(vii') The Inspectorate decides at the beginning of the reference time interval, i.e., at time point t_{k+1}, at which time point(s) it will perform its k interim inspection(s).

The Operator decides at the beginning of the reference time interval whether to start the illegal activity immediately at time point t_{k+1} or to postpone its start; in the latter case he decides again after the first interim inspection, whether to start the illegal activity immediately at that time point or to postpone its start again; and so on. While in Section 10.1 the Operator starts the illegal activity latest immediately at the time point of the last interim inspection, in Section 10.3 he can behave legally throughout the game.

Further assumptions will be specified in Sections 10.1 and 10.3, while the remaining assumptions of Chapter 8 hold throughout this chapter.

10.1 Any number of interim inspections; errors of the second kind

The inspection game analysed in this section is based on the following specifications:

(iv') The Operator starts once an illegal activity during the reference time interval $[t_{k+1}, t_0]$ in the only facility under consideration.

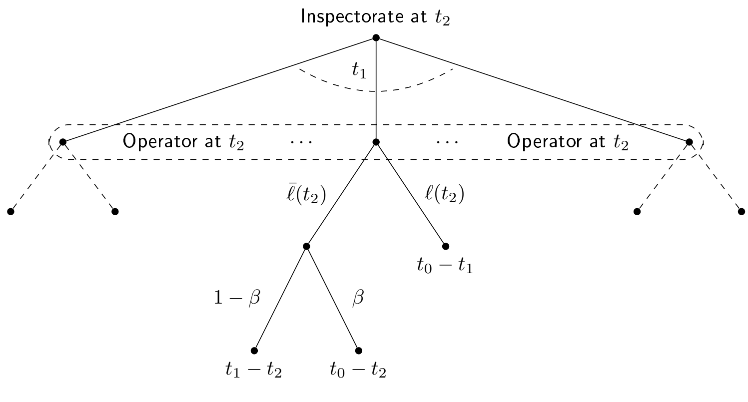
(v') During an interim inspection the Inspectorate may commit an error of the second kind with probability β, i.e., the illegal activity, see assumption (iv'), is not detected during the next interim inspection with probability β. Note that if there is no interim inspection left, then it is detected with certainty at the final PIV; see assumption (iii). This non-detection probability is the same for all k interim inspections.

(ix') The payoffs to the two players (Operator, Inspectorate) are linear functions of the detection time $\triangle t$, i.e., the time between start and detection of the illegal activity, and are given as follows

$$(\triangle t, -\triangle t) \qquad \text{for illegal behaviour and detection time} \, \triangle t \, .$$

Let us start with the case of $k = 1$ interim inspection in one facility. Figure 10.1 represents the extensive form of this inspection game. Due to the comment on p. 50, the extensive form games in this chapter start with the Inspectorate's decision at the beginning of the reference time interval. Note that the chance moves are not explicitly named, but can be identified via the probabilities $1 - \beta$ and β.

Figure 10.1 Extensive form of the Se-No inspection game with $k = 1$ interim inspection and with errors of the second kind.



In Figure 10.1 the Inspectorate chooses at the beginning of the reference time interval, i.e., at t_2, a time point t_1 for its interim inspection. The Operator decides at time point t_2 to start the illegal activity immediately ($\bar{\ell}(t_2)$), or to postpone its start ($\ell(t_2)$). According to assumption (viii) of Chapter 8, both players decide independently of each other which implies that the

Operator does not know the time point of the interim inspection. This lack of knowledge is indicated by his information set. At the interim inspection a chance move takes place: In case the Operator starts the illegal activity at t_2, it will be detected at t_1 with probability $1-\beta$ and it will not be detected at t_1 with probability β. In the first case the payoff to the Operator is $t_1 - t_2$ and in the second case $t_0 - t_2$, because the Inspectorate will detect the illegal activity with certainty at the end of the reference time interval during the PIV; see assumption (iii) of Chapter 8. Also, if the Operator postpones the start of the illegal activity to the first interim inspection, his payoff will be $t_0 - t_1$, again because it will be detected with certainty during the PIV.

In accordance with (9.1), the Inspectorate's set of pure strategies is given by

$$\mathcal{T}_1 := \{t_1 \in \mathbb{R} : t_2 < t_1 < t_0\}\,, \tag{10.1}$$

see Figure 8.1 for $k = 1$ interim inspection. Let g_2 be the Operator's probability to postpone the start of the illegal activity at time point t_2. Thus, his behavioural strategy set is given by (4.7), which is introduced here again for easy reference

$$G_1 := \{g_2 : g_2 \in [0, 1]\}\ . \tag{10.2}$$

Note that due to assumption (iv'), the Operator must behave illegally. Thus, like in the discrete time Se-No inspection game in Section 4.2 but in contrast to the Se-Se inspection game analysed in Section 12.1, here the probability g_1 of postponing the illegal activity at time point t_1, is zero and thus, not included in the Operator's strategy set. Using Figure 10.1, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $g_2 \in G_1$ and any $t_1 \in \mathcal{T}_1$, given by

$$Op_1(g_2, t_1) := (1 - g_2)\,[(1 - \beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2)] + g_2\,(t_0 - t_1)\,. \tag{10.3}$$

If we define the conditional detection times $H_2(t_2, t_1)$ and $H_1(t_1)$, i.e., the detection times when the illegal activity is started at time point t_2 resp. t_1 by

$$H_2(t_2, t_1) := (1 - \beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2) \qquad \text{and} \qquad H_1(t_1) := t_0 - t_1\,, \tag{10.4}$$

then the payoff (10.3) to the Operator can be expressed as

$$Op_1(g_2, t_1) = (1 - g_2)\,H_2(t_2, t_1) + g_2\,H_1(t_1)\,. \tag{10.5}$$

The game theoretical solution of this inspection game, see Avenhaus et al. (2009a), Avenhaus et al. (2010) and Avenhaus and Krieger (2013b), is presented in

Lemma 10.1. *Given the Se-No inspection game on the reference time interval $[t_2, t_0]$ with $k = 1$ interim inspection, and with errors of the second kind. The sets of behavioural resp. pure strategies are given by (10.2) and (10.1), and the payoff to the Operator by (10.3).*

Then an optimal strategy of the Operator is given by

$$g_2^* = \frac{1 - \beta}{2 - \beta}\,, \tag{10.6}$$

and an optimal strategy of the Inspectorate by

$$t_1^* - t_2 = \frac{1 - \beta}{2 - \beta}\,(t_0 - t_2)\,. \tag{10.7}$$

The optimal payoff to the Operator is

$$Op_1^* := Op_1(g_2^*, t_1^*) = t_0 - t_1^* = \frac{t_0 - t_2}{2 - \beta}\,. \tag{10.8}$$

Proof. We need to show that, in analogy to (19.10), the saddle point criterion

$$Op_1(g_2, t_1^*) \leq Op_1^* \leq Op_1(g_2^*, t_1) \tag{10.9}$$

is fulfilled for any $g_2 \in G_1$ and any $t_1 \in \mathcal{T}_1$. Using (10.4), (10.7) and (22.7) for $k = 1$, we get

$$H_2(t_2, t_1^*) = (t_0 - t_2)\left((1-\beta)\frac{1-\beta}{2-\beta} + \beta\right) = \frac{t_0 - t_2}{2-\beta} = t_0 - t_1^* = H_1(t_1^*)\,. \tag{10.10}$$

Thus, we have by (10.5)

$$Op_1(g_2, t_1^*) = \frac{t_0 - t_2}{2-\beta} \tag{10.11}$$

for any $g_2 \in G_1$, i.e., the left hand inequality of (10.9) is fulfilled as equality. On the other hand, (10.3) and (10.6) imply

$$(2-\beta)\, Op_1(g_2^*, t_1) = [(1-\beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2)] + (1-\beta)\,(t_0 - t_1) = t_0 - t_2\,,$$

for any $t_1 \in \mathcal{T}_1$, i.e., the right hand inequality of (10.9) is also fulfilled as equality. □

Let us comment the results of Lemma 10.1: First, it should be emphasized that our analysis leads to an explicit dependence of the optimal interim inspection time point t_1^* on β. Whereas for $\beta = 0$ the common sense point of view would lead to the results (10.6) – (10.8), for $\beta > 0$ one would hardly arrive at these results without a quantitative analysis. According to (10.10), the optimal strategy t_1^* of the Inspectorate is determined such that the Operator is made indifferent between behaving illegally at time points t_2 or t_1^*, because the conditional detection times (10.4) are the same: $H_2(t_2, t_1^*) = H_1(t_1^*) = t_0 - t_1^*$. Also it is interesting to note that the optimal interim inspection time point t_1^* depends on the length $t_0 - t_2$ of the reference time interval and β, while the optimal strategy of the Operator g_2^* is only a function of β. Further interesting properties of the optimal strategies are discussed after the proof of Theorem 10.1.

Second, due to (10.8) the optimal expected detection time is the time between the optimal interim inspection time point and the final PIV. This is a remarkable property which holds for quite a lot of the inspections games in this monograph; see also the comment on p. 232. The Operator would have also obtained the payoff $t_0 - t_1^*$ if he would have started the illegal activity with certainty right after the interim inspection, i.e., $g_2^* = 1$. In that case, however, g_2^* and t_1^* given by (10.7) do not constitute a pair of optimal strategies: We have by (10.5) and (10.11)

$$Op_1(g_2, t_1^*) = t_0 - t_1^*\,, \qquad Op_1(1, t_1^*) = t_0 - t_1^* \qquad \text{and} \qquad Op_1(1, t_1) = t_0 - t_1\,,$$

i.e., the right hand inequality in (10.9) is not fulfilled for any $t_1 \in \mathcal{T}_1$ but only for $t_1 \leq t_1^*$. Thus, (g_2^*, t_1) does not constitute a saddle point.

Third, there exist interesting relations between the solution of Lemma 10.1 and the solution of the corresponding Se-Se inspection game; see the comments after Lemma 12.1.

Finally and most importantly, the optimal strategy of the Inspectorate is a *pure* strategy, i.e., t_1^* is deterministic. In other words, the Inspectorate can announce the time point of its interim inspection if it wishes so and which the Operator knows anyhow because in case he also performs a game theoretical analysis, it is based on the same strategy sets, the same payoff and also the saddle point criterion (common knowledge). Remember that in the discrete time Se-No inspection game with $k = 1$ interim inspection the existence of an optimal *pure* strategy of the Inspectorate can only be assured for special values of β; see Section 6.3. Nevertheless, an optimal pure strategy may pose a problem to practitioners since they argue sometimes that a randomization might have some advantage, somehow in the sense of letting the adversary unclear about the own intention. Let us expand this a bit. Due to the linearity of the expected detection time in t_1, the Inspectorate can also choose the time point t_1 for its interim inspection using an arbitrary density function $f(t_1)$ concentrated on $[t_2, t_0]$ such that

$$t_1^* = \int_{t_2}^{t_0} t_1\, f(t_1)\, dt_1\,,$$

where t_1^* is the deterministic interim inspection time point given by (10.7). However, this way the Inspectorate does not improve its optimal payoff, i.e., does not shorten the optimal payoff to the Operator, thus, in the framework of this model the Inspectorate does not gain anything indeed. Remember also, that in case of the discrete time Se-No and Se-Se inspection games, Theorem 4.1 and Lemma 4.4 as well as Theorems 5.2 and 5.3 show that the set of optimal strategies of the Inspectorate is fully characterized by the uniquely determined expected interim inspection time points; see the comment on p. 68.

Before turning to the general case of any number k of interim inspections we consider the case $k = 2$ interim inspections in one facility, since thereafter it is easier to understand the somewhat more advanced formalism and the results for the general case. The extensive form of the Se-No inspection game with $k = 2$ interim inspections is represented in Figure 10.2.

At the beginning of the reference time interval, i.e., at time point t_3, the Inspectorate chooses the time points (t_2, t_1) for the first and the second interim inspection. Note knowing the Inspectorate's decision at t_3, indicated by the information set, the Operator decides to start the illegal activity immediately at t_3 ($\bar{\ell}(t_3)$) or to postpone its start (ℓ_3). In case the Operator starts the illegal activity at t_3 ($\bar{\ell}(t_3)$), it will be detected at the first interim inspection at t_2 with probability $1-\beta$, or it will be detected at the second interim inspection at t_1 with probability $\beta\,(1-\beta)$, or it will be detected at the PIV at t_0 with probability β^2, leading to the detection time $t_2 - t_3$, $t_1 - t_3$ and $t_0 - t_3$, respectively. In case the Operator does not start the illegal activity at t_3 (ℓ_3) he decides at t_2 whether to start an illegal activity now ($\bar{\ell}(t_2)$) or to postpone its start again ($\ell(t_2)$). In the latter case he has to start at t_1. The information set (there is one for any $t_3 < t_2 < t_0$) indicates that at time point t_2 the Operator knows t_2, of course, but not the time point t_1 of the second interim inspection. If he starts the illegal activity at t_2, it is detected at t_1 with probability $1-\beta$ or at the final PIV at t_0 with probability β, leading to the detection times $t_1 - t_2$ and $t_0 - t_2$, respectively. If the Operator postpones the start of the illegal activity until time point t_1 ($\ell(t_2)$), the detection time is $t_0 - t_1$.

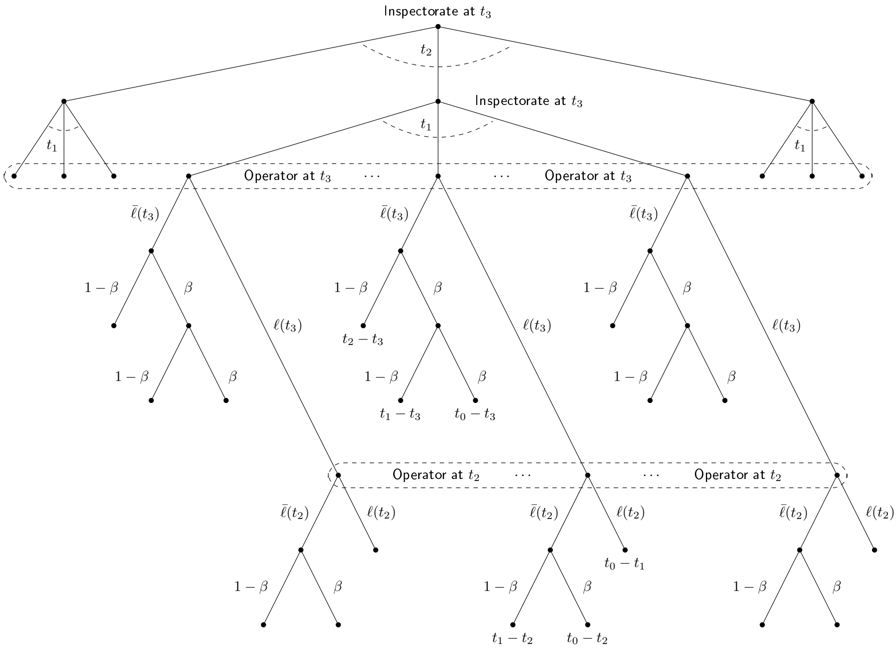
As mentioned, the Inspectorate chooses at time point t_3 the two time points for the interim inspections, i.e., its set of pure strategies is, see also (9.18) for $k = 2$ interim inspections and $t_3 = 0$, given by

$$\mathcal{T}_2 := \{\mathbf{t} := (t_2, t_1) \in \mathbb{R}^2 : t_3 < t_2 < t_1 < t_0\}\,, \tag{10.12}$$

see Figure 8.1 for $k = 2$ interim inspections. Let $1 - g_3$ be the Operator's probability to start the illegal activity immediately at time point t_3. In case he postpones the start (with probability g_3), he decides at the first inspection at t_2 to start the illegal activity immediately (with probability $1 - g_2(t_2)$) or to postpone its start again (with probability $g_2(t_2)$). In the latter case he has to start it at t_1. Thus, his set of behavioural strategies is given by

$$G_2 := \{\mathbf{g} := (g_3, g_2) : g_3 \in [0,1]\,, g_2 : (t_3, t_0) \to [0,1]\}\,. \tag{10.13}$$

Figure 10.2 Extensive form of the Se-No inspection game with $k = 2$ interim inspections and with errors of the second kind.



Note that while the strategy set G_1 coincide in the discrete and continuous Se-No inspection game with $k = 1$ interim inspection, see (4.7) and (10.2), G_k for $k \geq 2$ differs; see (4.15) and (10.13).

Using Figure 10.2, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $\mathbf{g} \in G_2$ and any $\mathbf{t} \in \mathcal{T}_2$, given by

$$\begin{aligned} Op_2(\mathbf{g}, \mathbf{t}) := &(1 - g_3)\,[(1-\beta)\,(t_2 - t_3) + \beta\,(1-\beta)\,(t_1 - t_3) + \beta^2\,(t_0 - t_3)] \\ &+ g_3\,[(1 - g_2(t_2))\,((1-\beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2)) + g_2(t_2)\,(t_0 - t_1)]\,. \end{aligned} \tag{10.14}$$

Introducing the conditional detection time $H_3(t_3, t_2, t_1)$, i.e., the detection time when the illegal activity is started at time point t_3,

$$H_3(t_3, t_2, t_1) := (1-\beta)\,(t_2 - t_3) + \beta\,(1-\beta)\,(t_1 - t_3) + \beta^2\,(t_0 - t_3)\,, \tag{10.15}$$

then (10.14) can be written as, using (10.4),

$$Op_2(\mathbf{g}, \mathbf{t}) = (1 - g_3)\,H_3(t_3, t_2, t_1) + g_3\,[(1 - g_2(t_2))\,H_2(t_2, t_1) + g_2(t_2)\,H_1(t_1)]\,. \tag{10.16}$$

The game theoretical solution of this inspection game, see Avenhaus et al. (2009a) and Avenhaus et al. (2010), is presented in

Lemma 10.2. *Given the Se-No inspection game on the reference time interval $[t_3, t_0]$ with $k = 2$ interim inspections, and with errors of the second kind. The sets of behavioural resp. pure strategies are given by (10.13) and (10.12), and the payoff to the Operator by (10.14).*

Then an optimal strategy of the Operator is given by

$$g_3^* = \frac{2\,(1-\beta)}{3-2\,\beta} \qquad \textit{and} \qquad g_2^*(t_2) = \frac{1}{2} \quad \textit{for all} \quad t_3 < t_2 < t_0\,, \tag{10.17}$$

and an optimal strategy of the Inspectorate by

$$t_2^* - t_3 = \frac{1-\beta}{3-2\,\beta}\,(t_0 - t_3) \qquad \textit{and} \qquad t_1^* - t_2^* = \frac{1-\beta}{2-\beta}\,(t_0 - t_2^*)\,. \tag{10.18}$$

The optimal payoff to the Operator is

$$Op_2^* := Op_2(\mathbf{g}^*, \mathbf{t}^*) = t_0 - t_1^* = \frac{t_0 - t_3}{3-2\,\beta}\,. \tag{10.19}$$

Proof. We show that $Op_2(\mathbf{g}, \mathbf{t}^*) = Op_2^* = Op_2(\mathbf{g}^*, \mathbf{t})$ for any $\mathbf{g} \in G_2$ and any $\mathbf{t} \in \mathcal{T}_2$, i.e., the saddle point condition is fulfilled as equality.

Because (10.18) fulfils the recursive relations (22.2) for $k = 2$, we get by (22.1) and (10.15)

$$\begin{aligned} H_3(t_3, t_2^*, t_1^*) &= (t_0 - t_3)\,\left((1-\beta)^2\,\frac{1}{1+2\,(1-\beta)} + \beta\,(1-\beta)^2\,\frac{2}{1+2\,(1-\beta)} + \beta^2\right) \\ &= \frac{t_0 - t_3}{3-2\,\beta}\left((1-\beta)^2 + \beta\,(1-\beta)^2\,2 + \beta^2\,(3-2\,\beta)\right) = \frac{t_0 - t_3}{3-2\,\beta}\,. \end{aligned}$$

For H_2 and H_1 given by (10.4) we obtain analogously to (10.10), using (10.18) and (22.7) for $k = 2$,

$$\begin{aligned} H_2(t_2^*, t_1^*) &= (1-\beta)\,(t_1^* - t_2^*) + \beta\,(t_0 - t_2^*) \\ &= \frac{t_0 - t_2^*}{2-\beta} = \frac{t_0 - t_3 - (t_2^* - t_3)}{2-\beta} = \frac{(t_0 - t_3)}{2-\beta}\,\left(1 - \frac{1-\beta}{3-2\,\beta}\right) \\ &= \frac{t_0 - t_3}{3-2\,\beta} = H_1(t_1^*)\,. \end{aligned}$$

Thus, we have $H_3(t_3, t_2^*, t_1^*) = H_2(t_2^*, t_1^*) = H_1(t_1^*)$ and get by (10.16) for any $\mathbf{g} \in G_2$

$$Op_2(\mathbf{g}, \mathbf{t}^*) = \frac{t_0 - t_3}{3-2\,\beta} = Op_2^*\,.$$

Rearranging (10.14), we get for any $\mathbf{g} \in G_2$ and any $\mathbf{t} = (t_2, t_1) \in \mathcal{T}_2$

$$\begin{aligned} Op_2(\mathbf{g}, \mathbf{t}) &= (1-g_3)\,[(-t_3) + \beta^2\,t_0] + g_3\,[(1-g_2(t_2))\,\beta\,t_0 + g_2(t_2)\,t_0] \\ &\quad + t_2\,\Big[(1-g_3)\,(1-\beta) - g_3\,(1-g_2(t_2))\Big] \\ &\quad + t_1\,\Big[(1-g_3)\,\beta\,(1-\beta) + g_3\,[(1-g_2(t_2))\,(1-\beta) - g_2(t_2)]\Big]\,. \end{aligned}$$

Using (10.17), we see that the coefficient of t_2 and t_1 are zero and thus, we obtain for any $\mathbf{t} = (t_2, t_1) \in \mathcal{T}_2$

$$Op_2(\mathbf{g}^*, \mathbf{t}) = \frac{1}{3 - 2\,\beta} \left[(-t_3) + \beta^2\, t_0 + t_0\,(1 - \beta^2) \right] = \frac{t_0 - t_3}{3 - 2\,\beta} = Op_2^*\,,$$

which completes the proof. □

Let us comment the results of Lemma 10.2: First, technically speaking, these solutions are more simple then the corresponding ones for the discrete time Se-No inspection game, see Lemma 6.5, in the sense that no distinction of cases with respect to β are necessary.

Second, since according to the right hand equation of (10.18) and (22.1) we get $t_1^* - t_3 = 2\,(t_2^* - t_3)$, we obtain that for $\beta < 1$ the second interim inspection takes place after double the time than the first one. For $\beta = 0$ we get

$$t_2^* - t_3 = \frac{1}{3}\,(t_0 - t_3) \qquad \text{and} \qquad t_1^* - t_3 = \frac{2}{3}\,(t_0 - t_3)\,.$$

As in the case of $k = 1$ interim inspection, for $\beta = 0$ the common sense point of view would lead to this result, for $\beta > 0$ one would hardly arrive at this result without quantitative analysis. The same holds for the Operator's optimal strategy (g_3^*, g_2^*): Since the Operator is confronted at t_3 with three inspection intervals of equal length he chooses $g_3^* = 2/3$. After the first inspection however, only two intervals of equal length are left. Thus, he chooses $g_2^* = 1/2$.

Third, like in the case of $k = 1$ interim inspection, and as shown in the proof of Lemma 10.2, the optimal strategy of the Inspectorate $\mathbf{t}^* = (t_2^*, t_1^*)$ is chosen such that the Operator is indifferent as regards to the start of the illegal activity at t_3, t_2^* or t_1^*, because the conditional detection times are the same: $H_3(t_3, t_2^*, t_1^*) = H_2(t_2^*, t_1^*) = H_1(t_1^*) = t_0 - t_1^*$.

Finally, and again like in the case of $k = 1$ interim inspection, see p. 189, the Inspectorate may announce the optimal interim inspection time points, if it wishes so. It might – instead of using the deterministic pure strategies (10.18) – also randomize as follows: Let $f_2(t_2)$ and $f_1(t_1)$ be two density functions concentrated on $[t_3, t_0]$ such that

$$\sup_{t_3 < t_2 < t_0} \{t_2 : f_2(t_2) > 0\} < \inf_{t_3 < t_1 < t_0} \{t_1 : f_1(t_1) > 0\}\,,$$

i.e., we always have $t_2 < t_1$ for any realization, and, using (10.18),

$$t_2^* = \int_{t_3}^{t_0} t_2\, f_2(t_2)\, dt_2 \qquad \text{and} \qquad t_1^* = \int_{t_3}^{t_0} t_1\, f_1(t_1)\, dt_1\,.$$

Then the linearity of the expected detection time (10.14) in t_2 and t_1 shows, that (f_2, f_1) are also optimal strategies. The Inspectorate, however, does not gain anything.

Let us consider now the general case of k of interim inspections in one facility. According to assumption (ii) of Chapter 8 the Inspectorate chooses k interim inspection time points $t_k, \ldots, t_1$ with $t_{k+1} < t_k < \ldots < t_1 < t_0$ in the reference time interval $[t_{k+1}, t_0]$, see Figure 10.3, i.e., its set of pure strategies is given by

$$\mathcal{T}_k := \{\mathbf{t} := (t_k, \ldots, t_1) \in \mathbb{R}^k : t_{k+1} < t_k < \ldots < t_1 < t_0\}\,. \tag{10.20}$$

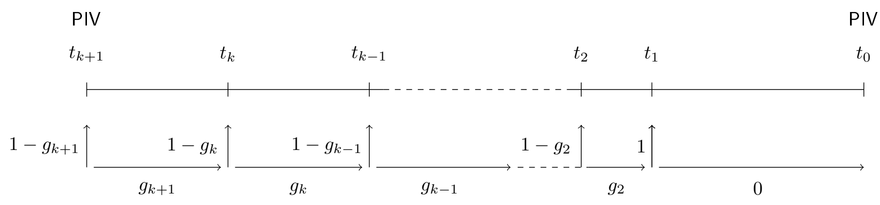
Assumption (vii') means here that the Operator starts the illegal activity at t_{k+1} with probability $1 - g_{k+1}$ or he postpones its start with probability g_{k+1}; in the latter case he starts it at t_k

with probability $1 - g_k(t_k)$ which depends on t_k or he postpones its start again with probability $g_k(t_k)$. If the Operator postpones the start of the illegal activity until time point t_n, he starts it there with probability $1 - g_n(t_n)$ and postpones its start again with probability $g_n(t_n)$. If he does not start the illegal activity before he has to do it at t_1, i.e., $g_1(t_1) = 0$.

Note that we assume – like in the corresponding discrete time Se-No and Se-Se inspection games treated in Part I – that g_n depends only on t_n and not on the whole history $t_k, \ldots, t_n$, i.e., $g_n = g_n(t_n)$ for all $n = 2, \ldots, k$. This is a plausible assumption because the Operator's payoff in the remaining game, i.e., the game starting at time point t_n, is not influenced by the time points $t_k, \ldots, t_{n+1}$. Furthermore, it turns out in the proof of Theorem 10.1 that a dependence of g_n also on $t_k, \ldots, t_{n+1}$, i.e., $g_n = g_n(t_k, \ldots, t_{n+1}, t_n)$ leads to the same optimal strategies; see also p. 65 for the discrete time Se-No inspection game.

Figure 10.3 presents the time line of the interim inspections and probabilities for starting or postponing the illegal activity.

Figure 10.3 Time line of the interim inspections and probabilities for starting or postponing the illegal activity for the Se-No inspection game with k interim inspections. For reasons of clarity we write g_n instead of $g_n(t_n)$, $n = 2, \ldots, k$.



In analogy to (10.13) we define the Operator's set of behavioural strategies to be

$$G_k := \{\mathbf{g} := (g_{k+1}, g_k, \ldots, g_2) : g_{k+1} \in [0,1] \,, \\ g_n : (t_{k+1}, t_0) \to [0,1] \,, \quad n = 2, \ldots, k\} \,. \quad (10.21)$$

Expanding the definitions in (10.4) and (10.15), we define for all $n = 1, \ldots, k+1$ and any $(t_n, \ldots, t_1) \in \mathbb{R}^n$ with $t_n < \ldots < t_1 < t_0$ the conditional detection time $H_n(t_n, \ldots, t_1)$ by

$$H_n(t_n, \ldots, t_1) := (1 - \beta) \sum_{m=1}^{n-1} \beta^{n-1-m} \, (t_m - t_n) + \beta^{n-1} \, (t_0 - t_n) \,, \quad (10.22)$$

where $\sum_{m=1}^{0} \ldots = 0$. Thus, the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $\mathbf{g} \in G_k$ and any $\mathbf{t} \in \mathcal{T}_k$, with $H_n = H_n(t_n, \ldots, t_1)$ and $g_n = g_n(t_n)$, given by

$$\begin{aligned} Op_k(\mathbf{g}, \mathbf{t}) := {} & (1 - g_{k+1}) \, H_{k+1} \\ & + g_{k+1} \, [(1 - g_k) \, H_k \\ & \quad + g_k \, [(1 - g_{k-1}) \, H_{k-1} \ldots \\ & \quad\quad + (1 - g_3) \, H_3 + g_3 \, [(1 - g_2) \, H_2 + g_2 \, H_1] \ldots]] \,, \end{aligned}$$

or, in closed form, by

$$Op_k(\mathbf{g},\mathbf{t}) = (1-g_{k+1})\,H_{k+1} + \sum_{n=2}^{k} (1-g_n(t_n))\,H_n\,g_{k+1} \prod_{\ell=n+1}^{k} g_\ell(t_\ell)$$

$$+\,(t_0-t_1)\,g_{k+1}\prod_{\ell=2}^{k} g_\ell(t_\ell)\,, \tag{10.23}$$

with $\prod_{\ell=k+1}^{k} g_\ell(j_\ell) := 1$. A comparison between (4.31) and (10.23) shows how similar the payoffs to the Operator in both Se-No inspection games are.

The game theoretical solution of this inspection game, see Avenhaus and Krieger (2013b), is presented in

Theorem 10.1. *Given the Se-No inspection game on the reference time interval $[t_{k+1}, t_0]$ with k interim inspections, and with errors of the second kind. The sets of behavioural resp. pure strategies are given by (10.21) and (10.20), and the payoff to the Operator by (10.23).*

Then an optimal strategy of the Operator is given by

$$g^*_{k+1} = \frac{k\,(1-\beta)}{1+k\,(1-\beta)} \qquad \textit{and} \tag{10.24}$$

$$g^*_n(t_n) = \frac{n-1}{n} \quad \textit{for all} \quad t_{k+1} < t_n < t_0 \quad \textit{and all} \quad n = 2,\ldots,k\,, \tag{10.25}$$

and an optimal strategy of the Inspectorate, with $t^*_{k+1} := t_{k+1}$, by

$$t^*_n - t^*_{n+1} = \frac{1-\beta}{1+n\,(1-\beta)}\,(t_0 - t^*_{n+1}) \qquad \textit{for} \qquad n = 1,\ldots,k\,. \tag{10.26}$$

The optimal payoff to the Operator is

$$Op^*_k := Op_k(\mathbf{g}^*,\mathbf{t}^*) = t_0 - t^*_1 = \frac{t_0 - t_{k+1}}{1+k\,(1-\beta)}\,. \tag{10.27}$$

Proof. As in the proofs of Lemmata 10.1 and 10.2 we show that the saddle point condition $Op_k(\mathbf{g},\mathbf{t}^*) \le Op^*_k \le Op_k(\mathbf{g}^*,\mathbf{t})$ for any $\mathbf{g} \in G_k$ and any $\mathbf{t} \in \mathcal{T}_k$ is fulfilled as equality:

$$Op_k(\mathbf{g},\mathbf{t}^*) = Op^*_k = Op_k(\mathbf{g}^*,\mathbf{t})\,. \tag{10.28}$$

Using (10.24) and (10.25), we get for all $n = 1,\ldots,k-1$

$$g^*_{k+1}\prod_{\ell=n+1}^{k} g^*_\ell(t_\ell) = \frac{k\,(1-\beta)}{1+k\,(1-\beta)}\,\frac{k-1}{k}\,\cdots\,\frac{n}{n+1} = \frac{n}{1+k\,(1-\beta)}\,.$$

Thus, we get, using (10.23) and if we omit the arguments $(t_n,\ldots,t_1)$ in H_n,

$$Op_k(\mathbf{g}^*,\mathbf{t}) = \frac{1}{1+k\,(1-\beta)}\,H_{k+1} + \sum_{n=1}^{k} \frac{1}{n}\,H_n\,\frac{k\,(1-\beta)}{1+k\,(1-\beta)}\,\frac{k-1}{k}\cdots\frac{n+1}{n+2}\,\frac{n}{n+1}$$

$$= \frac{1}{1 + k\,(1-\beta)} \left(H_{k+1} + (1-\beta) \sum_{n=1}^{k} H_n \right) . \tag{10.29}$$

We show by induction that for any $k \in \mathbb{N}$ and any $t_0, \ldots, t_{k+1} \in \mathbb{R}$

$$H_{k+1} + (1-\beta) \sum_{n=1}^{k} H_n = t_0 - t_{k+1} \tag{10.30}$$

holds. For $k = 1$ we get by (10.4)

$$H_2 + (1-\beta)\,H_1 = (1-\beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2) + (1-\beta)\,(t_0 - t_1) = t_0 - t_2\,,$$

i.e., (10.30) is fulfilled. Let (10.30) be true for an arbitrary k. Then (10.30) implies

$$\begin{aligned}
H_{k+2} + (1-\beta) \sum_{n=1}^{k+1} H_n &= H_{k+2} + (1-\beta)\,H_{k+1} + t_0 - t_{k+1} - H_{k+1} \\
&= H_{k+2} - \beta\,H_{k+1} + t_0 - t_{k+1} \\
&= (1-\beta) \sum_{m=1}^{k+1} \beta^{k+1-m}\,(t_m - t_{k+2}) + \beta^{k+1}\,(t_0 - t_{k+2}) \\
&\qquad - \beta \left((1-\beta) \sum_{m=1}^{k} \beta^{k-m}\,(t_m - t_{k+1}) + \beta^k\,(t_0 - t_{k+1}) \right) + t_0 - t_{k+1} \\
&= (1-\beta)\,t_{k+1} - t_{k+2}\,(1-\beta) \sum_{m=1}^{k+1} \beta^{k+1-m} + \beta^{k+1}\,(t_{k+1} - t_{k+2}) \\
&\qquad + t_{k+1}\,\beta\,(1-\beta) \sum_{m=1}^{k} \beta^{k-m} + t_0 - t_{k+1} \\
&= (1-\beta)\,t_{k+1} - t_{k+2}\,(1-\beta^{k+1}) + \beta^{k+1}\,(t_{k+1} - t_{k+2}) + t_{k+1}\,\beta\,(1-\beta^k) + t_0 - t_{k+1} \\
&= t_0 - t_{k+2}\,,
\end{aligned}$$

which was to be shown. Thus, (10.29) yields

$$Op_k(\mathbf{g}^*, \mathbf{t}) = \frac{t_0 - t_{k+1}}{1 + k\,(1-\beta)}$$

for any $\mathbf{t} \in \mathcal{T}_k$, i.e., the right hand side of (10.28). In order to prove the left hand side of (10.28) note that (10.26) fulfils the recursive relations (22.2). Thus, (22.1) yields

$$t_m^* - t_n^* = (n-m)\,\frac{1-\beta}{1 + k\,(1-\beta)}\,(t_0 - t_{k+1}) \tag{10.31}$$

for all $1 \le m < n \le k$, and for all $n = 1, \ldots, k$,

$$t_0 - t_n^* = t_0 - t_{k+1} - (t_n^* - t_{k+1})$$

$$= \left(1 - (k+1-n)\,\frac{1-\beta}{1+k\,(1-\beta)}\right)\,(t_0 - t_{k+1})\,. \tag{10.32}$$

Using (10.22), (10.31) and (10.32), we get for $n = 2, \ldots, k+1$ with $t^*_{k+1} := t_{k+1}$

$$\begin{aligned}
&H_n(t^*_n, \ldots, t^*_1)\\
&= (1-\beta)\sum_{m=1}^{n-1} \beta^{n-1-m}\,(t^*_m - t^*_n) + \beta^{n-1}\,(t_0 - t^*_n)\\
&= (t_0 - t_{k+1})\left((1-\beta)\sum_{m=1}^{n-1}\beta^{n-1-m}\,(n-m)\,\frac{1-\beta}{1+k\,(1-\beta)} + \beta^{n-1}\,\frac{1+(1-\beta)\,(n-1)}{1+k\,(1-\beta)}\right)\\
&= \frac{t_0 - t_{k+1}}{1+k\,(1-\beta)}\left((1-\beta)^2\sum_{m=1}^{n-1}\beta^{n-1-m}\,(n-m) + \beta^{n-1}\,(1+(1-\beta)\,(n-1))\right)\\
&= \frac{t_0 - t_{k+1}}{1+k\,(1-\beta)}\left(-(1-\beta)^2\sum_{m=1}^{n-1}\beta^{n-1-m}\,m + \beta^n + n\,(1-\beta)\right). \qquad (10.33)
\end{aligned}$$

Because we have for $n = 2, \ldots, k+1$

$$\begin{aligned}
\sum_{m=1}^{n-1}\beta^{-1-m}\,m &= (-1)\sum_{m=1}^{n-1}\frac{d}{d\beta}\left(\beta^{-m}\right) = (-1)\,\frac{d}{d\beta}\left(\sum_{m=1}^{n-1}\beta^{-m}\right)\\
&= \frac{d}{d\beta}\left(\frac{1-\beta^{-n+1}}{1-\beta}\right) = \frac{\beta^{-n}\,(1-\beta)\,(n-1) + 1 - \beta^{-n+1}}{(1-\beta)^2}\,,
\end{aligned}$$

we obtain by (10.33)

$$\begin{aligned}
H_n(t^*_n, \ldots, t^*_1) &= \frac{t_0 - t_{k+1}}{1+k\,(1-\beta)}\Big(-(1-\beta)\,(n-1) - \beta^n + \beta + \beta^n + n\,(1-\beta)\Big)\\
&= \frac{t_0 - t_{k+1}}{1+k\,(1-\beta)}\,.
\end{aligned}$$

Thus, we get together with (22.7)

$$H_{k+1}(t_{k+1}, t^*_k, \ldots, t^*_1) = H_k(t^*_k, \ldots, t^*_1) = \ldots = H_2(t^*_2, t^*_1) = H_1(t^*_1)\,, \tag{10.34}$$

which finally leads, using (10.23), for any $\mathbf{g} \in G_k$ to

$$Op_k(\mathbf{g}, \mathbf{t}^*) = \frac{t_0 - t_{k+1}}{1+k\,(1-\beta)}\,,$$

i.e., the left hand side of (10.28). □

Let us comment the results of Theorem 10.1: First, of course, Lemmata 10.1 and 10.2 are special cases of this Theorem if one only considers (10.24) for $k = 1$ interim inspection. Like in the comments to these Lemmata we mention that the optimal strategy of the Inspectorate, i.e., $\mathbf{t}^* = (t^*_k, \ldots, t^*_1)$, is determined such that the Operator is indifferent as regards to the start of the illegal activity at $t_{k+1}, t_k^*, \ldots, t_1^*$: (10.34) indicates that the conditional detection times are the same again.

Second, the optimal interim inspection time points t_n^* depend on the length $t_0 - t_{k+1}$ of the reference time interval and β. It is interesting to note that according to (10.31) the time differences $t_n^* - t_{n+1}^*$, $n = 1, \ldots, k$ with $t_{k+1}^* := t_{k+1}$, between two subsequent interim inspection time points are the same. For $\beta = 0$ we have $t_n^* - t_{n+1}^* = (t_0 - t_{k+1})/(k+1)$ for $n = 1, \ldots, k$ which, as mentioned after Lemma 10.1, could have been guessed with common sense arguments. Also, for $\beta = 0$ and $t_0 - t_{k+1} = N + 1$ we get by (10.26) and (22.3)

$$t_n^* - t_{n+1}^* = \frac{t_0 - t_{n+1}^*}{n+1} = \frac{t_0 - t_{k+1}}{k+1} = \frac{N+1}{k+1}\,, \tag{10.35}$$

i.e., the differences between the optimal interim inspection time points are the same as the differences between the optimal *expected* interim inspection time points as given in (4.47).

Third, while the component g_{k+1}^* of the Operator's optimal strategy is a function of β, the components $g_n^*(t_n)$, $n = 2, \ldots, k$, given by (10.25) only depend on the number of interim inspections left. Again, $1 - g_2^*(t_2), \ldots, 1 - g_k^*(t_k)$ as given by (10.25) form a harmonic progression; see also Table 4.1 on p. 72 for an overview of inspection games with this property.

Fourth, it is intuitive that $t_n^* \to t_{k+1}$, $n = 1, \ldots, k$, and $g_{k+1}^* \to 0$ with increasing β, see (10.32) and (10.24): For β close to 1 the Operator starts the illegal activity with probability close to 1 at time point t_{k+1}. Consequently, the Inspectorate will perform its interim inspections also very early.

Finally, like in the case of one and two interim inspections, the Inspectorate can also choose the interim inspection time points according to some density function such that the expected time points coincides with the one given by (10.26); see p. 192.

Using (10.24) and (10.25), we obtain for the optimal expected time point S for the start of the illegal activity (recall $\prod_{\ell=k+1}^{k} g_\ell(t_\ell) := 1$)

$$\begin{aligned}
\mathbb{E}_{(\mathbf{g}^*,\mathbf{t}^*)}(S) &:= (1 - g_{k+1}^*)\, t_{k+1} + g_{k+1}^* \left(\sum_{n=2}^{k} (1 - g_n^*(t_n^*))\, t_n^* \prod_{\ell=n+1}^{k} g_\ell^*(t_\ell^*) + t_1^* \prod_{\ell=2}^{k} g_\ell^*(t_\ell^*) \right) \\
&= \frac{1}{1 + k\,(1-\beta)}\, t_{k+1} + \sum_{n=1}^{k} \frac{1}{n}\, t_n^*\, \frac{k\,(1-\beta)}{1 + k\,(1-\beta)}\, \frac{k-1}{k} \cdots \frac{n+1}{n+2}\, \frac{n}{n+1} \\
&= \frac{1}{1 + k\,(1-\beta)} \left(t_{k+1} + (1-\beta) \sum_{n=1}^{k} t_n^* \right) \\
&= \frac{1}{1 + k\,(1-\beta)} \left(t_{k+1} + (1-\beta) \sum_{n=1}^{k} (t_n^* - t_{k+1}) + k\,(1-\beta)\, t_{k+1} \right).
\end{aligned}$$

This expression simplifies, using (22.1), to

$$\mathbb{E}_{(\mathbf{g}^*,\mathbf{t}^*)}(S) = \frac{1}{1 + k\,(1-\beta)} \left(t_{k+1} + (1-\beta)^2 \sum_{n=1}^{k} \frac{k+1-n}{1 + k\,(1-\beta)}\, (t_0 - t_{k+1}) \right.$$

$$+ \; k\,(1-\beta)\,t_{k+1}\Big)$$

$$= \frac{1}{1+k\,(1-\beta)}\,\left(t_{k+1}\,(1+k\,(1-\beta)) + \frac{(1-\beta)^2}{1+k\,(1-\beta)}\,(t_0 - t_{k+1})\,\frac{k\,(k+1)}{2}\right)$$

$$= t_{k+1} + \frac{(1-\beta)^2}{(1+k\,(1-\beta))^2}\,\frac{k\,(k+1)}{2}\,(t_0 - t_{k+1})\,.$$

For $\beta = 0$, $t_{k+1} = 0$ and $t_0 = N + 1$ this is the same form as (4.49) in Lemma 4.4.

Note that Derman (1961) considers the following inspection model in the context of reliability studies, see p. 155: Each interim inspection permits the detection of a preceding illegal activity only with a certain fixed probability, and in addition to the time until detection, there is a cost to be paid for each inspection. Derman describes a deterministic minimax schedule for the inspection time points.

10.2 Applications to Nuclear Safeguards

Let us come back to the practical application which has already been discussed in Section 6.6. However, here we assume that interim inspections can be performed at any point of time within the reference time interval.

Using (10.27), Figure 10.4 illustrates the optimal expected detection times Op_k^* as a function of the non-detection probability β for $k = 1, \ldots, 7$ interim inspection(s) from top to bottom. In contrast to Section 6.6, however, we choose $t_0 = 12$, i.e., the reference time interval is measured here in month.

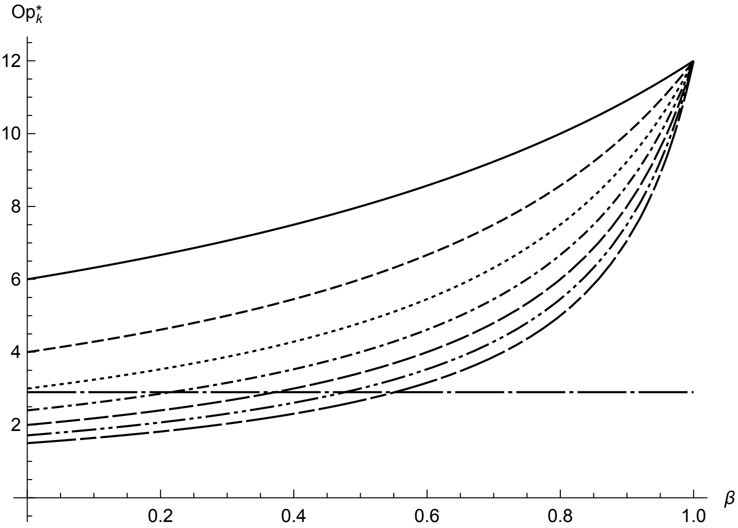
Figure 10.4 implies that if the required optimal expected detection time is set, e.g., to 2.9 month, then it cannot be achieved with $k = 1, 2, 3$ interim inspection(s). However, it can be achieved with $k \geq 4$ interim inspections. The resulting non-detection probabilities β are given in the second column of Table 10.1. The optimal interim inspections time points are given by (10.26).

Table 10.1 The non-detection probabilities (rounded to four digits), the sample size and the total sample size for $k = 4, 5, 6, 7$ interim inspections, and for a required optimal expected detection time of 2.9 month.

	non-detection probability β	sample size n per interim inspection	total sample size
$k = 4$	0.2155	40	160
$k = 5$	0.3724	28	140
$k = 6$	0.4770	22	132
$k = 7$	0.5517	18	126

In order to arrive at a more complex example compared to the one in Section 6.6, we assume now that $r = 3$ (instead of $r = 1$) out of $M = 100$ items have to be falsified to illegally

Figure 10.4 The optimal expected detection times Op_k^* in month as a function of the non-detection probability β for $k = 1, \ldots, 7$ interim inspection(s). The horizontal line is at 2.9.



acquire one significant quantity. Then, using (6.43), the sample size is the smallest n fulfilling the inequality

$$1 - \left(1 - \frac{n}{M}\right)\left(1 - \frac{n}{M-1}\right)\left(1 - \frac{n}{M-2}\right) \geq 1 - \beta\,,$$

which leads to the third and fourth column in Table 10.1. Which number of interim inspections should be recommended to the Inspectorate? Using the cost model from p. 132, we get

$$\begin{aligned}
4 \cdot a + 160 \cdot b \quad &\text{for} \quad k = 4\\
5 \cdot a + 140 \cdot b \quad &\text{for} \quad k = 5\\
6 \cdot a + 132 \cdot b \quad &\text{for} \quad k = 6\\
7 \cdot a + 126 \cdot b \quad &\text{for} \quad k = 7\,.
\end{aligned}$$

It can be seen that, e.g., $k = 5$ interim inspections is the cost-optimal solution if and only if $8\,b < a < 20\,b$; see Figure 10.5.

Figure 10.5 Cost-optimal number of interim inspections.

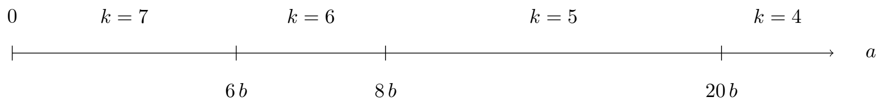


Figure 10.5 illustrates that the optimal number of interim inspections depends on the ratio of the overhead cost a per interim inspection and the cost b for checking one seal/item.

10.3 Two interim inspections; errors of the first and second kind

In this section we consider the Se-No inspection game with non-vanishing errors of the first and second kind. Because there is no difference between the Se-No and the Se-Se inspection game in case of $k = 1$ interim inspection, we refer the reader for its analysis to Section 12.1 and immediately consider the case of $k = 2$ interim inspections.

The inspection game analysed in this section is based on the following specifications:

(iv') The Operator may start at most once an illegal activity during the reference time interval $[t_3, t_0]$ in the only facility under consideration.

(v') During an interim inspection the Inspectorate may commit an error of the first and second kind with probabilities α and β. While during an interim inspection which is performed before the start of the illegal activity only an error of the first kind (false alarm) may occur, during an interim inspection which is performed after the start of the illegal activity only an error of the second kind (non-detection) may occur. The "game" continues after an error of the first kind. The error probabilities α and β are the same for all interim inspections.

Let us comment assumption (v'): It is assumed that – like in Section 9.4 – the value α of the false alarm probability is fixed a priori, and that the test procedure used for either of the two interim inspections is unbiased, i.e., $\alpha + \beta < 1$. The additional assumption that a false alarm is not possible during an interim inspection if prior to that interim inspection an illegal activity was started, is not a trivial assumption. Depending on the details of the inspection scheme alternative assumptions would have to be formulated; see p. 282. Note that inspections with errors of the first and second kind have already been considered in Section 7.4 and in more detail in Section 9.4.

Note that because a false alarm causes costs to both players, the zero-sum assumption has to be given up, and more than that, payoff parameters have to be introduced which evaluate the different outcomes of the game; see (8.1) and (8.2) with $k = 2$ interim inspections. In case the Operator behaves legally throughout the game, two false alarms may occur with the corresponding costs of $(-2\,f, -2\,g)$.

In the following we present the Se-No inspection game and its game theoretical solution along the lines of the papers Avenhaus and Krieger (2010) and Avenhaus and Krieger (2011b).

While the Inspectorate has the same strategy set $\mathcal{T}_2$ like in the Se-No inspection game in Section 10.1, see (10.12), the Operator's strategy set needs to be amended: Because he does not need to behave illegally at all throughout the game, he starts the illegal activity at time point t_1 with probability $1 - g_1(t_1)$ and behaves legally at time point t_1 with probability $g_1(t_1)$. Thus, we have instead of (10.13) the new set of behavioural strategies

$$G_2 := \{\mathbf{g} := (g_3, g_2, g_1) : g_3 \in [0,1], g_2, g_1 : (t_3, t_0) \to [0,1]\} \ . \qquad (10.36)$$

With the same arguments as on p. 193, we assume that g_1 only depends on t_1 and not on the whole history (t_2, t_1).

Instead of presenting the extensive form of this inspection game, see Avenhaus and Krieger (2010) and Avenhaus and Krieger (2011b), we derive the payoffs to both players explicitly:

Because the Inspectorate decides at the beginning of the reference time interval t_3 when to perform its two interim inspections, let $(t_2, t_1) \in \mathcal{T}_2$ be fixed. If the Operator starts the illegal activity at t_3 (i.e., $\bar{\ell}(t_3)$ with probability $1 - g_3$) then it is detected with probability $1 - \beta$ at t_2 or with probability $\beta\,(1-\beta)$ at t_1 or with probability β^2 at the final PIV at t_0. Because in this case – as mentioned above – a false alarm is excluded at time points t_2 and t_1, the Operator's payoff is, using (8.1), given by

$$d\,((1-\beta)\,(t_2 - t_3) + \beta\,(1-\beta)\,(t_1 - t_3) + \beta^2\,(t_0 - t_3)) - b\,.$$

If the Operator does not start the illegal activity at t_3 (i.e., $\ell(t_3)$ with probability g_3), then two cases may occur:

Case 1: Suppose a false alarm is risen at t_2. Then, by assumption (v'), this false alarm is clarified and at t_2 a new proper (sub)game starts with the reference time interval $[t_2, t_0]$ and one interim inspection. If the Operator starts the illegal activity then at t_2 (i.e., $\bar{\ell}(t_2)$ with probability $(1 - g_2(t_2))$), it is detected with probability $1-\beta$ at t_1 or with probability β at the final PIV at t_0, then with certainty. Thus, his payoff is in this case, using (8.1),

$$d\,((1-\beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2)) - b - f\,\alpha\,. \tag{10.37}$$

If he does not start the illegal activity at t_2 (i.e., $\ell(t_2)$ with probability $g_2(t_2)$), he starts it at time point t_1 (i.e., $\bar{\ell}(t_1)$ with probability $1 - g_1(t_1)$) or does behave legally throughout the game with probability $g_1(t_1)$, i.e., $\ell(t_1)$. In either case two false alarms can occur which result in the costs $-2\,f$. Thus, his payoff is

$$\left\{ \begin{array}{lll} d\,(t_0 - t_1) - b - 2\,f\,\alpha & \text{for} & \bar{\ell}(t_1) \\ -2\,f\,\alpha & \text{for} & \ell(t_1) \end{array} \right. . \tag{10.38}$$

Case 2: Suppose no alarm is risen at t_2. Then the same holds as derived in case 1 with the exception that – from a modelling point of view – different probabilities $g_2'(t_2)$ and $g_1'(t_1)$ at t_2 and t_1 for the Operator have to be assumed, because he might behave differently after the event "no false alarm" (case 2) compared to the event "false alarm" (case 1). Since, however, the decision between $\bar{\ell}(t_2)$ and $\ell(t_2)$ resp. $\bar{\ell}(t_1)$ and $\ell(t_1)$ is based on the same payoff alternatives (10.37) resp. (10.38), it is sufficient to introduce the same behavioural strategy $g_2(t_2)$ and $g_1(t_1)$. This statement does not hold for the corresponding Se-Se inspection game treated in Section 12.2.

In sum, the (expected) payoff to the Operator is, for any $\mathbf{g} \in G_2$ and any $\mathbf{t} \in \mathcal{T}_2$, given by

$$\begin{aligned} Op_2(t_3; \mathbf{g}, \mathbf{t}) := {} & (1 - g_3)\,\Big[d\,((1-\beta)\,(t_2 - t_3) + \beta\,(1-\beta)\,(t_1 - t_3) + \beta^2\,(t_0 - t_3)) - b\Big] \\ & + g_3\,\Big[(1 - g_2(t_2))\,\Big[d\,((1-\beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2)) - b - f\,\alpha\Big] \\ & \quad + g_2(t_2)\,\Big[(1 - g_1(t_1))\,(d\,(t_0 - t_1) - b - 2\,f\,\alpha) + g_1(t_1)\,(-2\,f\,\alpha)\Big]\Big]\,. \end{aligned} \tag{10.39}$$

Using (8.1), the (expected) payoff to the Inspectorate is obtained from that to the Operator by replacing d by $-a$ and f by g and setting $b = 0$, i.e., it is, for any $\mathbf{g} \in G_2$ and any $\mathbf{t} \in \mathcal{T}_2$, given by

$$In_2(t_3; \mathbf{g}, \mathbf{t}) := (1 - g_3)\,\Big[(-a)\,((1-\beta)\,(t_2 - t_3) + \beta\,(1-\beta)\,(t_1 - t_3) + \beta^2\,(t_0 - t_3))\Big]$$

$$+ g_3 \Big[(1 - g_2(t_2)) \Big[(-a)\,((1-\beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2)) - g\,\alpha\Big] \qquad (10.40)$$

$$+ \; g_2(t_2) \Big[(1 - g_1(t_1))\,((-a)\,(t_0 - t_1) - 2\,g\,\alpha) + g_1(t_1)\,(-2\,g\,\alpha)\Big]\Big] .$$

Note that we use the time point t_3 of the beginning PIV in the notation of Op_2 and In_2, because the Inspectorate's equilibrium strategies can be formulated more symmetrically using t_3 instead of 0, and the usage of t_3 is more appropriate for a generalization to any number k of interim inspections; see Chapter 12.

For $d = 1$, $b = 0$, $\alpha = 0$ and $g_1(t_1) = 0$, the payoff (10.39) reduces to the expected detection time (10.14), as expected. Note that the Inspectorate's payoff parameter g should not be confused with the Operator's strategies g_3, $g_2(t_1)$ and $g_1(t_1)$.

The game theoretical solution of this inspection game, see Avenhaus and Krieger (2010) and Avenhaus and Krieger (2011b), is presented in

Lemma 10.3. *Given the Se-No inspection game on the reference time interval $[t_3, t_0]$ with $k = 2$ interim inspections, errors of the first and second kind, and an unbiased test procedure. The sets of behavioural resp. pure strategies are given by (10.36) and (10.12), and the payoffs to both players by (10.39) and (10.40).*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op_2^*(t_3) := Op_2(t_3; \mathbf{g}^*, \mathbf{t}^*)$ and $In_2^*(t_3) := In_2(t_3; \mathbf{g}^*, \mathbf{t}^*)$:

(i) For

$$\frac{1}{3 - 2\,\beta} \left(t_0 - t_3 + \frac{f}{d}\,\alpha\,(3 - \beta)\right) > \frac{b}{d} \quad \textit{and} \quad \frac{f\,\alpha}{d\,(t_0 - t_3)} < \frac{1 - \beta}{3 - 3\,\beta + \beta^2} \qquad (10.41)$$

the Operator behaves illegally and an equilibrium strategy of the Operator is given by

$$g_3^* = 1 - \frac{1}{3 - 2\,\beta}\,, \qquad g_2^*(t_2) = \frac{1}{2} \qquad \textit{and} \qquad g_1^*(t_1) = 0 \qquad (10.42)$$

for all $t_3 < t_n < t_0$, $n = 1, 2$. An equilibrium strategy of the Inspectorate is given by

$$\begin{aligned} t_2^* - t_3 &= \frac{1 - \beta}{3 - 2\,\beta}\,(t_0 - t_3) - \frac{f}{d}\,\alpha\,\frac{3 - 3\,\beta + \beta^2}{3 - 2\,\beta} \\ t_1^* - t_2^* &= \frac{1 - \beta}{2 - \beta}\,(t_0 - t_2^*) - \frac{f}{d}\,\alpha\,\frac{1}{2 - \beta}\,. \end{aligned} \qquad (10.43)$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$\begin{aligned} Op_2^*(t_3) &= d\,\frac{1}{3 - 2\,\beta}\,(t_0 - t_3) - f\,\alpha\,\frac{3\,(1 - \beta)}{3 - 2\,\beta} - b \qquad \textit{and} \\ In_2^*(t_3) &= -\,a\,\frac{1}{3 - 2\,\beta}\,(t_0 - t_3) - g\,\alpha\,\frac{3\,(1 - \beta)}{3 - 2\,\beta}\,. \end{aligned} \qquad (10.44)$$

(ii) For

$$\frac{1}{3 - 2\,\beta} \left(t_0 - t_3 + \frac{f}{d}\,\alpha\,(3 - \beta)\right) < \frac{b}{d} \qquad (10.45)$$

the Operator behaves legally, i.e., $g_3^* = g_2^*(t_2) = g_1^*(t_1) = 1$ *for all* $t_3 < t_n < t_0$, $n = 1, 2$, *and the Inspectorate's set of equilibrium strategies is given by*

$$\begin{aligned} \frac{b}{d} - \frac{2\,f}{d}\,\alpha &\geq (1-\beta)\,(t_2^* - t_3) + \beta\,(1-\beta)\,(t_1^* - t_3) + \beta^2\,(t_0 - t_3) \\ \frac{b}{d} - \frac{f}{d}\,\alpha &\geq (1-\beta)\,(t_1^* - t_2^*) + \beta\,(t_0 - t_2^*) \\ \frac{b}{d} &\geq t_0 - t_1^*\,. \end{aligned} \tag{10.46}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_2^*(t_3) = -2\,f\,\alpha \qquad \textit{and} \qquad In_2^*(t_3) = -2\,g\,\alpha\,. \tag{10.47}$$

Proof. The proof is presented in Chapter 21. □

Let us comment the results of Lemma 10.3: First, setting $\alpha = 0$, $d = a = 1$ and $b = f = g = 0$ we arrive at the Se-No inspection game for $k = 2$ interim inspections in Section 10.1, and thus – not surprisingly at all – the optimal strategies and payoffs of Lemma 10.2 correspond with the Nash equilibrium strategies in (i) of Lemma 10.3. Furthermore, it will turn out in Lemma 12.2 that the Inspectorate's equilibrium strategies (10.43) and (10.46) coincide with the corresponding ones (12.25) and (12.27) of the Se-Se inspection game, and that the equilibrium payoffs to both players are the same. The Operator's equilibrium strategies, however, are different in case of illegal behaviour, because (10.42) and (12.24) imply for any $t_3 < t_2 < t_0$

$$g_2^*(t_2) = \begin{cases} \dfrac{1}{2} & \text{for the Se-No inspection game} \\ 1 - \dfrac{2\,(1-\alpha) - \beta}{2\,(1-\alpha)}\,\dfrac{1}{2-\beta} & \text{for the Se-Se inspection game} \end{cases}.$$

This result is discussed on p. 248.

Second, the equilibrium time points (t_2^*, t_1^*) given by (10.43) do not necessarily fulfil (10.46). Why? As remarked on p. 141 we need to assure that (t_2^*, t_1^*) is a meaningful expression under condition (10.45), i.e., if $(t_2^*, t_1^*) \in \mathcal{T}_2$. In Chapter 21 it is shown that $(t_2^*, t_1^*) \in \mathcal{T}_2$ if and only if the right hand inequality of (10.41) is fulfilled. Thus, (t_2^*, t_1^*) is not a robust equilibrium strategy, because the Inspectorate cannot just play (t_2^*, t_1^*) but has to check whether the right hand inequality of (10.41) is fulfilled; see also the example below. For $\alpha = 0$, however, the right hand inequality of (10.41) vanishes, and (t_2^*, t_1^*) is a robust equilibrium strategy; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

For illustration, consider two numerical examples with $t_3 = 0$ and $t_1 = 1$ each, and

$$\frac{b}{d} = 0.75\,, \qquad \beta = 0.5\,, \qquad \frac{f}{d}\,\alpha = 0.1\,, \qquad \alpha > 2/15\,, \qquad \text{and}$$

$$\frac{b}{d} = 0.8\,, \qquad \beta = 0.3\,, \qquad \frac{f}{d}\,\alpha = 0.32\,, \qquad \alpha > 4/10\,.$$

Both examples fulfil the left hand side of (8.2) with $k = 2$, and because

$$\frac{1}{3 - 2\,\beta}\,\left(t_0 - t_3 + \frac{f}{d}\,\alpha\,(3-\beta)\right) = \begin{cases} \dfrac{1}{2}\,(1 + 0.1 \cdot 2.5) = 0.625 & \text{first example} \\ \dfrac{1}{2.4}\,(1 + 0.32 \cdot 2.7) = \dfrac{233}{300} & \text{second example} \end{cases} < \frac{b}{d}\,,$$

(10.45) is also fulfilled. The right hand inequality of (10.41), however, is only fulfilled for the first example ($0.1 < 0.286$) whereas for the second one it is not ($0.32 > 0.319$). According to (10.43) the *illegal strategy* (t_2^*, t_1^*) of the Inspectorate is given by

$$(t_2^*, t_1^*) \approx (0.16, 0.37) \qquad \text{and} \qquad (t_2^*, t_1^*) = (-0.0003, 0.223)\,. \tag{10.48}$$

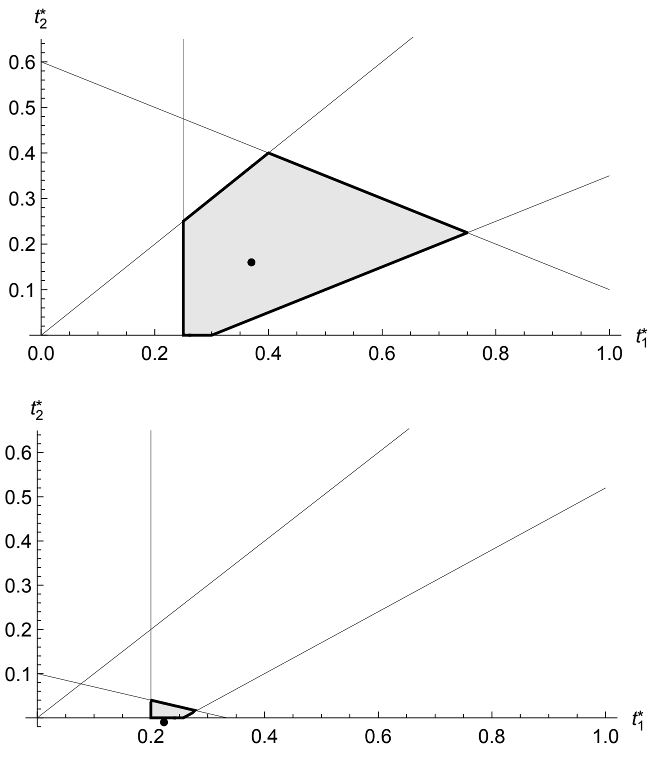
The negative value of t_2^* in the second example is not surprising as the proof in Chapter 21 shows that a violation of the right hand inequality of (10.41) is equivalent to $t_2^* < 0 = t_3$.

According to (10.46) the equilibrium strategies (t_2^*, t_1^*) of the Inspectorate in case of legal behaviour of the Operator, i.e., case (ii), are given by the inequalities

$$\begin{aligned} 0.6 &\geq t_2^* + 0.5\,t_1^*\,, & 0.15 &\geq 0.5\,t_1^* - t_2^*\,, & t_1^* &\geq 0.25\,, \quad \text{and} \\ 0.07 &\geq 0.7\,t_2^* + 0.21\,t_1^*\,, & 0.18 &\geq 0.7\,t_1^* - t_2^*\,, & t_1^* &\geq 0.2\,. \end{aligned} \tag{10.49}$$

Figure 10.6 illustrates both examples. We see the rather complicated regions for the legal equilibria (shaded area). In a similar case Kilgour (1992) called this area *cone of deterrence*.

Figure 10.6 Illustration of the cone of deterrence (shaded area) according to (10.49). The dot indicates (t_2^*, t_1^*) according to (10.48), which constitutes only in the upper example an equilibrium strategy.



Third, we did not consider the case

$$\frac{1}{3-2\beta}\left(t_0 - t_3 + \frac{f}{d}\,\alpha\,(3-\beta)\right) > \frac{b}{d} \qquad \text{and} \qquad \frac{f\,\alpha}{d\,(t_0-t_3)} > \frac{1-\beta}{3-3\,\beta+\beta^2}\,,$$

which, for the Se-Se inspection game, was considered in Avenhaus and Canty (2005) and will be treated in Lemma 12.3 of Section 12.2, where we will anyhow come back to the results of this section. There, it led to the equilibrium strategy $t_2^* = 0$ of the Inspectorate which is practically not feasible. Let us mention that the case $t_2 = 0$ is excluded in the inspection model, since we assumed a priori $t_3 < t_2 < t_1 < t_0$. However, without this assumption we think that the same would happen here.

Fourth, whereas the equilibrium strategies of the Operator in Lemma 10.3 do not depend on α, that of the Inspectorate does. It enters the equilibrium inspection time points (t_2^*, t_1^*) in the order $\alpha\, f/d$, which is supposed to be very small compared to the other terms. Therefore, we may conclude that even though errors of the first kind may occur, and the subsequent clarification of false alarms may cause technical and organizational problems, for *planning purposes* they may be ignored.

Fifth, like in the Se-No inspection game for $\alpha = 0$, the Inspectorate may announce the equilibrium interim inspection time points, if it wishes so; see the comment on p. 192.

Finally, the question of the appropriate choice of the false alarm probability α may be raised again. Starting with the Operator's payoff (10.44) and that for legal behaviour, $-2\,f\,\alpha$, we can proceed in the same way as in Section 9.5 and get a determinate for the optimal value α^* of the false alarm probability. Since, however, we will not make use of it, we do not discuss it here in more detail.

Chapter 11

Se-No inspection game for more facilities: Krieger-Avenhaus model

The inspection models which are presented in this chapter and which are stimulated by demands from the practitioners' side, see the comment on p. 146, differ from all other ones in this monograph insofar as several facilities are considered.

In this chapter, assumptions (iv), (vii) and (ix) of Chapter 8 are specified as follows:

(iv') The Operator starts once an illegal activity during the reference time interval $[t_{k+1}, t_0]$ in any of the N facilities.

(vii') The Inspectorate decides at the beginning of the reference time interval, i.e., at time point t_{k+1}, in which facilities and at which time points it performs its interim inspections.

The Operator decides at the beginning of the reference time interval whether and in which facility to start the illegal activity immediately at time point t_{k+1} or to postpone its start; in the latter case he decides again after the first interim inspection, whether and in which facility to start the illegal activity immediately at that time point or to postpone its start again; and so on. Because of assumption (iv'), the Operator starts the illegal activity latest immediately at the time point of the last interim inspection in either facility.

(ix') The payoffs to the two players (Operator, Inspectorate) are linear functions of the detection time $\triangle t$, i.e., the time between start and detection of the illegal activity, and are given as follows

$$(\triangle t, -\triangle t) \qquad \text{for illegal behaviour and detection time} \, \triangle t \, .$$

Assumptions (v) will be specified in the following sections, while the remaining assumptions of Chapter 8 hold throughout this chapter. The reason why in assumption (iv') the Operator is assumed to behave illegally once in one of the N facilities is that this is the worst case from the detection view point of the Inspectorate. The symbol N used in this chapter should not be confused with the meaning of N in Part I.

Compared to the inspection games treated in Sections 11.2 and 11.3, the inspection game analysed in Section 11.1 is more general on the one hand, because the detection probabilities, i.e., the probability of detecting the illegal activity, are assumed to be facility-dependent, and more special on the other, because only $k = 1$ interim inspection in N facilities is considered. Section 11.1 is based on Avenhaus and Krieger (2012).

In Section 11.2 the inspection game with $k = 2$ interim inspections is discussed for the sake of illustrating the general case which is then treated in Section 11.3. This general case deals with k interim inspections in N facilities, and facility-independent detection probabilities, and has been treated by Avenhaus and Krieger (2013b), where, however, optimal strategies and payoffs have been formulated only as a conjecture since its proof appeared infeasible at that time. In between, however, the proof has been achieved and is published in this monograph for the first time.

11.1 One interim inspection; facility-dependent errors of the second kind

Let us assume that there are N facilities – not necessarily of the same type – in a State. There is a reference time interval $[t_2, t_0]$, e.g., one calendar year, at the beginning and end of which Physical Inventory Verifications (PIVs) take place in all facilities which permit to detect with certainty any illegal activity which has been started during the reference time interval in one of these facilities. Note that it is assumed that the PIVs in the single facilities are carried through at the same time points t_2 respectively t_0. Inspection games with shifted PIVs are only analysed so far for special cases in Avenhaus and Krieger (2012).

The inspection game analysed in this section is based on the following specification:

(v') During the interim inspection the Inspectorate may commit a *facility-dependent* error of the second kind with probability β_i, $i = 1, \ldots, N$, i.e., the illegal activity, see assumption (iv'), started in facility i is not detected during the next interim inspection in that facility with probability β_i. Note that if there is no interim inspection left, then it is detected with certainty at the final PIV; see assumption (iii).

According to assumption (vii'), the Inspectorate chooses at the beginning of the reference time interval, i.e., at t_2, the facility i, $i = 1, \ldots, N$, for the interim inspection. In order to assure the existence of optimal strategies we have to assign a probability for choosing facility i at t_2: Let q_i, $i = 1, \ldots, N$, be the probability that facility i is inspected and define

$$Q_{N,1} := \Big\{ \mathbf{q} := (q_1, \ldots, q_N) \in [0,1]^N : \sum_{i=1}^{N} q_i = 1 \Big\} . \tag{11.1}$$

The meaning of q_i should not be confused with the meaning of q_j in Part I, see p. 26, because there it denotes the Inspectorate's probability to perform the only interim inspection at time point j, where here it means the probability to select facility i for inspection.

Depending on the choice i of the facility to be inspected, the Inspectorate chooses the time point $t_1(i)$ for the interim inspection. Because at time points t_2 and t_0 regular inspections (PIVs) are performed, the interim inspection can only be scheduled in the open interval (t_2, t_0), i.e., the set $\mathcal{T}_{N,1}$ of interim inspection time points for a selected facility is given by

$$\mathcal{T}_{N,1} := \Big\{ \mathbf{t} = t_1 : \{1, \ldots, N\} \to \mathbb{R} : t_2 < t_1(i) < t_0 \quad \text{for all} \quad i = 1, \ldots, N \Big\} . \tag{11.2}$$

In case of $N = 1$ facility, (11.2) simplifies to $\mathcal{T}_1$ as given by (9.1). Although in this section only the case of $k = 1$ interim inspection is considered, we could have avoided the use of the backward numbering of the interim inspection time points. However, to be consistent with Sections 11.2 and 11.3, we also apply it here. (11.1) and (11.2) together define the Inspectorate's strategy set

$$Q_{N,1} \times \mathcal{T}_{N,1}\,. \tag{11.3}$$

Let us now formalize the Operator's behaviour: Because of assumption (vii'), the Operator chooses the facility i in which he starts the illegal activity at t_2 with probability $g_{2,i}$, $i = 1,\ldots,N$, or he postpones the start with probability g_2. In the latter case he starts it in either facility at time point $t_1(i)$. Define for all $n = 2,3,\ldots$ the set

$$G_N^{(n)} := \left\{ \mathbf{g}_n := (g_{n,1},\ldots,g_{n,N},g_n) \in [0,1]^{N+1} : \sum_{i=1}^{N} g_{n,i} + g_n = 1 \right\}, \tag{11.4}$$

then the Operator's strategy set $G_{N,1}$ is given by

$$G_{N,1} := G_N^{(2)}\,. \tag{11.5}$$

In case of only $N = 1$ facility, $g_{2,1}$ is the probability to start the illegal activity immediately at t_2, and g_2 the probability to postpone its start to time point t_1, where $g_{2,1} + g_2 = 1$. Thus, $G_{N,1}$ can be identified with G_1 as given by (10.2).

The payoff to the Operator can be derived as follows: If the Inspectorate performs its interim inspection in facility i, $i = 1,\ldots,N$, (with probability q_i and at time point $t_1(i)$) and if the Operator starts the illegal activity at t_2 (with probability $g_{2,i}$) in the same facility, then the illegal activity will be detected with probability $1 - \beta_i$ at time point $t_1(i)$ with the resulting detection time $t_1(i) - t_2$; see assumption (v'). It will not be detected at time point $t_1(i)$ with probability β_i but at the final PIV with certainty, and the detection time is $t_0 - t_2$. Thus, the conditional detection time $H_i(\mathbf{t})$, i.e., the detection time when the illegal activity is started at time point t_2 in facility i and the interim inspection is performed in the same facility, is, in analogy to p. 187, defined by

$$H_i(\mathbf{t}) := (1-\beta_i)\,(t_1(i) - t_2) + \beta_i\,(t_0 - t_2)\,, \qquad i = 1,\ldots,N\,. \tag{11.6}$$

If the Operator starts the illegal activity at t_2 in facility j, $j \neq i$ (with probability $g_{2,j}$), the detection time is $t_0 - t_2$. If he postpones the start to time point $t_1(i)$ (with probability g_2), the detection time is $t_0 - t_1(i)$.

Therefore, the (expected) payoff to the Operator, i.e., the expected detection time, is, using (11.6), for any $\mathbf{g} \in G_{N,1}$ and any $(\mathbf{q},\mathbf{t}) \in Q_{N,1} \times \mathcal{T}_{N,1}$, given by

$$Op_{N,1}(\mathbf{g},(\mathbf{q},\mathbf{t})) := \sum_{i=1}^{N} q_i \left[g_{2,i}\,H_i(\mathbf{t}) + \sum_{\substack{j=1\\ j\neq i}}^{N} g_{2,j}\,(t_0 - t_2) + g_2\,(t_0 - t_1(i)) \right]. \tag{11.7}$$

Note that in case only $N = 1$ facility is considered, (11.7) simplifies to $Op_1(g_2,t_1)$ given by (10.3). Also note that the meaning of the index N in $Op_{N,1}(\mathbf{g},(\mathbf{q},\mathbf{t}))$ given by (11.7) and $Op_{N,1}(g_2,\mathbf{q})$ given by (4.8) differ: While in the former case it refers to the the number of facilities, in the latter one it refers to the possible number of time points for interim inspection(s).

This inspection game was developed to determine optimal inspection strategies, i.e., the optimal interim inspection time point and optimal facility selection probabilities, in European spent fuel storages where there was the additional constraint that a storage is inspected with at least 20% probability per year; see Avenhaus and Krieger (2012). In case of N facilities there is at least one facility i with $q_i \leq 1/N$ which implies that for $N \geq 6$ the 20% postulate can never be fulfilled no matter which value β_i can take. We will come back to this point in the comments on Theorem 11.1.

The game theoretical solution of this inspection game, see Avenhaus and Krieger (2012), is presented in

Theorem 11.1. *Given the Se-No inspection game on the reference time interval $[t_2, t_0]$ with $k = 1$ interim inspection in $N \geq 2$ facilities, and with an facility-dependent error of the second kind, i.e., $\beta_i \geq 0$ for $i = 1, \dots, N$. The Operator's set of behavioural strategies is given by (11.5), the Inspectorate's strategy set by (11.3), and the payoff to the Operator by (11.7).*

Then an optimal strategy of the Operator is, for $i = 1, \dots, N$, given by

$$g^*_{2,i} = \frac{1}{1-\beta_i}\left(1 + \sum_{\ell=1}^{N} \frac{1}{1-\beta_\ell}\right)^{-1} \quad \text{and} \quad g^*_2 = \left(1 + \sum_{\ell=1}^{N} \frac{1}{1-\beta_\ell}\right)^{-1}, \tag{11.8}$$

and an optimal strategy of the Inspectorate by the facility selection probabilities

$$q^*_i = \frac{1}{1-\beta_i}\left(\sum_{\ell=1}^{N} \frac{1}{1-\beta_\ell}\right)^{-1} \quad \text{for} \quad i = 1, \dots, N, \tag{11.9}$$

and by the interim inspection time point

$$\frac{t_0 - t_2}{t_0 - t^*_1(i)} = 1 + \left(\sum_{\ell=1}^{N} \frac{1}{1-\beta_\ell}\right)^{-1} \quad \text{for} \quad i = 1, \dots, N. \tag{11.10}$$

The optimal payoff to the Operator is

$$Op^*_{N,1} := Op_{N,1}(\mathbf{g}^*, (\mathbf{q}^*, \mathbf{t}^*)) = t_0 - t^*_1(i) = \frac{t_0 - t_2}{1 + \left(\sum_{\ell=1}^{N} \frac{1}{1-\beta_\ell}\right)^{-1}}. \tag{11.11}$$

Proof. It is clear from (11.8) – (11.10) that $\mathbf{g}^* \in G_{N,1}$ and $(\mathbf{q}^*, \mathbf{t}^*) \in Q_{N,1} \times \mathcal{T}_{N,1}$.

Again, we have to show that, in analogy to (19.10), the saddle point criterion

$$Op_{N,1}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) \leq Op^*_{N,1} \leq Op_{N,1}(\mathbf{g}^*, (\mathbf{q}, \mathbf{t})) \tag{11.12}$$

for any $\mathbf{g} \in G_{N,1}$ and any $(\mathbf{q}, \mathbf{t}) \in Q_{N,1} \times \mathcal{T}_{N,1}$ is fulfilled. We first prove the right hand inequality of (11.12). Let

$$C := \left(\sum_{\ell=1}^{N} \frac{1}{1-\beta_\ell}\right)^{-1}.$$

Then we have for all $i = 1, \ldots, N$, using (11.6), (11.8) and (11.11),

$$g_{2,i}^* \, H_i(\mathbf{t}) + \sum_{j \neq i} g_{2,j}^* \, (t_0 - t_2) + g_2^* \, (t_0 - t_1(i))$$

$$= \left(1 + \frac{1}{C}\right)^{-1} \left[t_1(i) - t_2 + \frac{\beta_i}{1 - \beta_i} \, (t_0 - t_2) + \left(\frac{1}{C} - \frac{1}{1 - \beta_i}\right) \, (t_0 - t_2) + (t_0 - t_1(i)) \right]$$

$$= \left(1 + \frac{1}{C}\right)^{-1} \left[t_1(i) - t_2 + \left(\frac{1}{C} - 1\right) \, (t_0 - t_2) + (t_0 - t_1(i)) \right]$$

$$= \left(1 + \frac{1}{C}\right)^{-1} \frac{1}{C} \, (t_0 - t_2)$$

$$= \frac{t_0 - t_2}{C + 1} = Op_{N,1}^* \, ,$$

i.e., by (11.7) the right hand inequality of (11.12) is fulfilled as equality. In order to prove the left hand inequality of (11.12) we first rearrange (11.7) by using (11.5) and get

$$Op_{N,1}(\mathbf{g}, (\mathbf{q}, \mathbf{t})) \tag{11.13}$$

$$= \sum_{i=1}^{N} g_{2,i} \left[q_i \, H_i(\mathbf{t}) + (t_0 - t_2) \sum_{\substack{j=1 \\ j \neq i}}^{N} q_j - \sum_{j=1}^{N} q_j \, (t_0 - t_1(i)) \right] + \sum_{i=1}^{N} q_i \, (t_0 - t_1(i)) \, .$$

By (11.10) we obtain for all $i = 1, \ldots, N$

$$t_1^*(i) - t_2 = -(t_0 - t_1^*) + (t_0 - t_2) = \frac{C}{1 + C} \, (t_0 - t_2)$$

and, using (11.6),

$$H_i(\mathbf{t}^*) = (1 - \beta_i) \, (t_1^*(i) - t_2) + \beta_i \, (t_0 - t_2) = \frac{C + \beta_i}{1 + C} \, (t_0 - t_2) \, . \tag{11.14}$$

Thus, (11.9), (11.10) and (11.14) imply for all $i = 1, \ldots, N$

$$q_i^* \, H_i(\mathbf{t}^*) + (t_0 - t_2) \sum_{j \neq i} q_j^* - \sum_{j=1}^{N} q_j^* \, (t_0 - t_1^*(i))$$

$$= \frac{C}{1 - \beta_i} \, \frac{C + \beta_i}{1 + C} \, (t_0 - t_2) + (t_0 - t_2) \, \left(1 - \frac{C}{1 - \beta_i}\right) - (t_0 - t_2) \, \frac{1}{1 + C} = 0 \, ,$$

which yields, using (11.13),

$$Op_{N,1}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) = \sum_{i=1}^{N} q_i^* \, (t_0 - t_1^*(i)) = \frac{t_0 - t_2}{C + 1} = Op_{N,1}^* \, ,$$

i.e., the left hand inequality of (11.12) is also fulfilled as equality. □

Let us comment the results of Theorem 11.1: First, the 20% requirement mentioned on p. 210 leads for $N \leq 5$, using (11.9), to conditions on β_i, which may look complicated . For $N = 2$ facilities we obtain by (11.9) for the Inspectorate's optimal facility selection probabilities

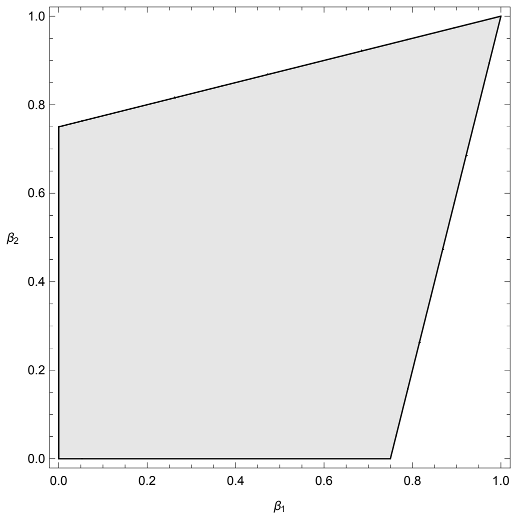
$$q_1^* = \frac{1-\beta_2}{2-\beta_1-\beta_2} \qquad \text{and} \qquad q_2^* = \frac{1-\beta_1}{2-\beta_1-\beta_2}\,.$$

The requirement $q_1^* \geq 0.2$ and $q_2^* \geq 0.2$ is equivalent to

$$4\,\beta_1 - 3 \leq \beta_2 \leq \frac{1}{4}\,(3+\beta_1)\,. \tag{11.15}$$

All pairs (β_1, β_2) fulfilling (11.15) are depicted in the grey area of Figure 11.1. One can see, that for instance in case of $\beta_1 = 0$ and $\beta_2 = 0.8$, the 20% requirement cannot be fulfilled. Consequently, the non-detection probability in facility 2 has to be decreased in order to fulfil it. Strangely enough, one can also increase the non-detection probability in facility 1, which seems to be counter-intuitive. The reason for this strange property is that we consider one interim inspection; however, we do not fix the inspection effort which influences β_1 and β_2. Of course, the optimal expected detection time grows with growing β's.

Figure 11.1 Pairs (β_1, β_2) fulfilling the 20% requirement in case of $N = 2$ facilities.



Second, in case only $N = 1$ facility is considered, we have $\beta_1 = \beta$ and $q_1^* = 1$, and the results for Theorem 11.1 simplify to that of Lemma 10.1. Like in Lemma 10.1, our analysis leads to an explicit dependence of the optimal interim inspection time point $t_1^*(i)$ and the optimal facility selection probability q_i^* on β_i, $i = 1, \ldots, N$. Because of (11.10), the optimal interim inspection time point $t_1^*(i)$ does not depend on i, which is surprising, since the non-detection probabilities are facility-dependent. Note that like in Lemma 10.1 the optimal expected detection time is the time between the interim inspection and the final PIV. To start the illegal activity after the interim inspection at time point $t_1^*(i)$, however, would not be an optimal strategy; see p. 188 and the comment on p. 232.

Third, note that both the optimal facility selection probabilities q_i^* and the probabilities $g_{2,i}^*$ of the Operator for starting the illegal activity immediately at t_2 are inversely proportional to the detection probabilities $1-\beta_i$, $i=1,\ldots,N$, in these facilities. In other words, both players put their emphases on facilities with small detection probabilities. This feature is well known from other inspection games; see Avenhaus and Canty (1996), Chapter 6.

Fourth, again the optimal strategy of the Inspectorate is a pure strategy, i.e., $t_1^*(i)$ is deterministic, and the Inspectorate can announce this time point if it wishes so. In analogy to what has been said on p. 189, the Inspectorate can also randomize the interim inspection time point.

Fifth, the more facilities are considered the earlier the interim inspection will be performed, because (11.10) yields for all $i=1,\ldots,N$ and any $\beta_i \in [0,1)$

$$t_0 - t_2 = (t_0 - t_1^*(i)) \left(1 + \left(\sum_{\ell=1}^{N} \frac{1}{1-\beta_\ell}\right)^{-1}\right) \leq (t_0 - t_1^*(i)) \left(1 + \frac{1}{N}\right),$$

i.e., the optimal interim inspection time point $t_1^*(i)$ tends to t_2 for all $i=1,\ldots,N$ and $N\to\infty$. This is plausible, since the Operator will start his illegal activity with probability one right at the beginning of the reference time interval: (11.8) leads for any $\beta_i \in [0,1)$ to

$$g_2^* = \left(1 + \sum_{\ell=1}^{N} \frac{1}{1-\beta_\ell}\right)^{-1} \leq \frac{1}{N+1},$$

i.e., the optimal probability g_2^* tend to zero for $N\to\infty$. The inspector reacts on his part with an early interim inspection time point $t_1^*(i)$.

Sixth, if the non-detection probabilities are facility-independent, i.e., $\beta_i=\beta$ for all $i=1,\ldots,N$, then we have

$$\sum_{\ell=1}^{N} \frac{1}{1-\beta_\ell} = \frac{N}{1-\beta},$$

and (11.8) – (11.10) imply for the Operator's and the Inspectorate's optimal strategies:

$$g_{2,i}^* = \frac{1}{N+1-\beta} \quad \text{for} \quad i=1,\ldots,N \quad \text{and} \quad g_2^* = \frac{1-\beta}{N+1-\beta} \tag{11.16}$$

and

$$t_0 - t_1^*(i) = \frac{N}{N+1-\beta}(t_0 - t_2) \quad \text{and} \quad q_i^* = \frac{1}{N} \quad \text{for} \quad i=1,\ldots,N. \tag{11.17}$$

Both results coincide as expected with the results of Theorem 11.2 for $k=1$ interim inspection, with the results in Avenhaus and Krieger (2013b), and for $\beta=0$ with the results in Krieger (2010).

Finally, and just in order to demonstrate the variety of models which have already been analysed, let us mention another variant in which the Inspectorate's set of strategies is given by (11.2), whereas the Operator selects first the facility for the illegal activity and only thereafter decides to start the latter one immediately or not. A justification for this assumption could be the following: While in the game discussed in this chapter the Operator is assumed to be able to start the illegal activity immediately, in this variant the Operator needs time to prepare the illegal activity in the facility chosen at first. This variant has been analysed for two facilities with different detection probabilities; see Avenhaus and Krieger (2012). Although both models are different from a strategic point of view, it turned out that the optimal interim inspection time points are the same for both variants.

11.2 Two interim inspections; facility-independent errors of the second kind

Because the case of one interim inspection in N facilities with *facility-independent* detection probabilities $1-\beta$ is a special case of Theorem 11.1 leading to the optimal strategies (11.16) and (11.17), we consider now the inspection game with $k=2$ interim inspections in N facilities, which is based on the following specification:

(v') During an interim inspection the Inspectorate may commit an *facility-independent* error of the second kind with probability β, i.e., the illegal activity, see assumption (iv'), started in facility i is not detected during the next interim inspection in that facility with probability β. Note that if there is no interim inspection left, then it is detected with certainty at the final PIV; see assumption (iii).

The model analysed in this section has its origin in the work of Krieger (2010) who considered the case of one and two interim inspections in N facilities with no errors of the second kind, i.e., $\beta_1 = \ldots = \beta_N = 0$. In contrast to his work, we will include here errors of the second kind that are assumed to be *facility-independent*, i.e., $\beta_1 = \ldots = \beta_N =: \beta \geq 0$. The results of this section have been published by Avenhaus and Krieger (2013b).

Note that the comment made after (11.1) on the different meaning of q_j in Part I and this chapter, the comment after (11.2) regarding the relation between $\mathcal{T}_{N,1}$ and $\mathcal{T}_1$, and the comment after (11.7) on the different meaning of $Op_{N,1}$ in Part I and this chapter, apply – accordingly modified – also here, and are not repeated.

According to assumption (vii'), the Inspectorate chooses at the beginning of the reference time interval, i.e., at t_3, the facilities $(i_2, i_1) \in \{1, \ldots, N\}^2$ with probability $q_{(i_2,i_1)}$, where i_2 indicates the facility for the first and i_1 the for the second interim inspection. Define

$$Q_{N,2} := \Big\{ \mathbf{q} := (q_{(1,1)}, \ldots, q_{(N,N)}) \in [0,1]^{N^2} : \sum_{(i_2,i_1)\in\{1,\ldots,N\}^2} q_{(i_2,i_1)} = 1 \Big\} . \tag{11.18}$$

Depending on the choice (i_2, i_1) of the two facilities to be inspected, the Inspectorate chooses the time point $t_2(i_2, i_1)$ for the first and $t_1(i_2, i_1)$ for the second interim inspection. From the modelling point of view it is important that the time points are supposed to be dependent on (i_2, i_1), since it cannot be excluded a priori that, e.g., $t_2(1,1)$ is different from $t_2(1,2)$. Keeping in mind that due to assumption (ii) of Chapter 8 the Inspectorate cannot perform the two interim inspections at the same time – yet, we will come back to this point on p. 230 –, the strategy set concerning the time points of the interim inspections is given by

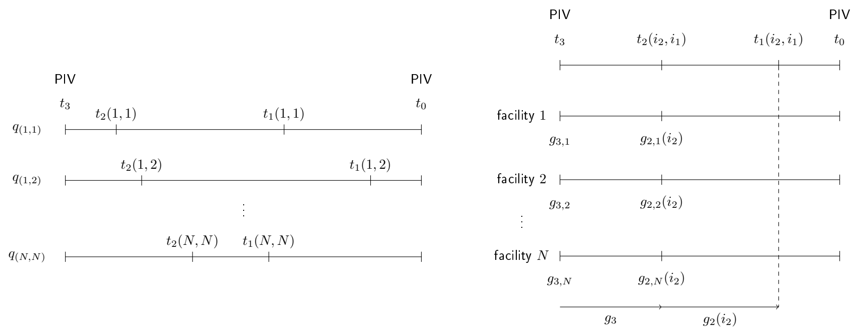
$$\mathcal{T}_{N,2} := \Big\{ \mathbf{t} = (t_2, t_1) : \{1, \ldots, N\}^2 \to \mathbb{R}^2 : t_3 < t_2(i_2, i_1) < t_1(i_2, i_1) < t_0 \quad \text{for any} \quad (i_2, i_1) \in \{1, \ldots, N\}^2 \Big\} . \tag{11.19}$$

(11.18) and (11.19) together define the Inspectorate's strategy set

$$Q_{N,2} \times \mathcal{T}_{N,2}\,. \tag{11.20}$$

According to assumption (vii'), the Operator's behaviour is formalized as follows: Let $g_{3,i}$ and g_3 be the probabilities that the Operator starts at t_3 the illegal activity in facility i, $i = 1, \ldots, N$, or postpones its start. Using (11.4) we have $\mathbf{g}_3 := (g_{3,1}, \ldots, g_{3,N}, g_3) \in G_N^{(3)}$. In case the illegal activity is postponed, the Operator observes the time point $t_2 = t_2(i_2, i_1)$ of the first interim inspection and the respective facility i_2 in which it takes place; of course he does not know the time point i_1 of the second interim inspection. Depending on i_2 and t_2 he chooses the probabilities $g_{2,i}(i_2, t_2)$, $i = 1, \ldots, N$, for starting the illegal activity in facility i at t_2 and $g_2(i_2, t_2)$ to postpone its start again. The Operator starts the illegal activity with certainty at $t_1(i_2, i_1)$ in either facility if he does not do so before. The time lines of the interim inspection time points and the facility selection probabilities as well as the probabilities for starting or postponing the illegal activity in different facilities are presented in Figure 11.2, where for brevity $g_{2,i}(i_2, t_2)$ and $g_2(i_2, t_2)$ are abbreviated by $g_{2,i}(i_2)$ and $g_2(i_2)$, respectively.

Figure 11.2 Time lines of the interim inspection time points and the facility selection probabilities (left) as well as the probabilities for starting or postponing the illegal activity in different facilities (right).



Note that while in Section 10.13 the probability $g_2(t_2)$ is only a function of t_2 (because only one facility is considered), here is becomes a function of the facility of the first interim inspection i_2 and the time point t_2: $g_2(i_2, t_2)$.

Using (11.4), define for any $t \in \mathbb{R}$ with $t < t_0$ and all $n = 2, 3, \ldots$ the set $G_n(t)$ of all functions mapping a pair $\{1, \ldots, N\} \times (t, t_0)$, i.e., a facility and a time point, onto $G_N^{(n)}$:

$$G_n(t) := \left\{ \mathbf{g}_n : \{1, \ldots, N\} \times (t, t_0) \to G_N^{(n)} \right\}\,. \tag{11.21}$$

Then the Operator's strategy set is given by

$$G_{N,2} := G_N^{(3)} \times G_2(t_3)\,. \tag{11.22}$$

An element of $G_{N,2}$ is denoted as $\mathbf{g} := (\mathbf{g}_3, \mathbf{g}_2)$. Defining the indicator function by

$$\mathbb{1}_i(j) := \left\{ \begin{array}{lcl} 0 & : & i \neq j \\ 1 & : & i = j \end{array} \right., \tag{11.23}$$

then the payoff to the Operator can be derived, using Figure 11.2 and (11.23), as follows: Let $(i_2, i_1) \in \{1, \ldots, N\}^2$ be a pair of facilities to be inspected. The following types of detection times, i.e., differences between interim inspection time points, can occur:

- In order to get the detection time $t_0 - t_1(i_2, i_1)$, the illegal activity has to be started at time point $t_1(i_2, i_1)$. The probability of this event is given by
$$g_3\, g_2(i_2, t_2)\,.$$
- In order to get the detection time $t_0 - t_2(i_2, i_1)$, the illegal activity has to be started at time point $t_2(i_2, i_1)$ in one of the N facilities, say the r-th one, and is not detected at the next interim inspection at time points t_1 in case it is performed in the r-th facility. Thus, the probability that the illegal activity is only detected at the final PIV is given by
$$g_3 \sum_{r=1}^{N} g_{2,r}(i_2, t_2)\, \beta^{\mathbb{1}_{i_1}(r)}\,.$$
- In order to get the detection time $t_0 - t_3$, the illegal activity has to be started at time point t_3 in one of the N facilities, say the r-th one, and is not detected at any of the two interim inspections at time points t_2 and t_1. Because $\sum_{j=1}^{2} \mathbb{1}_{i_j}(r)$ is the number of interim inspections taking place in facility r and because of the stochastic independence of the non-detection events, the probability that the illegal activity is only detected at the final PIV is given by
$$\sum_{r=1}^{N} g_{3,r}\, \beta^{\sum_{j=1}^{2} \mathbb{1}_{i_j}(r)}\,.$$
- In order to get the detection time $t_1(i_2, i_1) - t_2(i_2, i_1)$, the illegal activity has to be started at time point t_2 in facility i_1 and has to be detected at time point t_1 (with probability $1 - \beta$). The probability of this event is given by
$$g_3\, g_{2,i_1}(i_2, t_2)\, (1 - \beta)\,.$$
- In order to get the detection time $t_1(i_2, i_1) - t_3$, the illegal activity has to be started at time point t_3 in facility i_1 and has to be detected at time point t_1 (with probability $1 - \beta$). Because $\mathbb{1}_{i_2}(i_1)$ indicates if the first interim inspection is performed in facility i_1, the probability of this event is given by
$$g_{3,i_1}\, \beta^{\mathbb{1}_{i_2}(i_1)}\, (1 - \beta)\,.$$
- In order to get the detection time $t_2(i_2, i_1) - t_3$, the illegal activity has to be started at time point t_3 in facility i_2 and has to be detected at time point t_2 (with probability $1 - \beta$). The probability of this event is given by
$$g_{3,i_2}\, (1 - \beta)\,.$$

If we write for the sake of clarity in the following equation only t_2 and t_1 instead of $t_2(i_2, i_1)$ and $t_1(i_2, i_1)$ and only $g_{2,i}(i_2)$ and $g_2(i_2)$ instead of $g_{2,i}(i_2, t_2)$ and $g_2(i_2, t_2)$, then the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $\mathbf{g} \in G_{N,2}$ and any $(\mathbf{q}, \mathbf{t}) \in Q_{N,2} \times \mathcal{T}_{N,2}$, given by

$$
\begin{aligned}
Op_{N,2}(\mathbf{g}, (\mathbf{q}, \mathbf{t})) := \sum_{(i_2,i_1) \in \{1,\ldots,N\}^2} q_{(i_2,i_1)} \Big[& (t_0 - t_1)\, g_3\, g_2(i_2) \\
& + (t_0 - t_2)\, g_3 \sum_{r=1}^{N} g_{2,r}(i_2)\, \beta^{\mathbb{1}_{i_1}(r)} + (t_0 - t_3) \sum_{r=1}^{N} g_{3,r}\, \beta^{\sum_{j=1}^{2} \mathbb{1}_{i_j}(r)} \\
& + (t_1 - t_2)\, g_3\, g_{2,i_1}(i_2)\, (1-\beta) + (t_1 - t_3)\, g_{3,i_1}\, \beta^{\mathbb{1}_{i_2}(i_1)}\, (1-\beta) \\
& + (t_2 - t_3)\, g_{3,i_2}\, (1-\beta) \Big] \,.
\end{aligned} \tag{11.24}
$$

The game theoretical solution of this inspection game, see Krieger (2010) for the case $\beta = 0$ and Avenhaus and Krieger (2013b) for the case $\beta \geq 0$, is presented in Lemma 11.1. Even though this Lemma is a special case of Theorem 11.2, we present its proof in detail as it is helpful for understanding the proof of Theorem 11.2.

Lemma 11.1. *Given the Se-No inspection game on the reference time interval $[t_3, t_0]$ with $k = 2$ interim inspections in $N \geq 2$ facilities, and with facility-independent errors of the second kind, i.e., $\beta_1 = \beta_2 = \ldots = \beta_N =: \beta \geq 0$. The Operator's set of behavioural strategies is given by (11.22), the Inspectorate's strategy set by (11.20), and the payoff to the Operator by (11.24).*

Then an optimal strategy of the Operator is given by

$$
g_{3,i}^* = \frac{1}{N + 2\,(1-\beta)} \qquad \textit{for} \qquad i = 1, \ldots, N\,, \qquad g_3^* = \frac{2\,(1-\beta)}{N + 2\,(1-\beta)}\,, \tag{11.25}
$$

and, for all $i = 1, \ldots, N$, for all $i_2 = 1, \ldots, N$, and for all $t_3 < t_2 < t_0$, by

$$
g_{2,i}^*(i_2, t_2) = \frac{1}{2}\, \mathbb{1}_{i_2}(i) \qquad \textit{and} \qquad g_2^*(i_2, t_2) = \frac{1}{2}\,. \tag{11.26}
$$

An optimal strategy of the Inspectorate is for any $(i_2, i_1) \in \{1, \ldots, N\}^2$ given by the facility selection probabilities

$$
q_{(i_2,i_1)}^* = N^{-2} \tag{11.27}
$$

and by the interim inspection time points

$$
\begin{aligned}
t_2^*(i_2, i_1) - t_3 &= \frac{1-\beta}{N + 2\,(1-\beta)}\, (t_0 - t_3) \qquad \textit{and} \\
t_1^*(i_2, i_1) - t_2^*(i_2, i_1) &= \frac{1-\beta}{N + 1 - \beta}\, (t_0 - t_2^*(i_2, i_1))\,.
\end{aligned} \tag{11.28}
$$

The optimal payoff to the Operator is

$$
Op_{N,2}^* := Op_{N,2}(\mathbf{g}^*, (\mathbf{q}^*, \mathbf{t}^*)) = t_0 - t_1^*(i_2, i_1) = \frac{N}{N + 2\,(1-\beta)}\, (t_0 - t_3)\,. \tag{11.29}
$$

Proof. We have to show that, in analogy to (19.10), the saddle point criterion

$$Op_{N,2}(\mathbf{g},(\mathbf{q}^*,\mathbf{t}^*)) \leq Op^*_{N,2} \leq Op_{N,2}(\mathbf{g}^*,(\mathbf{q},\mathbf{t})) \tag{11.30}$$

is fulfilled for any $\mathbf{g} \in G_{N,2}$ and any $(\mathbf{q},\mathbf{t}) \in Q_{N,2} \times \mathcal{T}_{N,2}$. Because $\sum_{r=1}^N \mathbb{1}_{i_2}(r)\,\beta^{\mathbb{1}_{i_1}(r)} = \beta^{\mathbb{1}_{i_1}(i_2)}$, (11.24) implies, using (11.25) and (11.26),

$$\begin{aligned}
&(N+2\,(1-\beta))\,Op_{N,2}(\mathbf{g}^*,(\mathbf{q},\mathbf{t}))\\
&= \sum_{(i_2,i_1)\in\{1,\ldots,N\}^2} q_{(i_2,i_1)}\,\Big[(t_0-t_1)\,(1-\beta)\\
&+(t_0-t_2)\,(1-\beta)\sum_{r=1}^N \mathbb{1}_{i_2}(r)\,\beta^{\mathbb{1}_{i_1}(r)} + (t_0-t_3)\sum_{r=1}^N \beta^{\sum_{j=1}^2 \mathbb{1}_{i_j}(r)}\\
&+(t_1-t_2)\,\mathbb{1}_{i_2}(i_1)\,(1-\beta)^2 + (t_1-t_3)\,\beta^{\mathbb{1}_{i_2}(i_1)}\,(1-\beta) + (t_2-t_3)\,(1-\beta)\Big]\,.
\end{aligned} \tag{11.31}$$

Let $(i_2,i_1) \in \{1,\ldots,N\}^2$ be fixed. Then (11.31) yields for the coefficient A_1 of t_1

$$A_1 = -\,(1-\beta) + \mathbb{1}_{i_2}(i_1)\,(1-\beta)^2 + \beta^{\mathbb{1}_{i_2}(i_1)}\,(1-\beta) \tag{11.32}$$

$$= \begin{cases} -(1-\beta)+(1-\beta)=0 & \text{for } \; i_1 \neq i_2 \\ -(1-\beta)+(1-\beta)^2+\beta\,(1-\beta)=0 & \text{for } \; i_1 = i_2 \end{cases}\,. \tag{11.33}$$

For the coefficient A_2 of t_2 we obtain by making use of (11.31) and (11.32)

$$\begin{aligned}
A_2 &= -(1-\beta)\sum_{r=1}^N \mathbb{1}_{i_2}(r)\,\beta^{\mathbb{1}_{i_1}(r)} - \mathbb{1}_{i_2}(i_1)\,(1-\beta)^2 + (1-\beta)\\
&= -(1-\beta)\,\beta^{\mathbb{1}_{i_1}(i_2)} - \mathbb{1}_{i_2}(i_1)\,(1-\beta)^2 + (1-\beta) = -A_1\,.
\end{aligned}$$

The coefficient A_0 of t_0 is, using (11.33), given by

$$A_0 = (1-\beta) + (1-\beta)\,\beta^{\mathbb{1}_{i_1}(i_2)} + \sum_{r=1}^N \beta^{\sum_{j=1}^2 \mathbb{1}_{i_j}(r)}\,, \tag{11.34}$$

which evaluates to

$$A_0 = \begin{cases} (1-\beta)+(1-\beta)+(N-2)+2\,\beta = N & \text{for } \; i_1 \neq i_2 \\ (1-\beta)+(1-\beta)\,\beta+(N-1)+\beta^2 = N & \text{for } \; i_1 = i_2 \end{cases}\,.$$

Finally, (11.31) and (11.34) yield for the coefficient A_3 of t_3

$$A_3 = -\sum_{r=1}^N \beta^{\sum_{j=1}^2 \mathbb{1}_{i_j}(r)} - \beta^{\mathbb{1}_{i_2}(i_1)}\,(1-\beta) - (1-\beta) = -A_0\,.$$

Therefore, we have $A_3 = -A_0 = -N$, and (11.31) implies

$$(N+2\,(1-\beta))\,Op_{N,2}(\mathbf{g}^*,(\mathbf{q},\mathbf{t})) = \sum_{(i_2,i_1)\in\{1,\ldots,N\}^2} q_{(i_2,i_1)}\,\Big[A_0\,t_0 - A_3\,t_3\Big] = N\,(t_0-t_3)$$

for any $(\mathbf{q},\mathbf{t}) \in Q_{N,2} \times \mathcal{T}_{N,2}$, i.e., the right hand side of (11.30) is fulfilled as equality.

To prove the left hand inequality of (11.30), we first note that (11.28) – again suppressing (i_2, i_1) in t_n^*, $n = 1, 2$ – implies

$$t_2^* - t_3 = t_1^* - t_2^* = \frac{1-\beta}{N + 2\,(1-\beta)}\,(t_0 - t_3)\,.$$

Thus, we get

$$t_1^* - t_3 = t_1^* - t_2^* + t_2^* - t_3 = 2\,(t_1^* - t_2^*) = 2\,(t_0 - t_3)\,\frac{1-\beta}{N + 2\,(1-\beta)}\,, \tag{11.35}$$

$$t_0 - t_2^* = t_0 - t_3 - (t_2^* - t_3) = (t_0 - t_3)\,\frac{N + 1 - \beta}{N + 2\,(1-\beta)}\,,$$

and furthermore, by (11.35),

$$t_0 - t_1^* = t_0 - t_3 - (t_1^* - t_3) = (t_0 - t_3)\left(1 - \frac{2\,(1-\beta)}{N + 2\,(1-\beta)}\right) = (t_0 - t_3)\,\frac{N}{N + 2\,(1-\beta)}\,.$$

Using these relations and (11.27), (11.24) leads to

$$\begin{aligned}
&\frac{N^2\,(N + 2\,(1-\beta))}{(t_0 - t_3)}\,Op_{N,2}(\mathbf{g},(\mathbf{q}^*,\mathbf{t}^*)) = \sum_{(i_2,i_1)\in\{1,\ldots,N\}^2} \Big[N\,g_3\,g_2(i_2) \\
&+ (N + 1 - \beta)\,g_3 \sum_{r=1}^{N} g_{2,r}(i_2)\,\beta^{\mathbb{1}_{i_1}(r)} + (N + 2\,(1-\beta)) \sum_{r=1}^{N} g_{3,r}\,\beta^{\sum_{j=1}^{2}\mathbb{1}_{i_j}(r)} \\
&+ g_3\,g_{2,i_1}(i_2)\,(1-\beta)^2 + 2\,g_{3,i_1}\,\beta^{\mathbb{1}_{i_2}(i_1)}\,(1-\beta)^2 + g_{3,i_2}\,(1-\beta)^2\Big]\,,
\end{aligned} \tag{11.36}$$

where instead of $g_{2,i}(i_2, t_2)$ and $g_2(i_2, t_2)$ only write $g_{2,i}(i_2)$ and $g_2(i_2)$. We first consider the sum over all terms containing $g_{2,\cdot}(i_2)$ and $g_2(i_2)$ and get because they only depend on i_2 and because $\sum_{i_1=1}^{N} \beta^{\mathbb{1}_{i_1}(r)} = N - 1 + \beta$

$$\begin{aligned}
&g_3 \sum_{(i_2,i_1)\in\{1,\ldots,N\}^2} \Big[N\,g_2(i_2) + (N + 1 - \beta) \sum_{r=1}^{N} g_{2,r}(i_2)\,\beta^{\mathbb{1}_{i_1}(r)} + g_{2,i_1}(i_2)\,(1-\beta)^2\Big] \\
&= g_3 \sum_{i_2=1}^{N} \Big[N^2\,g_2(i_2) + (N + 1 - \beta) \sum_{r=1}^{N} g_{2,r}(i_2) \sum_{i_1=1}^{N} \beta^{\mathbb{1}_{i_1}(r)} + (1 - g_2(i_2))\,(1-\beta)^2\Big] \\
&= g_3 \sum_{i_2=1}^{N} \Big[N^2\,g_2(i_2) + (N + 1 - \beta)\,(N - 1 + \beta)\,(1 - g_2(i_2)) + (1 - g_2(i_2))\,(1-\beta)^2\Big] \\
&= g_3\,N^3\,.
\end{aligned} \tag{11.37}$$

Thus, the right hand side of (11.36) simplifies by (11.37) to

$$g_3\,N^3 + \sum_{(i_2,i_1)\in\{1,\dots,N\}^2} \Big[(N+2\,(1-\beta))\sum_{r=1}^{N} g_{3,r}\,\beta^{\sum_{j=1}^{2}\mathbb{1}_{i_j}(r)} + 2\,g_{3,i_1}\,\beta^{\mathbb{1}_{i_2}(i_1)}\,(1-\beta)^2 + g_{3,i_2}\,(1-\beta)^2\Big]. \quad (11.38)$$

Because we have for fixed $r\in\{1,\dots,N\}$

$$\sum_{j=1}^{2}\mathbb{1}_{i_j}(r) = \begin{cases} 0 & \text{for } \; i_2\neq r \text{ and } i_1\neq r \\ 1 & \text{for } \; i_2=r\neq i_1 \text{ or } i_1=r\neq i_2 \\ 2 & \text{for } \; i_2=i_1=r \end{cases}, \quad (11.39)$$

there are $(N-1)^2$ pairs (i_2,i_1) leading to β^0, $2\,(N-1)$ pairs leading to β^1, and only one pair, namely (r,r), leading to β^2. Thus, we get

$$\sum_{(i_2,i_1)\in\{1,\dots,N\}^2}\sum_{r=1}^{N} g_{3,r}\,\beta^{\sum_{j=1}^{2}\mathbb{1}_{i_j}(r)} = \sum_{r=1}^{N} g_{3,r}\,\left[(N-1)^2+2\,(N-1)\,\beta+\beta^2\right] = (1-g_3)\,(N-1+\beta)^2\,.$$

The two remaining terms in (11.38) simplify to

$$\sum_{(i_2,i_1)\in\{1,\dots,N\}^2} g_{3,i_1}\,\beta^{\mathbb{1}_{i_2}(i_1)} = \sum_{i_2=1}^{N} g_{3,i_2}\,\beta + \sum_{i_2=1}^{N}\sum_{\substack{i_1=1\\ i_1\neq i_2}}^{N} g_{3,i_1}$$

$$= (1-g_3)\,\beta + \sum_{i_2=1}^{N}(1-g_3-g_{3,i_2}) = (1-g_3)\,\beta + N\,(1-g_3) - (1-g_3)$$

$$= (1-g_3)\,(N-1+\beta)$$

and

$$\sum_{(i_2,i_1)\in\{1,\dots,N\}^2} g_{3,i_2} = \sum_{i_1=1}^{N}(1-g_3) = N\,(1-g_3)\,.$$

Thus, we finally get by (11.36) and (11.38)

$$\frac{N^2\,(N+2\,(1-\beta))}{(t_0-t_3)}\,Op_{N,2}(\mathbf{g},(\mathbf{q}^*,\mathbf{t}^*)) = N^3$$

for any $\mathbf{g}\in G_{N,2}$, i.e., the left hand side of (11.30) is also fulfilled as equality. □

Note that interesting properties of the optimal strategies, such as the Operator starts the illegal activity at time point t_2 only in the facility inspected at that time point, see (11.26), and of the optimal payoff are discussed after the proof of Theorem 11.2.

Let us comment the results of Lemma 11.1: First, the optimal expected detection time (11.29) is for $N=2$ facilities the same as that for the case of one facility and $k=1$ interim inspection, i.e., $N = k = 1$; see (10.8). This property suggests that the Inspectorate may consider in case $N = k = 2$ the two facilities independently and place the two interim inspections in the two facilities at the same time point t^* given by (10.7), of course with reference to $[t_3, t_0]$. This result is intuitive, but it represents a modelling problem: If the two interim inspections take place at the same time, then it makes no sense that they are performed in the same facility. This possibility, however, is not yet excluded from the strategy set of the Inspectorate. In Lemma 11.3 it will be shown that this time point t^* given by (10.7) is also an optimal strategy, if we do not maintain assumption (ii) of Chapter 8 that the two interim inspections cannot take place at the same time.

Second, there is another reason for mentioning this additional optimal strategy in the appropriately modified game: Since we consider the N facilities to be of the same type, one might guess that the optimal facility selection probability given by (11.27) is obvious, in other words, that this equal distribution strategy might be taken a priori, thus reducing the Inspectorate's strategy set, which, however, would not really reduce the complexity of the problem. We have demonstrated in this monograph that one is well advised to consider a rather general set of strategies of the Inspectorate, since one might miss other optimal strategies which may be interesting from a practitioners' point of view; see the discrete time Se-No inspection game in Section 4.2, where, using (4.35) and Lemma 4.4, the equal distribution $\binom{N}{k}^{-1}$ on the set $J_{N,k}$, see (4.28), is only one element of the uncountable set of optimal strategies of the Inspectorate.

Third, let us consider an analyst who wants to generalize the result obtained for the case of $k = 2$ interim inspections in $N = 1$ facility as given in Lemma 10.2 to any number N of facilities.[1] How could he proceed? He could argue that because all facilities are of the same type, the probability to perform inspections in the facilities (i_2, i_1) should be N^{-2} as in this case none of the facilities is preferred. This argument yields (11.27). He could also assume that the time differences between interim inspections $t_k^* - t_{k+1}, t_{k-1}^* - t_k^*, \ldots, t_1^* - t_2^*$ are all equal. However, he would have to take into account that the detection probability at the next interim inspection is now $(1 - \beta)/N$, because the events "detection of the illegal activity" and "selection of the facility in which the illegal activity has been started at t_3 resp. t_2" are independent. Thus, instead of (10.18), he would consider the time points – replace $1 - \beta$ in (10.18) by $(1 - \beta)/N$ –

$$t_2^* - t_3 = \frac{(1-\beta)/N}{1 + 2\,(1-\beta)/N}\,(t_0 - t_3) \qquad \text{and} \qquad t_1^* - t_2^* = \frac{(1-\beta)/N}{1 + (1-\beta)/N}\,(t_0 - t_2^*)$$

as optimal interim inspection time points, that coincides with (11.28). The same argument leads, using the left hand equation of (10.17), to g_3^* in (11.25). Furthermore, he distributes the probability $1 - g_3^*$ of starting the illegal activity at time point t_3 equally to all N facilities, i.e., the practitioner assumes, using again the left hand equation of (10.17), that

$$g_{3,i}^* = \frac{1}{N}\,(1 - g_3^*) = \frac{1}{N}\,\frac{1}{1 + 2\,(1-\beta)/N} = \frac{1}{N + 2\,(1-\beta)}\,,$$

i.e., the left hand equation of (11.25). Furthermore, it is easy to see that (10.19) with $(1-\beta)/N$ instead of $1 - \beta$ yields (11.29). Finally, the practitioner needs to define the probabilities $g_{2,i}^*(i_2, t_2)$ and $g_2^*(i_2, t_2)$. If he proceeds in the same way as before, then the right hand equality of (10.17) yields, we use $\tilde{g}_{2,i}^*$ and $\tilde{g}_2^*$ to avoid confusion, for all $i = 1, \ldots, N$, for all

[1] The following heuristic considerations have been provided by an unknown Referee to whom we thank here.

$i_2 = 1, \ldots, N$, and for all $t_3 < t_2 < t_0$,

$$\tilde{g}^*_{2,i}(i_2, t_2) = \frac{1}{2}\frac{1}{N} \qquad \text{and} \qquad \tilde{g}^*_2(i_2, t_2) = \frac{1}{2}\,, \tag{11.40}$$

which is obviously different from (11.26). In sum, the practitioner considers (11.25), (11.40), (11.27) and (11.28) as his heuristically derived optimal strategies.

Does his somehow convincing heuristic derivation lead to a saddle point of the game? As in proof the of Lemma 11.1, the left hand inequality in (11.30) is fulfilled as equality, i.e., $Op_{N,2}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) = Op^*_{N,2}$ for any $\mathbf{g} \in G_{N,2}$, because the heuristically derived $\mathbf{q}^*$ and $\mathbf{t}^*$ coincide with (11.27) and (11.28). With $\tilde{\mathbf{g}}^* := (\mathbf{g}^*_3, \tilde{\mathbf{g}}^*_2)$, where $\mathbf{g}^*_3$ resp. $\tilde{\mathbf{g}}^*_2$ are given by (11.25) resp. (11.40), we obtain

$$(N + 2\,(1-\beta))\, Op_{N,2}(\tilde{\mathbf{g}}^*, (\mathbf{q}, \mathbf{t})) =$$
$$N\,(t_0 - t_3) - (1-\beta)^2\,(t_0 - t_1)\left(\frac{1}{N} - \sum_{i_2=1}^{N} q_{(i_2,i_2)}\right)\,, \tag{11.41}$$

the derivation of which is provided in Section 23.1. Now, using (11.41), the right hand inequality of (11.30) is, because of $t_1 < t_0$ for any $\mathbf{t} = (t_2, t_1) \in \mathcal{T}_{N,2}$, equivalent to

$$\frac{1}{N} \le \sum_{i_2=1}^{N} q_{(i_2,i_2)}\,,$$

which is obviously not fulfilled for any $\mathbf{q} \in Q_{N,2}$. Thus, $(\mathbf{q}^*, \mathbf{t}^*)$ is not a best reply against $\tilde{\mathbf{g}}^*$, and therefore, the heuristic construction given above does *not* lead to a saddle point.

Wrapping up, caution has to be given when generalizing results from special cases to general ones or from simpler games to more complex ones even if the heuristic arguments seem convincing.

11.3 Any number of interim inspections; facility-independent errors of the second kind

Because the inspection game treated in this section is a generalization of that analysed in Section 11.2, it is also based on specification (v') on p. 214.

Looking once more at the payoff (11.24), one realizes that it would become very cumbersome to try to formulate it for the general case of k interim inspections in N facilities. Therefore, the proof of the conjecture formulated in Avenhaus and Krieger (2013b), see Table 11.1, seemed at that time out of reach. We will show now that with the help of a new formulation of the payoff indeed the conjecture can be proven.

Again, the comments made after (11.1), (11.2) and (11.7) apply – accordingly modified – also here; see p. 214.

According to assumption (vii'), the Inspectorate chooses at the beginning t_{k+1} of the reference time interval $[t_{k+1}, t_0]$ first the facilities $(i_k, \ldots, i_1) \in \{1, \ldots, N\}^k$ to be inspected during the k interim inspections, where i_n denotes the facility inspected at the $(k-n+1)$-th interim

Table 11.1 Normalized optimal expected detection times $Op^*_{N,k}/(t_0 - t_{k+1})$ as function of N, k and β; see Avenhaus and Krieger (2013b). Entries in the shaded area were conjectured.

k \ N	1	2	3	4	5	$\cdots$
1	$\frac{1}{2-\beta}$	$\frac{2}{3-\beta}$	$\frac{3}{4-\beta}$	$\frac{4}{5-\beta}$	$\frac{5}{6-\beta}$	$\cdots$
2	$\frac{1}{3-2\beta}$	$\frac{2}{4-2\beta}$	$\frac{3}{5-2\beta}$	$\frac{4}{6-2\beta}$	$\frac{5}{7-2\beta}$	$\cdots$
3	$\frac{1}{4-3\beta}$	$\frac{2}{5-3\beta}$	$\frac{3}{6-3\beta}$	$\frac{4}{7-3\beta}$	$\frac{5}{8-3\beta}$	$\cdots$
4	$\frac{1}{5-4\beta}$	$\frac{2}{6-4\beta}$	$\frac{3}{7-4\beta}$	$\frac{4}{8-4\beta}$	$\frac{5}{9-4\beta}$	$\cdots$
5	$\frac{1}{6-5\beta}$	$\frac{2}{7-5\beta}$	$\frac{3}{8-5\beta}$	$\frac{4}{9-5\beta}$	$\frac{5}{10-5\beta}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

inspection, $n = 1, \ldots, k$. Obviously, there are N^k possibilities to place the interim inspections in the facilities, i.e., in the space.

Let $q_{(i_k,\ldots,i_1)}$, $(i_k, \ldots, i_1) \in \{1, \ldots, N\}^k$, be the Inspectorate's probabilities of choosing the facilities $(i_k, \ldots, i_1)$ for the interim inspections. These probabilities are collected in the set $Q_{N,k}$ as follows

$$Q_{N,k} := \Big\{ \mathbf{q} := (q_{(1,\ldots,1)}, \ldots, q_{(N,\ldots,N)}) \in [0,1]^{N^k} : \sum_{(i_k,\ldots,i_1)\in\{1,\ldots,N\}^k} q_{(i_k,\ldots,i_1)} = 1 \Big\} . \tag{11.42}$$

Depending on the choice $(i_k, \ldots, i_1)$ of the facilities to be inspected, the Inspectorate chooses now – still at the beginning of the reference time interval – the facility-dependent time points $t_n(i_k, \ldots, i_1)$, $n = 1, \ldots, k$, for the interim inspections. Because of assumption (ii) of Chapter 8, the Inspectorate performs only one interim inspection at some time point, i.e., the $t_n(i_k, \ldots, i_1)$ are subject to the condition $t_{k+1} < t_k(i_k, \ldots, i_1) < \ldots < t_1(i_k, \ldots, i_1) < t_0$ for any $(i_k, \ldots, i_1) \in \{1, \ldots, N\}^k$. These time points are combined in $\mathbf{t} = (t_k, \ldots, t_1) : \{1, \ldots, N\}^k \to \mathbb{R}^k$ which assigns to each outcome $(i_k, \ldots, i_1)$ of the random experiment the interim inspection time points $t_k, \ldots, t_1$ fulfilling the above condition. Let the set of all functions mapping $\{1, \ldots, N\}^k$ onto $\mathbb{R}^k$ fulfilling the above condition be defined by

$$\begin{aligned} \mathcal{T}_{N,k} := \Big\{ & \mathbf{t} = (t_k, \ldots, t_1) : \{1, \ldots, N\}^k \to \mathbb{R}^k \quad \text{with} \\ & t_{k+1} < t_k(i_k, \ldots, i_1) < \ldots < t_1(i_k, \ldots, i_1) < t_0 \\ & \text{for any} \quad (i_k, \ldots, i_1) \in \{1, \ldots, N\}^k \Big\} . \end{aligned} \tag{11.43}$$

Therefore, using (11.42) and (11.43), the Inspectorate's strategy set is given by

$$Q_{N,k} \times \mathcal{T}_{N,k} \,. \tag{11.44}$$

The Operator's behaviour is, using assumption (vii'), formalized as follows: Let $g_{k+1,i}$ resp. g_{k+1} be the probabilities that the Operator starts at t_{k+1} the illegal activity in facility i, $i = 1, \dots, N$, resp. postpones its start. These probabilities are collected in the set $G_N^{(k+1)}$; see (11.4). Suppose the start of the illegal activity is postponed until time point t_n, $n = 2, \dots, k$, i.e., the $(k-n+1)$-th interim inspection. At that time point the Operator has full knowledge of the facilities $(i_k, \dots, i_n)$ which has been inspected so far and the respective time points $(t_k, \dots, t_n)$. Based on this information he starts the illegal activity in facility i with probability $g_{n,i}(i_n, t_n)$, $i = 1, \dots, N$, and postpones its start again with probability $g_n(i_n, t_n)$. Note that we assume that the probabilities $g_{n,i}$ and g_n only depend on i_n and t_n and not on the whole history $(i_k, \dots, i_n)$ and $(t_k, \dots, t_n)$; see the discussion on p. 193 for the case of $N = 1$ facility. Again, in case the Operator does not start the illegal activity before, he will do so with certainty at $t_1(i_k, \dots, i_1)$ in either facility. In sum, using (11.21), the Operator's set of behavioural strategies is given by

$$G_{N,k} := G_N^{(k+1)} \times_{n=2}^{k} G_n(t_{k+1}) \,, \tag{11.45}$$

which is a generalization of (11.22). An element of $G_{N,k}$ is denoted as $\mathbf{g} := (\mathbf{g}_{k+1}, \mathbf{g}_k, \dots, \mathbf{g}_2)$.

After having introduced the strategy set for both players we now determine the payoff to the Operator, i.e., according to (ix) the expected detection time $Op_{N,k}(\mathbf{g}, (\mathbf{q}, \mathbf{t}))$ in the following Lemma. For that purpose we define the indeterminate form 0^0 to be 1; see Knuth (1992).

Lemma 11.2. *Given the Se-No inspection game on the reference time interval $[t_{k+1}, t_0]$ with k interim inspections in $N \geq 2$ facilities, and with facility-independent errors of the second kind, i.e., $\beta_1 = \beta_2 = \ldots = \beta_N =: \beta \geq 0$. The Operator's set of behavioural strategies is given by (11.45) and the Inspectorate's strategy set by (11.44).*

Then the (expected) payoff to the Operator, i.e., the expected detection time, is, for any $\mathbf{g} \in G_{N,k}$ and any $(\mathbf{q}, \mathbf{t}) \in Q_{N,k} \times \mathcal{T}_{N,k}$, using (11.23), given by

$$\begin{aligned} Op_{N,k}(\mathbf{g}, (\mathbf{q}, \mathbf{t})) := \sum_{(i_k, \dots, i_1) \in \{1, \dots, N\}^k} & q_{(i_k, \dots, i_1)} \Big[(t_0 - t_1)\, w_2 + \\ & + \sum_{\ell=1}^{k} (t_\ell - t_{\ell+1})\, w_{\ell+2}\, g_{\ell+1, i_\ell}\, (1-\beta) \\ & + \sum_{m=2}^{k+1} (t_0 - t_m)\, w_{m+1} \sum_{r=1}^{N} g_{m,r}\, \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r)} \\ & + \sum_{\ell=1}^{k-1} \sum_{m=\ell+2}^{k+1} (t_\ell - t_m)\, w_{m+1}\, g_{m,i_\ell}\, \beta^{\sum_{j=\ell+1}^{m-1} \mathbb{1}_{i_j}(i_\ell)}\, (1-\beta) \Big] , \end{aligned} \tag{11.46}$$

where $t_n = t_n(i_k, \dots, i_1)$, $n = 1, \dots, k$, and where $w_\ell = w_\ell(i_k, \dots, i_\ell, t_k, \dots, t_\ell)$, $\ell = 2, \dots, k+1$, is the probability that the start of the illegal activity is postponed at time points

$t_{k+1}, \ldots, t_\ell$:

$$w_\ell := \begin{cases} g_{k+1} \prod_{n=\ell}^{k} g_n(i_n, t_n) & \textit{for} \quad \ell = 2, \ldots, k \\ g_{k+1} & \textit{for} \quad \ell = k+1 \\ 1 & \textit{for} \quad \ell = k+2 \end{cases} . \tag{11.47}$$

Proof. We choose a fixed combination of facilities $(i_k, \ldots, i_1) \in \{1, \ldots, N\}^k$ to be inspected and determine for all possible differences of detection times $t_\ell - t_m$, $0 \leq \ell < m \leq k+1$ the corresponding probabilities, i.e., we determine the distribution of the random variables *time between start and detection of the illegal activity*. Indeed, this new approach turned out to be the path to proving the conjecture formulated in Avenhaus and Krieger (2013b).

Four types of differences $t_\ell - t_m$ of interim inspection time points t_ℓ and t_m have to be considered which correspond to the four lines in (11.46) and which are represented in Table 11.2. Because $(i_k, \ldots, i_1)$ is fixed throughout the proof, we omit these arguments in the t_ns and in the w_ℓs.

Table 11.2 The four types of differences $t_\ell - t_m$ of interim inspection time points t_ℓ and t_m in case of k interim inspections.

ℓ \ m	1	2	3	4	$\cdots$	$k-2$	$k-1$	k	$k+1$
0	$\langle 1\rangle$	$\langle 3\rangle$	$\langle 3\rangle$	$\langle 3\rangle$	$\cdots$	$\langle 3\rangle$	$\langle 3\rangle$	$\langle 3\rangle$	$\langle 3\rangle$
1		$\langle 2\rangle$	$\langle 4\rangle$	$\langle 4\rangle$	$\cdots$	$\langle 4\rangle$	$\langle 4\rangle$	$\langle 4\rangle$	$\langle 4\rangle$
2			$\langle 2\rangle$	$\langle 4\rangle$	$\cdots$	$\langle 4\rangle$	$\langle 4\rangle$	$\langle 4\rangle$	$\langle 4\rangle$
3				$\langle 2\rangle$	$\cdots$	$\langle 4\rangle$	$\langle 4\rangle$	$\langle 4\rangle$	$\langle 4\rangle$
$\vdots$					$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-3$						$\langle 2\rangle$	$\langle 4\rangle$	$\langle 4\rangle$	$\langle 4\rangle$
$k-2$							$\langle 2\rangle$	$\langle 4\rangle$	$\langle 4\rangle$
$k-1$								$\langle 2\rangle$	$\langle 4\rangle$
k									$\langle 2\rangle$

First line in (11.46): In order to get the detection time $t_0 - t_1$, i.e., entry $\langle 1\rangle$ in Table 11.2, the illegal activity has to be started at time point t_1. The probability of this event is, using (11.47), given by

$$g_{k+1}\, g_k(i_k, t_k)\, \ldots\, g_2(i_2, t_2) = w_2\,.$$

Second line in (11.46): In order to get the detection time $t_\ell - t_{\ell+1}$, $\ell = 1, \ldots, k-1$, i.e., entries $\langle 2\rangle$ in Table 11.2, the illegal activity has to be started at time point $t_{\ell+1}$ in facility i_ℓ and has to be detected at time point t_ℓ (with probability $1-\beta$). The probability of this event is, using (11.47), given by

$$g_{k+1}\, g_k(i_k, t_k)\, \ldots\, g_{\ell+2}(i_{\ell+2}, t_{\ell+2})\, g_{\ell+1, i_\ell}(i_{\ell+1}, t_{\ell+1})\, (1-\beta)$$

$$= w_{\ell+2}\, g_{\ell+1,i_\ell}(i_{\ell+1}, t_{\ell+1})\,(1-\beta)\,.$$

In case of $\ell = k$, the illegal activity has to be started at time point t_{k+1} in facility i_k and has to be detected at time point t_k (with probability $1-\beta$). Using (11.47) again, the probability of this event is given by

$$g_{k+1,i_k}\,(1-\beta) = w_{k+2}\, g_{k+1,i_k}\,(1-\beta)\,.$$

Third line in (11.46): In order to get the detection time $t_0 - t_m$, $m = 2,\ldots,k$, i.e., entries $\langle 3\rangle$ in Table 11.2, the illegal activity has to be started at time point t_m in one of the N facilities, say the r-th one, and is not detected at any of the remaining interim inspections at time points $t_{m-1},\ldots,t_1$. Because $\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r)$ is the number of interim inspections after time point t_m taking place in facility r and because of the stochastic independence of the non-detection events, the probability, that the illegal activity is only detected at the final PIV, is given by

$$g_{m,r}(i_m, t_m)\,\beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r)}\,.$$

Thus, we finally get by (11.47)

$$w_{m+1} \sum_{r=1}^{N} g_{m,r}(i_m, t_m)\,\beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r)}\,. \tag{11.48}$$

For $m = k+1$ the probability w_{m+1} vanishes and we also get (11.48), because $w_{k+2} = 1$ by definition.

Fourth line in (11.46): In order to get the detection time $t_\ell - t_m$, $\ell = 1,\ldots,k-1, m = \ell+2,\ldots,k$, i.e., entries $\langle 4\rangle$ in Table 11.2, the illegal activity has to be started at time point t_m in facility i_ℓ and has to be detected at time point t_ℓ (with probability $1-\beta$). Because $\beta^{\sum_{j=\ell+1}^{m-1} \mathbb{1}_{i_j}(i_\ell)}$ gives the probability of not detecting the illegal activity during the interim inspections at time points $t_{\ell+1},\ldots,t_{m-1}$ which are carried out in facility i_ℓ, the probability to start the illegal activity at time point t_m in facility i_ℓ and to detect is at time point t_ℓ is, using (11.47), given by

$$w_{m+1}\, g_{m,i_\ell}(i_m, t_m)\,\beta^{\sum_{j=\ell+1}^{m-1} \mathbb{1}_{i_j}(i_\ell)}\,(1-\beta)\,. \tag{11.49}$$

Because $w_{k+2} = 1$, the case $m = k+1$ leads also to (11.49), which completes the proof. □

The game theoretical solution of this inspection game, which was conjectured in Avenhaus and Krieger (2013b), is presented in Theorem 11.2. It includes two Lemmata which are the subject of Section 23.2 and it is, admittedly, not easily to be understood. In view of the simple structure of the solution we do not exclude that one day a simpler and perhaps more intuitive proof can be found.

Theorem 11.2. *Given the Se-No inspection game on the reference time interval $[t_{k+1}, t_0]$ with k interim inspections in $N \geq 2$ facilities, and with facility-independent errors of the second kind, i.e., $\beta_1 = \beta_2 = \ldots = \beta_N =: \beta \geq 0$. The Operator's set of behavioural strategies is given by (11.45), the Inspectorate's strategy set by (11.44), and the payoff to the Operator by (11.46).*

Then an optimal strategy of the Operator is given by

$$g^*_{k+1,i} = \frac{1}{N + k\,(1-\beta)} \quad \textit{for} \quad i = 1,\ldots,N \quad \textit{and} \quad g^*_{k+1} = \frac{k\,(1-\beta)}{N + k\,(1-\beta)}\,, \tag{11.50}$$

and, for all $n = 2, \ldots, k$, for all $i = 1, \ldots, N$, for all $i_n = 1, \ldots, N$, and for all $t_{k+1} < t_n < t_0$, by

$$g^*_{n,i}(i_n, t_n) = \frac{1}{n}\, \mathbb{1}_{i_n}(i) \qquad \textit{and} \qquad g^*_n(i_n, t_n) = 1 - \frac{1}{n}\,. \tag{11.51}$$

An optimal strategy of the Inspectorate is for any $(i_k, \ldots, i_1) \in \{1, \ldots, N\}^k$ given by the facility selection probabilities

$$q^*_{(i_k,\ldots,i_1)} = N^{-k} \tag{11.52}$$

and, for all $n = 1, \ldots, k$, by the interim inspection time points

$$t^*_n(i_k, \ldots, i_1) - t^*_{n+1}(i_k, \ldots, i_1) = \frac{1-\beta}{N + n\,(1-\beta)}\,(t_0 - t^*_{n+1}(i_k, \ldots, i_1))\,, \tag{11.53}$$

where $t^*_{k+1}(i_k, \ldots, i_1) := t_{k+1}$.

The optimal payoff to the Operator is

$$Op^*_{N,k} := Op_{N,k}(\mathbf{g}^*, (\mathbf{q}^*, \mathbf{t}^*)) = t_0 - t^*_1(i_k, \ldots, i_1) = \frac{N}{N + k\,(1-\beta)}\,(t_0 - t_{k+1})\,. \tag{11.54}$$

Proof. In order to show that $\mathbf{g}^*$ and $(\mathbf{q}^*, \mathbf{t}^*)$ are optimal strategies we have to verify, in analogy to (19.10), that the saddle point criterion

$$Op_{N,k}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) \le Op^*_{N,k} \le Op_{N,k}(\mathbf{g}^*, (\mathbf{q}, \mathbf{t})) \tag{11.55}$$

is fulfilled for any $\mathbf{g} \in G_{N,k}$ and any $(\mathbf{q}, \mathbf{t}) \in Q_{N,k} \times \mathcal{T}_{N,k}$.

We first show that $Op_{N,k}(\mathbf{g}^*, (\mathbf{q}, \mathbf{t})) = Op^*_{N,k}$ for any $(\mathbf{q}, \mathbf{t}) \in Q_{N,k} \times \mathcal{T}_{N,k}$. For the sake of brevity we suppress the arguments in the following equations. Let $(i_k, \ldots, i_1) \in \{1, \ldots, N\}^k$ be a fixed but arbitrary combination of facilities to be inspected. By (11.50) and (11.51) we obtain for all $\ell = 2, \ldots, k$

$$w^*_\ell = g^*_{k+1} \prod_{j=\ell}^{k} g^*_j = \frac{k\,(1-\beta)}{N + k\,(1-\beta)}\,\frac{k-1}{k}\,\cdots\,\frac{\ell-1}{\ell} = \frac{(1-\beta)\,(\ell-1)}{N + k\,(1-\beta)}\,. \tag{11.56}$$

Note that (11.56) is also valid in case of $\ell = k+1$, where $\prod_{j=k+1}^{k} g^*_j =: 1$. Furthermore, (11.56), (11.50) and (11.51) lead to

$$w^*_{\ell+2}\, g^*_{\ell+1,i_\ell} = \begin{cases} \dfrac{1-\beta}{N + k\,(1-\beta)}\, \mathbb{1}_{i_{\ell+1}}(i_\ell) & : \quad \ell = 1, \ldots, k-1 \\ \dfrac{1}{N + k\,(1-\beta)} & : \quad \ell = k \end{cases}$$

and

$$w^*_{m+1} \sum_{r=1}^{N} g^*_{m,r}\, \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r)} = \begin{cases} \dfrac{1-\beta}{N + k\,(1-\beta)}\, \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(i_m)} & : \quad m = 2, \ldots, k \\ \dfrac{1}{N + k\,(1-\beta)}\, \sum_{r=1}^{N} \beta^{\sum_{j=1}^{k} \mathbb{1}_{i_j}(r)} & : \quad m = k+1 \end{cases}$$

as well as

$$w^*_{m+1}\, g^*_{m,i_\ell}\, \beta^{\sum_{j=\ell+1}^{m-1} \mathbb{1}_{i_j}(i_\ell)} = \begin{cases} \dfrac{1-\beta}{N+k\,(1-\beta)}\, \mathbb{1}_{i_m}(i_\ell)\, \beta^{\sum_{j=\ell+1}^{m-1} \mathbb{1}_{i_j}(i_\ell)} & : \quad m = 3, \dots, k \\ \dfrac{1}{N+k\,(1-\beta)}\, \beta^{\sum_{j=\ell+1}^{k} \mathbb{1}_{i_j}(i_\ell)} & : \quad m = k+1 \end{cases} .$$

Define for any $k \in \mathbb{N}$ with $k \geq 2$, any $N \in \mathbb{N}$, any $(i_k, \dots, i_1) \in \{1, \dots, N\}^k$ and arbitrary $t_{k+1}, \dots, t_0 \in \mathbb{R}$ the function

$$\begin{aligned} R(i_k, \dots, i_1) := {} & (t_0 - t_1) + \sum_{\ell=1}^{k-1} (t_\ell - t_{\ell+1})\, \mathbb{1}_{i_{\ell+1}}(i_\ell)\, (1-\beta) + (t_k - t_{k+1}) \\ & + \sum_{m=2}^{k} (t_0 - t_m)\, \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(i_m)} + (t_0 - t_{k+1})\, \frac{1}{1-\beta} \sum_{r=1}^{N} \beta^{\sum_{j=1}^{k} \mathbb{1}_{i_j}(r)} \\ & + \sum_{\ell=1}^{k-2} \sum_{m=\ell+2}^{k} (t_\ell - t_m)\, \mathbb{1}_{i_m}(i_\ell)\, \beta^{\sum_{j=\ell+1}^{m-1} \mathbb{1}_{i_j}(i_\ell)}\, (1-\beta) \\ & + \sum_{\ell=1}^{k-1} (t_\ell - t_{k+1})\, \beta^{\sum_{j=\ell+1}^{k} \mathbb{1}_{i_j}(i_\ell)} , \end{aligned} \tag{11.57}$$

where for $k = 2$ the sum in the third row is put to zero. Using (11.46) and (11.57) as well as the result (23.2) of Lemma 23.1, we get for any $(\mathbf{q}, \mathbf{t}) \in Q_{N,k} \times \mathcal{T}_{N,k}$

$$\begin{aligned} \frac{N+k\,(1-\beta)}{1-\beta}\, Op_{N,k}(\mathbf{g}^*, (\mathbf{q}, \mathbf{t})) &= \sum_{(i_k, \dots, i_1) \in \{1, \dots, N\}^k} q_{(i_k, \dots, i_1)}\, R(i_k, \dots, i_1) \\ &= \frac{N}{1-\beta}\, (t_0 - t_{k+1}) , \end{aligned} \tag{11.58}$$

which implies $Op_{N,k}(\mathbf{g}^*, (\mathbf{q}, \mathbf{t})) = Op^*_{N,k}$, i.e., the right hand inequality of (11.55) is fulfilled as equality.

We now prove that $Op_{N,k}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) = Op^*_{N,k}$ for any $\mathbf{g} \in G_{N,k}$. For a fixed but arbitrary combination of facilities $(i_k, \dots, i_1) \in \{1, \dots, N\}^k$ to be inspected we write for the sake of brevity only t^*_n instead of $t^*_n(i_k, \dots, i_1)$. We first prove that (11.53) implies

$$t^*_n = (k - n + 1)\, \frac{1-\beta}{N+k\,(1-\beta)}\, (t_0 - t_{k+1}) + t_{k+1} \qquad \text{for} \qquad n = 1, \dots, k , \tag{11.59}$$

which is shown by induction with respect to n. Because $t^*_{k+1}(i_k, \dots, i_1) := t_{k+1}$, (11.53) leads for $n = k$ directly to (11.59) for $n = k$. Suppose (11.59) is true for an $n = 1, \dots, k-1$, i.e., $n + 1 = 2, \dots, k$. Then (11.53) and (11.59) for $n \to n+1$ implies

$$\begin{aligned} t^*_n &= \frac{1-\beta}{N+n\,(1-\beta)}\, (t_0 - t^*_{n+1}) + t^*_{n+1} \\ &= \frac{(1-\beta)\,(t_0 - t_{k+1})}{N+n\,(1-\beta)} \left(1 - \frac{(1-\beta)\,(k-n)}{N+k\,(1-\beta)}\right) + \frac{(1-\beta)\,(k-n)}{N+k\,(1-\beta)}\, (t_0 - t_{k+1}) + t_{k+1} , \end{aligned}$$

which simplifies to (11.59). Also (11.59) yields

$$t_\ell^* - t_m^* = \frac{(m-\ell)\,(1-\beta)}{N + k\,(1-\beta)}\,(t_0 - t_{k+1}) \qquad \text{for} \qquad 1 \le \ell < m \le k+1 \tag{11.60}$$

and

$$t_0 - t_m^* = \frac{N + (m-1)\,(1-\beta)}{N + k\,(1-\beta)}\,(t_0 - t_{k+1}) \qquad \text{for} \qquad m = 1, \ldots, k\,. \tag{11.61}$$

Define for any $k \in \mathbb{N}$ with $k \ge 2$, any $N \in \mathbb{N}$ and any $\mathbf{g} \in G_{N,k}$ the function

$$\begin{aligned} L(\mathbf{g}) := \sum_{(i_k,\ldots,i_1)\in\{1,\ldots,N\}^k} \Big[& N\,w_2 + \sum_{\ell=1}^{k} w_{\ell+2}\, g_{\ell+1,i_\ell}\,(1-\beta)^2 \\ & + \sum_{m=2}^{k+1} (N + (m-1)\,(1-\beta))\,w_{m+1} \sum_{r=1}^{N} g_{m,r}\,\beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r)} \\ & + \sum_{\ell=1}^{k-1} \sum_{m=\ell+2}^{k+1} (m-\ell)\,w_{m+1}\,g_{m,i_\ell}\,\beta^{\sum_{j=\ell+1}^{m-1} \mathbb{1}_{i_j}(i_\ell)}\,(1-\beta)^2 \Big]\,. \end{aligned} \tag{11.62}$$

Then (11.46), (11.52), (11.60), (11.61) and (11.62) lead to

$$\frac{N^k\,(N + k\,(1-\beta))}{t_0 - t_{k+1}}\, Op_{N,k}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) = L(\mathbf{g})\,. \tag{11.63}$$

Making use of (23.18) in Lemma 23.2 we finally obtain $Op_{N,k}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) = Op_{N,k}^*$ for any $\mathbf{g} \in G_{N,k}$, i.e., the left hand inequality of (11.55) is also fulfilled as equality.

Using (11.54) and (11.61) for $m = 1$ leads to $Op_{N,k}^* = t_0 - t_1^*$, which completes the proof. □

Let us discuss the results of the Theorem 11.2, which also hold for Lemma 11.1: First, we think that the striking contrast between the complexity of the general inspection problem as expressed by the payoff (11.46) and the simplicity of its solution is particularly worth mentioning. Note that the case of $k = 1$ interim inspection in N facilities and $\beta_i = \beta$, $i = 1, \ldots, N$, that is treated in Theorem 11.1 leads to the same optimal strategies and optimal expected detection time as in Theorem 11.2, whereas (11.51) is omitted.

Second, the game theoretical analysis leads to an explicit dependence of the optimal interim inspection time points and the optimal facility selection probability on β, which could – at least in the case $\beta > 0$ and $N \ge 1$ – only be determined with the help of a quantitative analysis. The same holds for the Operator's optimal strategy: Who would have ever thought that the Operator starts the illegal activity at a time point t_n^* only in the facility inspected at that time point; see (11.51)? As in Theorem 11.1, the optimal interim inspection time points (11.53) do not depend on $(i_k, \ldots, i_1)$ which is less surprising here because of the facility-independent non-detection probability.

Third, it is interesting that the optimal interim inspection time point t_n^* depends on the length $t_0 - t_{k+1}$ of the reference time interval and β, while the optimal strategy of the Operator is only a function of β. It is intuitive, however, that $g_{k+1}^* \to 0$ and $t_n^* \to t_{k+1}$ with increasing β, see (11.50) and (11.59): For β close to 1 the Operator starts the illegal activity with probability close to 1 at time point t_{k+1} in one of the facilities. Consequently, the Inspectorate will perform its interim inspections also very early. Note that (11.59) implies

$$t_n^* - t_{n+1}^* = \frac{1-\beta}{N + k\,(1-\beta)}\,(t_0 - t_{k+1}) \qquad \text{for} \qquad n = 1, \ldots, k\,, \tag{11.64}$$

i.e., the time differences $t_n^* - t_{n+1}^*$, $n = 1, \ldots, k$, between two subsequent interim inspections are all equal. Note that the relation between (11.53) and (11.64) generalizes the results (22.2) and (22.3) of Lemma 22.1 to $N \geq 2$ facilities. From an application point of view it is interesting that the optimal interim inspection time points given by (11.53) are deterministic. In other words, the Inspectorate may announce the time points of its interim inspections – however, not the facilities to be inspected – if it wishes so; see the discussions on pp. 189 and 192.

Fourth, (11.54) implies again that the optimal expected detection time is the time between the last interim inspection and the end of the reference time interval; see also p. 232. The Operator could have started the illegal activity with certainty right after the last interim inspection at time point t_1, which, again, would not be an optimal strategy; see the discussion on p. 188.

Fifth, the expected number of interim inspections in one facility is by (11.52) given by

$$\sum_{i=1}^{k} i \binom{k}{i} \left(\frac{1}{N}\right)^i \left(1 - \frac{1}{N}\right)^{k-i} = \frac{k}{N}$$

as it was expected; see pp. 18 and 146.

Sixth, with (11.54) one sees that $Op_{N,k}^*$ only depends on the ratio k/N; for $k/N \ll 1$ it tends towards $t_0 - t_{k+1}$ whereas for $k/N \gg 1$ it tends towards zero, which is intuitive. Especially for $k = N$ one always get the same optimal expected detection time $Op_{N,k}^*$ as for the case $N = k = 1$. The latter property suggests that the Inspectorate may consider the N facilities independently and place the $k = N$ interim inspections in the facilities at the same time point. Indeed, we show that this strategy is also an optimal strategy. There is, however, a modelling problem which we mentioned already after Lemma 11.1. So far, two or more interim inspections cannot be performed at the same time point; see assumption (ii) of Chapter 8. If we admit it now, then we postulate also that they cannot be performed in the same facility. In the following we consider only the case that either all N interim inspections take place at different times, or else, all at the same time.

While the strategy set of the Operator remains the same, i.e., $G_{N,N}$, as given by (11.45), the Inspectorate's strategy set has to be modified: Let q be the probability that all interim inspections are performed at the same time point. Then we have

$$\widetilde{Q}_{N,N} := \Big\{(q, q_{(1,\ldots,1)}, \ldots, q_{(N,\ldots,N)}) \in [0,1]^{N^N+1} : q + \sum_{(i_N,\ldots,i_1)\in\{1,\ldots,N\}^N} q_{(i_N,\ldots,i_1)} = 1\Big\}\,.$$

For any $(q, \mathbf{q}) \in \widetilde{Q}_{N,N}$, any $t \in (t_{k+1}, t_0)$ and any $\mathbf{t} \in \mathcal{T}_{N,N}$, we get by (11.46) for the (expected) payoff to the Operator

$$\widetilde{Op}_{N,N}(\mathbf{g}, (q, \mathbf{q}, t, \mathbf{t})) = Op_{N,N}(\mathbf{g}, (\mathbf{q}, \mathbf{t}))$$
$$+\; q \left(\sum_{i=1}^{N} \Big[g_{N+1,i}\,((1-\beta)\,(t - t_{N+1}) + \beta\,(t_0 - t_{N+1})) \Big] + g_{N+1}\,(t_0 - t) \right)$$

$$= Op_{N,N}(\mathbf{g}, (\mathbf{q}, \mathbf{t}))$$

$$+ \; q\left[((1-\beta)\,(t-t_{N+1}) + \beta\,(t_0 - t_{N+1}))\,(1-g_{N+1}) + g_{N+1}\,(t_0 - t)\right]. \qquad (11.65)$$

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Lemma 11.3. *Given the Se-No inspection game on the reference time interval $[t_{N+1}, t_0]$ with $k = N$ interim inspections in $N \geq 2$ facilities, and with facility-independent errors of the second kind, i.e., $\beta_1 = \beta_2 = \ldots = \beta_N =: \beta \geq 0$. The Operator's set of behavioural strategies is given by (11.45) for $k = N$, the Inspectorate's strategy set by $\widetilde{Q}_{N,N} \times (t_{N+1}, t_0) \times \mathcal{T}_{N,N}$, and the payoff to the Operator by (11.65).*

Then optimal strategies of the Operator and of the Inspectorate are given by:

(i) $\mathbf{g}^*$, $\mathbf{q}^*$ and $\mathbf{t}^*$ as given by (11.50)–(11.53) together with $q^* = 0$ and arbitrary $t^* \in (t_{N+1}, t_0)$,

(ii) $\mathbf{g}^*$ as given by (11.50) and (11.51) and $(q^*, \mathbf{q}^*) = (1, 0, \ldots, 0)$ together with t^* given by

$$t^* - t_{N+1} = \frac{1-\beta}{2-\beta}\,(t_0 - t_{N+1})$$

and arbitrary $\mathbf{t}^* \in \mathcal{T}_{N,N}$.

The optimal payoff to the Operator is $\widetilde{Op}^*_{N,N} = (t_0 - t_{N+1})/(2-\beta)$.

Proof. We get by (11.58) and (23.2) for any $(\mathbf{q}, \mathbf{t}) \in Q_{N,N} \times \mathcal{T}_{N,N}$

$$Op_{N,N}(\mathbf{g}^*, (\mathbf{q}, \mathbf{t})) = \frac{t_0 - t_{N+1}}{2-\beta} \sum_{(i_k,\ldots,i_1)\in\{1,\ldots,N\}^N} q_{(i_k,\ldots,i_1)} = \frac{t_0 - t_{N+1}}{2-\beta}\,(1-q) \qquad (11.66)$$

and by (11.50)

$$q\left[((1-\beta)\,(t-t_{N+1}) + \beta\,(t_0 - t_{N+1}))\,(1-g^*_{N+1}) + g^*_{N+1}\,(t_0 - t)\right] = q\,\frac{t_0 - t_{N+1}}{2-\beta}\,. \qquad (11.67)$$

Thus, (11.65), (11.66) and (11.67) imply for any $(q, \mathbf{q}, t, \mathbf{t}) \in \widetilde{Q}_{N,N} \times (t_{N+1}, t_0) \times \mathcal{T}_{N,N}$ to $\widetilde{Op}_{N,N}(\mathbf{g}^*, (q, \mathbf{q}, t, \mathbf{t})) = \widetilde{Op}_{N,N}(\mathbf{g}^*, (q^*, \mathbf{q}^*, t^*, \mathbf{t}^*))$, i.e., the right hand saddle point inequality is fulfilled as equality for cases (i) and (ii). The left hand saddle point inequality is shown separately:

Ad (i): Using (11.63) we obtain for any $\mathbf{g} \in G_{N,N}$ that $Op_{N,N}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) = (t_0 - t_{N+1})/(2-\beta)$ and by (11.65) that $\widetilde{Op}_{N,N}(\mathbf{g}, (q^*, \mathbf{q}^*, t^*, \mathbf{t}^*)) = Op_{N,N}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) = (t_0 - t_{N+1})/(2-\beta)$ for any $\mathbf{g} \in G_{N,N}$.

Ad (ii): Using (11.46) we get for any $\mathbf{g} \in G_{N,N}$ that $Op_{N,N}(\mathbf{g}, (\mathbf{q}^*, \mathbf{t}^*)) = 0$ and consequently $\widetilde{Op}_{N,N}(\mathbf{g}, (q^*, \mathbf{q}^*, t^*, \mathbf{t}^*)) = (t_0 - t_{N+1})/(2-\beta)$ for any $\mathbf{g} \in G_{N,N}$.

Thus, in both cases the left hand saddle point inequality is fulfilled as equality, which completes the proof. □

Let us conclude our analysis with two remarks concerning the optimal expected interim inspection time points; see also Tables 13.1 and 13.2:

First, in the No-No inspection games in Sections 9.1 – 9.3 and the Se-No inspection game in Section 10.1 and Chapter 11, the optimal expected detection time is always the time remaining after the optimal time point of the last interim inspection and the end of the reference time interval. This property also holds for the discrete time Se-No and Se-Se inspection games in Sections 4.2, see (4.48), and 5.1, however, with regard to the optimal *expected* time point of the last interim inspection. For the discrete time No-No inspection game this property only holds approximately; see Table 13.1.

Second, the optimal expected detection time is for $\beta = 0$ in all No-No, Se-No and Se-Se inspection games considered in Parts I and II the time between the beginning of the reference time interval and the optimal (expected) time point of the first interim inspection, in the discrete time No-No inspection game in Section 3.1 the only one. Thus, as different as all these inspection models are, they share these remarkable properties.

Chapter 12

Se-Se inspection game: Avenhaus-Canty model

In this chapter the last of the four variants of the playing for time inspection game with continuous time, which have been introduced in Table 2.1, is considered. Let us repeat that in Se-Se inspection games both the Operator and the Inspectorate behave sequentially.

In this chapter, assumptions (iv), (v) and (vii) of Chapter 8 are specified as follows:

(iv') The Operator may start at most once an illegal activity during the reference time interval $[t_{k+1}, t_0]$ in the only facility under consideration.

(v') During an interim inspection the Inspectorate may commit an error of the first and second kind with probabilities α and β. While during an interim inspection which is performed before the start of the illegal activity only an error of the first kind (false alarm) may occur, during an interim inspection which is performed after the start of the illegal activity only an error of the second kind (non-detection) may occur. The "game" continues after an error of the first kind. The error probabilities α and β are the same for all interim inspections.

(vii') The Operator decides at the beginning of the reference time interval, i.e., at time point t_{k+1}, whether to start the illegal activity immediately at time point t_{k+1} or to postpone its start; in the latter case he decides again after the first interim inspection, whether to start the illegal activity immediately at that time point or to postpone its start again; and so on. Because of assumption (iv'), the Operator does not need to behave illegally.

The Inspectorate decides at the beginning of the reference time interval when to perform its first interim inspection. At the time point of its first interim inspection, it decides when to perform the second interim inspection, and so on.

The remaining assumptions of Chapter 8 hold throughout this chapter.

Two comments on the assumptions (v') and (vii'): First, it is assumed that – like in Sections 9.4 and 10.3 – the value α of the false alarm probability is fixed a priori, and that the test procedure used for interim inspections is unbiased, i.e., $\alpha + \beta < 1$. As mentioned in Section 10.3, the additional assumption that a false alarm is not possible during an interim inspection if prior to that interim inspection an illegal activity was started, is not a trivial assumption.

Depending on the details of the inspection scheme alternative assumptions would have to be formulated; see p. 282. Because errors of both kinds are taken into account, we describe the inspection problem as a non-zero-sum game; see Section 7.4. Like in Sections 7.4, 9.4 and 10.3, the payoffs to the two players (Operator, Inspectorate) are given by (8.1) and (8.2). In case of ℓ false alarms, $1 \leq \ell \leq k$, their costs are $(-\ell\, f, -\ell\, g)$.

Second, in the Se-Se inspection game described in this section and again in Chapter 16 in the context of critical time inspection games, the Operator decides to behave legally or illegally during the course of the game. In the generalized Thomas-Nisgav inspection game analysed in Section 17.1, which is also a Se-Se inspection game, however, the Operator/Smuggler makes the decision to behave legally or illegally at the beginning of the game.

The chapter is based on Avenhaus and Canty (2005) who first published the Se-Se inspection game and its game theoretical solution. Special cases, however, have been analysed before by Rothenstein (1997) and by Rothenstein and Zamir (2002): They treated the two versions $\alpha > 0, \beta > 0$ and one interim inspection, and furthermore $\alpha = \beta = 0$ and any number of interim inspections, however, as zero-sum games: According to our modelling ideas, this is not realistic since false alarms cause costs – eventually different ones – to both players.

Avenhaus and Canty (2005) also consider degenerated game theoretical solutions for the case of $k = 2$ interim inspections, such as the first resp. both interim inspection(s) coincide(s) with the initial PIV at t_3. Even though these solutions are of no practical value and even though they contradict the idea of an interim inspection, they are presented in Section 12.2 in order to demonstrate the intricacies of such models.

In the following we consider in Sections 12.1 and 12.2 first the special cases of $k = 1$ and $k = 2$ interim inspection(s), thereafter in Section 12.3 the general case of any number k of interim inspections. In all cases closed expressions for the equilibrium strategies and payoffs are given and, with them, the conditions which must be met in order to induce legal behaviour on the part of the Operator. In addition "saturated" equilibria are examined which arise when false alarm costs become excessive. A discussion of the choice of the false alarm probability in Section 12.4 concludes the chapter.

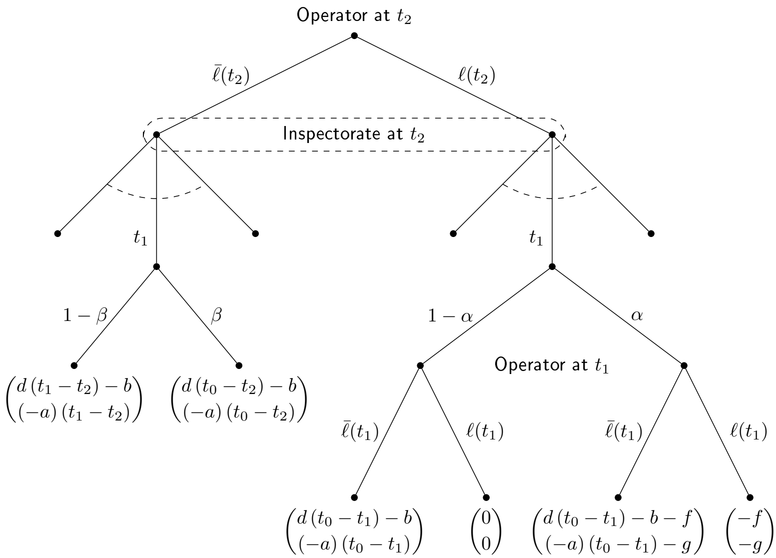
We mentioned on p. 139, that in practice payoff parameters are estimated and thus, they are always subject to uncertainty. Therefore, the cases in which a function of these parameters including α and β are exactly equal to a specified value are excluded from the following considerations. In particular, in the Lemmata and in the Theorem conditions to the parameters in form of equations are not taken into account.

In case of $k = 1$ interim inspection, the Se-No inspection game treated in Section 10.1 coincides with the Se-Se inspection game considered in this chapter for $\alpha = 0$ and $(d, b, a) = (1, 0, 1)$. Under these conditions it will turn out that for any number k of interim inspections the equilibria of the Se-Se inspection game, see Theorem 12.1, coincide with the optimal strategies of the Se-No inspection game; see Theorem 10.1. Therefore, we recommend to study the Se-No inspection game in Section 10.1 in some detail before turning to the Se-Se inspection game considered in this chapter.

12.1 One interim inspection; errors of the first and second kind

Let us consider the inspection game with $k = 1$ interim inspection the extensive form of which is represented in Figure 12.1. According to the comment on p. 50, all extensive form games in this chapter start with the Operator's decision at the beginning of the reference time interval. The chance moves are not explicitly named, but can be identified via the probabilities $1 - \beta$ and β as well as $1 - \alpha$ and α.

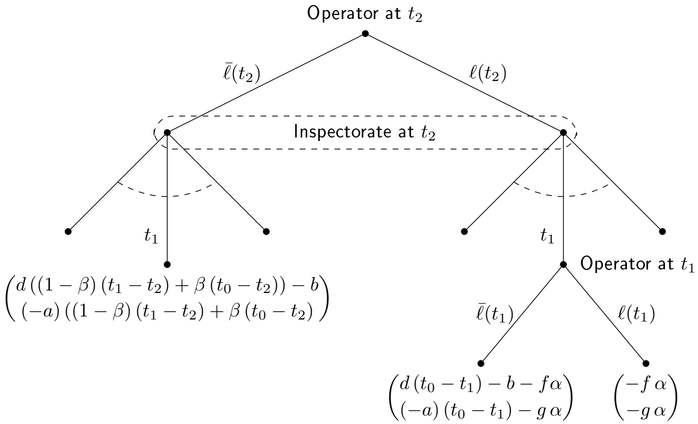
Figure 12.1 Extensive form of the Se-Se inspection game with $k = 1$ interim inspection and with errors of the first and second kind.



At the beginning of the reference time interval, i.e., at time point t_2, the Operator decides to start the illegal activity immediately ($\bar{\ell}(t_2)$) or to postpone its start ($\ell(t_2)$). The Inspectorate chooses also at the beginning of the reference time interval t_2, not knowing the Operator's decision at t_2, the time point t_1 for its interim inspection. This is indicated by its information set "Inspectorate at t_2". At the interim inspection a chance move takes place: In case the Operator starts the illegal activity at t_2, it will be detected with probability $1 - \beta$ at time point t_1, and detected only at the PIV with probability β. Thereafter the game ends with the payoffs given in the Figure. In case the Operator does not start the illegal activity at t_2, also a chance move takes place at t_1: With probability $1 - \alpha$ the Inspectorate will confirm the legal behaviour of the Operator, and with probability α it will raise a false alarm which will be clarified, but causes costs $-g$ and $-f$ to both players. In both cases the Operator will decide to start the illegal activity now ($\bar{\ell}(t_1)$) or to behave legally throughout the game ($\ell(t_1)$). The illegal activity will be detected by the Inspectorate with certainty at the end of the reference time interval during the final PIV. The payoffs are, using (8.1), given at the end nodes.

The subgames beginning at the chance nodes can be simplified by replacing the payoffs by their expected values, with respect to $1-\beta$ and β, respectively with respect to $1-\alpha$ and α. In particular the Operator's situations after his legal behaviour at t_1 are equivalent since all payoffs following a false alarm are reduced by the same amounts g respectively f. Figure 12.2 represents the reduced extensive form of the inspection game shown in Figure 12.1.

Figure 12.2 Reduced extensive form of the inspection game of Figure 12.1.



Let g_2 be the Operator's probability to postpone the start of the illegal activity at time point t_2. Because the Operator may behave legally throughout the game, see assumption (iv') for $k=1$ interim inspection, the probability $g_1(t_1)$ of postponing the illegal activity at time point t_1 needs to be taken into account. Thus, the Operator's set of behavioural strategies is an extension of (10.2), and given by

$$G_1 := \{\mathbf{g} := (g_2, g_1) : g_2 \in [0,1], g_1 : (t_2, t_0) \to [0,1]\} \ . \tag{12.1}$$

Note that in the Se-No inspection game discussed in Section 10.1 we have $g_1(t_1)=0$, because the Operator is assumed to behave illegally and he has to do it at t_1, if he does not start the illegal activity at t_2.

Note that according to Figure 12.1, the equilibrium probability $g_1^*(t_1)$ after a false alarm occurred is the same as in the case that not false alarm occurred, because of the constant shifts, $-f$ respectively $-g$, of the payoffs. The Inspectorate's strategy set is the same as that for the Se-No inspection game; see (10.1).

Using Figure 12.2, the (expected) payoff to the Operator is, for any $\mathbf{g} \in G_1$ and any $t_1 \in \mathcal{T}_1$, given by

$$\begin{aligned} Op_1(t_2;\mathbf{g},t_1) := {} & (1-g_2)\Big[d\,((1-\beta)\,(t_1-t_2)+\beta\,(t_0-t_2))-b\Big] \\ & + g_2\Big[(1-g_1(t_1))\,(d\,(t_0-t_1)-b-f\,\alpha)+g_1(t_1)\,(-f\,\alpha)\Big]\,, \end{aligned} \tag{12.2}$$

and that of the Inspectorate by

$$\begin{aligned} In_1(t_2; \mathbf{g}, t_1) := (1-g_2) \Big[(-a) \left((1-\beta)(t_1-t_2) + \beta (t_0 - t_2) \right) \Big] \\ + g_2 \Big[(1-g_1(t_1)) \left((-a)(t_0-t_1) - g\,\alpha \right) + g_1(t_1)(-g\,\alpha) \Big] . \end{aligned} \tag{12.3}$$

The reason why we include t_2 in the notation of the payoffs will become clear in Sections 12.2 and 12.3. Note that for $d = 1$, $b = f = 0$, $g(t_1) = 0$ for any $t_1 \in \mathcal{T}_1$, and $\alpha = 0$, (12.2) simplifies to (10.3) which was to be expected as the Se-No inspection game and the Se-Se inspection game in case of $k = 1$ interim inspection do not differ from a modelling point of view. Also note that the Inspectorate's payoff parameter g should not be confused with the Operator's probabilities g_2 and $g_1(t_1)$.

The game theoretical solution of this inspection game, see Avenhaus and Canty (2005), is presented in

Lemma 12.1. *Given the Se-Se inspection game on the reference time interval $[t_2, t_0]$ with $k = 1$ interim inspection, errors of the first and second kind, and an unbiased test procedure. The sets of behavioural resp. pure strategies are given by (12.1) and (10.1), and the payoffs to both players by (12.2) and (12.3).*

Define A_2 and B_2 by

$$A_2 = \frac{1}{2-\beta} \qquad \text{and} \qquad B_2 = \frac{1-\beta}{2-\beta}, \tag{12.4}$$

and furthermore,

$$L_2(t_2) := A_2 (t_0 - t_2) - \frac{f}{d} \alpha (B_2 - 1) = A_2 \left(t_0 - t_2 + \frac{f}{d} \alpha \right) . \tag{12.5}$$

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op_1^*(t_2) := Op_1(t_2; \mathbf{g}^*, t_1^*)$ and $In_1^*(t_2) := In_1(t_2; \mathbf{g}^*, t_1^*)$:

(i) For

$$L_2(t_2) > \frac{b}{d} \tag{12.6}$$

the Operator behaves illegally and an equilibrium strategy of the Operator is given by

$$g_2^* = 1 - A_2 = B_2 \qquad \text{and} \qquad g_1^*(t_1) = 0 \qquad \text{for all} \qquad t_2 < t_1 < t_0 . \tag{12.7}$$

An equilibrium strategy of the Inspectorate is given by

$$t_1^* - t_2 = (1-\beta) A_2 (t_0 - t_2) - \frac{f}{d} \alpha \left((1-\beta) B_2 + \beta \right) . \tag{12.8}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$\begin{aligned} Op_1^*(t_2) &= d\, A_2 (t_0 - t_2) - f\, \alpha\, B_2 - b \qquad \text{and} \\ In_1^*(t_2) &= (-a)\, A_2 (t_0 - t_2) - g\, \alpha\, B_2 . \end{aligned} \tag{12.9}$$

(ii) For

$$L_2(t_2) < \frac{b}{d} \tag{12.10}$$

the Operator behaves legally, i.e., $g_2^* = g_1^*(t_1) = 1$ *for all* $t_2 < t_1 < t_0$. *The Inspectorate's set of equilibrium strategies is given by*

$$\begin{aligned} \frac{b}{d} - \frac{f}{d}\,\alpha &\geq (1-\beta)\,(t_1^* - t_2) + \beta\,(t_0 - t_2) \\ \frac{b}{d} &\geq t_0 - t_1^*\,, \end{aligned} \tag{12.11}$$

where t_1^* *given by (12.8) fulfils (12.11).*

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_1^*(t_2) = -f\,\alpha \qquad \text{and} \qquad In_1^*(t_2) = -g\,\alpha\,. \tag{12.12}$$

Proof. In analogy to (19.5), the Nash equilibrium conditions

$$Op_1^*(t_2) \geq Op_1(t_2; \mathbf{g}, t_1^*) \quad \text{and} \quad In_1^*(t_2) \geq In_1(t_2; \mathbf{g}^*, t_1) \tag{12.13}$$

have to be proven for any $\mathbf{g} = (g_2, g_1) \in G_1$ and any $t_1 \in \mathcal{T}_1$.

The three strategies of the Operator, namely to start the illegal activity immediately at t_2, to start it at t_1, or to behave legally throughout the game, are equivalent to $\{g_2 = 0, g_1(t_1) \in [0,1]\}$, $\{g_2 = 1, g_1(t_1) = 0\}$ and $\{g_2 = 1, g_1(t_1) = 1\}$.[1] Using (12.2) and a similar argumentation as on p. 409, the Operator's Nash equilibrium condition (12.13) is equivalent to

$$\begin{aligned} Op_1^*(t_2) &\geq Op_1(t_2; (0, g_1), t_1) = d\,((1-\beta)\,(t_1^* - t_2) + \beta\,(t_0 - t_2)) - b \\ Op_1^*(t_2) &\geq Op_1(t_2; (1,0), t_1) = d\,(t_0 - t_1^*) - b - f\,\alpha \\ Op_1^*(t_2) &\geq Op_1(t_2; (1,0), t_1) = -f\,\alpha\,. \end{aligned} \tag{12.14}$$

Because $\alpha + \beta < 1$, the right hand expression in (8.2) implies

$$f\,\frac{\alpha}{1-\beta} < f < d\,(t_0 - t_2)\,,$$

which is, using (12.8) and the identity $(1-\beta)\,B_2 + \beta = A_2$, equivalent to $t_1^* - t_2 > 0$. The requirement $t_0 - t_1^* > 0$ is fulfilled because $(1-\beta)\,A_2 < 1$. Thus, $t_1^* \in \mathcal{T}_1$ independent whether (12.6) or (12.10) is fulfilled.

Ad (i): Obviously we have $(g_2^*, g_1^*) \in G_1$. Using (12.8) and $Op_1^*(t_2)$ from (12.9), a lengthy calculation shows that

$$\begin{aligned} Op_1^*(t_2) &= d\,((1-\beta)\,(t_1^* - t_2) + \beta\,(t_0 - t_2)) - b \\ Op_1^*(t_2) &= d\,(t_0 - t_1^*) - b - f\,\alpha\,, \end{aligned}$$

[1] The Operator has four *pure* strategies: $\bar{\ell}(t_2)\bar{\ell}(t_1), \bar{\ell}(t_2)\ell(t_1), \ell(t_2)\bar{\ell}(t_1)$ and $\ell(t_2)\ell(t_1)$, where the first two are combined in $\bar{\ell}(t_2)$.

i.e., the first two inequalities in (12.14) are fulfilled as equalities. Furthermore, because (12.6) is equivalent to

$$b < d\,A_2\,(t_0 - t_2) + f\,\alpha\,A_2 = d\,A_2\,(t_0 - t_2) + f\,\alpha\,(1 - B_2)$$

$$= d\,A_2\,(t_0 - t_2) - f\,\alpha\,B_2 + f\,\alpha\,,$$

we get by (12.9)

$$Op_1^*(t_2) = d\,A_2\,(t_0 - t_2) - f\,\alpha\,B_2 - b > -f\alpha\,,$$

i.e., the third inequality in (12.14) is also fulfilled but not as equality.

Using (12.3), (12.4), (12.7) and (12.9), we get for any $t_1 \in \mathcal{T}_1$

$$In_1(t_2; \mathbf{g}^*, t_1) = A_2\left[(-a)\left((1-\beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2)\right)\right] + B_2\left[(-a)\,(t_0 - t_1) - g\,\alpha\right]$$

$$= In_1^*(t_2)\,, \tag{12.15}$$

i.e., the right hand inequality of (12.13) is fulfilled as equality.

Ad (ii): Using Figure 12.2, the Operator behaves legally at t_1 if

$$d\,(t_0 - t_1^*) - b - f\,\alpha \le -f\,\alpha$$

which is equivalent to the second inequality in (12.11), and he will thus behave legally at t_2 if

$$d\,((1-\beta)\,(t_1^* - t_2) + \beta\,(t_0 - t_2)) - b \le -f\,\alpha\,,$$

which is equivalent to the first inequality in (12.11). The third inequality in (12.14) is fulfilled as equality. For the Inspectorate we get by (12.3) and (12.12) that $In_1(t_2; \mathbf{g}^*, t_1) = -g\,\alpha = In_1^*(t_2)$ for any $t_1 \in \mathcal{T}_1$, i.e., the right hand inequality of (12.13) is fulfilled as equality. Because (12.11) is equivalent to

$$t_0 - \frac{b}{d} < t_1^* < \frac{1}{1-\beta}\left(\frac{b}{d} - \frac{f}{d}\,\alpha - t_0\,\beta + t_2\right)\,,$$

subtracting its right hand side from its left hand side, we get, using (12.5),

$$\frac{1}{1-\beta}\left(\frac{b}{d} - \frac{f\,\alpha}{d} - t_0\,\beta + t_2 - (1-\beta)\,t_0 + (1-\beta)\frac{b}{d}\right)$$

$$= \frac{1}{1-\beta}\left(\frac{b}{d}\,(2-\beta) - \frac{f\,\alpha}{d} - (t_0 - t_2)\right)$$

$$= \frac{2-\beta}{1-\beta}\left(\frac{b}{d} - \frac{1}{2-\beta}\left[\frac{f\,\alpha}{d} + (t_0 - t_2)\right]\right) = \frac{2-\beta}{1-\beta}\left(\frac{b}{d} - L_2(t_2)\right)\,.$$

Thus, condition (12.10) guarantees the existence of t_1^*.

Substituting t_1^* as given by (12.8) in (12.11), and using (12.4), it is seen that both inequalities are fulfilled because of (12.10), which completes the proof. □

Let us comment the results of Lemma 12.1: First, because t_1^* given by (12.8) fulfils – under condition (12.10) – also the inequalities in (12.11), it is a robust equilibrium strategy because the

Inspectorate can just play t_1^* and does not need to check whether (12.6) or (12.10) is fulfilled. Note that for the Se-Se inspection game with $k \geq 2$ interim inspections a corresponding statement is only true in case of $\alpha = 0$; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

Second, for $\alpha = 0$ and $(d, b, a) = (1, 0, 1)$ the equilibrium strategies (12.7) and (12.8) coincide with the optimal strategies (10.6) and (10.7) of Lemma 10.1, and the equilibrium payoff (12.9) simplifies to (10.8). This is not surprising, because, as mentioned on p. 234, in this case both models are identical.

Third, the Inspectorate's equilibrium strategy in Lemma 12.1 is like that in Lemma 10.1 deterministic. Since the payoffs are linear this pure strategy could be replaced by any mixed strategy with expected value t_1^*, but there would be no advantage in doing so; see Rothenstein (1997) or the comment on p. 188. Also, like in the discrete time and continuous time Se-No inspection game, the set of optimal strategies of the Inspectorate is fully characterized by the uniquely determined (expected) interim inspection time point t_1^*; see the comments on pp. 68 and 189.

Fourth, the inequalities (12.11) are with $t_2 = 0$ exactly the same as that given by (9.64) for the No-No inspection game. This is, let us mention it again, so surprising since there the equilibrium strategies in case of illegal behaviour of the Operator are randomized ones; see (9.56). Note that here condition (12.6) guarantees the existence of t_1^*, while in the No-No inspection game condition (9.63) is required.

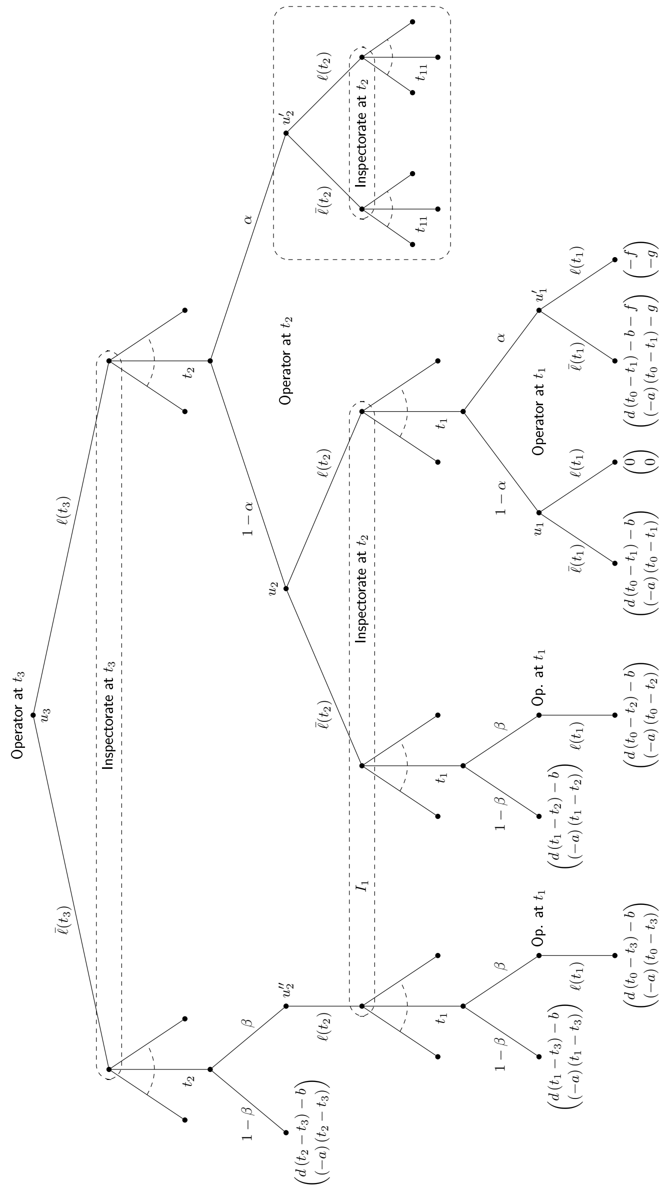
Finally and surprisingly, the Operator's equilibrium strategy depends on the error second kind probability β but neither on the error first kind probability α nor on the Inspectorate's utilities a and g.

12.2 Two interim inspections; errors of the first and second kind

Let us now consider the inspection game with $k = 2$ interim inspections the extensive form of which is represented in Figure 12.3.

As compared to Figure 12.1 some new features, similar to those given in Figure 6.6, can be observed. After the initial decisions of both players at the beginning of the reference time interval, i.e., at time point t_3, and after the first interim inspection at t_2, a chance move, error first or second kind, takes place, depending on the Operator's initial decision. In case the Operator does not start the illegal activity at t_3 ($\ell(t_3)$), he decides at t_2 to start it immediately ($\bar{\ell}(t_2)$) or to postpone its start again ($\ell(t_2)$). The Inspectorate decides at t_2 at which time point t_1 the second interim inspection will be performed, but now its state of knowledge is complicated: If the illegal activity is not detected at t_2, and if no false alarm is raised, then the Inspectorate does not know if it was started already at t_3 ($\bar{\ell}(t_3)$) or if it is started at t_2 ($\bar{\ell}(t_2)$), or if the Operator will behave legally again ($\ell(t_2)$). If at t_2 a false alarm occurs, which is clarified by assumption, then the Inspectorate knows that so far no illegal activity was started, and a new proper (sub)game starts with the reference time interval $[t_2, t_0]$ and $k = 1$ interim inspection. At the second interim inspection at time point t_1, again a chance move occurs, and the Operator decides again to start the illegal activity now ($\bar{\ell}(t_1)$) or to behave legally throughout the game ($\ell(t_1)$). Again, the payoffs are, using (8.1), given at the end nodes. Note that the decisions $\bar{\ell}(t_1)$ and $\ell(t_1)$ also occur in the proper subgame, see Figure 12.1, but this

Figure 12.3 Extensive form of the Se-Se inspection game with $k = 2$ interim inspections and with errors of the first and second kind. The proper subgame in the dashed box is identical to the game of Figure 12.2, except that all payoffs to the Operator and to the Inspectorate are reduced by amounts f and g, respectively.



is not illustrated in Figure 12.3. We need to include this possibility, however, in the following analysis.

Let g_3 be the Operator's probability to postpone the illegal activity at time point t_3 (node u_3), let $g_2(t_2)$ and g_{21} be the probabilities to postpone it at time point t_2 at nodes u_2 and u_2' respectively, and let $g_1(t_1)$ and $g_{11}(t_{11})$ be the probabilities to postpone the illegal activity at time point t_1 at nodes[2] u_1 and u_1', and in the proper subgame where the interim inspection is performed at time point t_{11} (see below), respectively. Note that the probabilities $g_1(t_1)$ and $g_{22}(t_{11})$ need to be taken into account again, because the Operator may behave legally throughout the game; see assumption (iv'). If we define

$$G_2 := \{\mathbf{g} := (g_3, g_2, g_1) : g_3 \in [0,1], g_2, g_1 : (t_3, t_0) \to [0,1]\} \,, \tag{12.16}$$

then the Operator's set of behavioural strategies is, using (12.1), given by

$$G_2 \times G_1 \,, \tag{12.17}$$

and an element of this set is denoted by $\mathbf{g} := (g_3, g_2, g_1, g_{21}, g_{11})$. As argued on p. 193, we assume that g_1 and g_{11} only depend on t_1 resp. t_{11} and not on the whole history (t_2, t_1) resp. (t_2, t_{11}), because the players payoffs if the game starts at time point t_1 resp. t_{11} is not influenced by t_2, i.e., $g_1 = g_1(t_1)$ and $g_{11} = g_{11}(t_{11})$. Even in case one assumes that they depend on the whole history, i.e., $g_1 = g_1(t_2, t_1)$ and $g_{11} = g_{11}(t_2, t_{11})$, the same Nash equilibrium strategy of the Operator are obtained; see the proof of Lemma 12.2.

The Inspectorate chooses the time point t_2 for the first interim inspection, time point t_1 for the second interim inspection and in case no false alarm is raised at t_2, and time point t_{11} in the proper subgame. Thus, its set of pure strategies is, using (10.1) and (10.12), given by

$$\mathcal{T}_2 \times \mathcal{T}_1 \,, \tag{12.18}$$

and an element of this set is denoted by $\mathbf{t} := (t_2, t_1, t_{11})$.

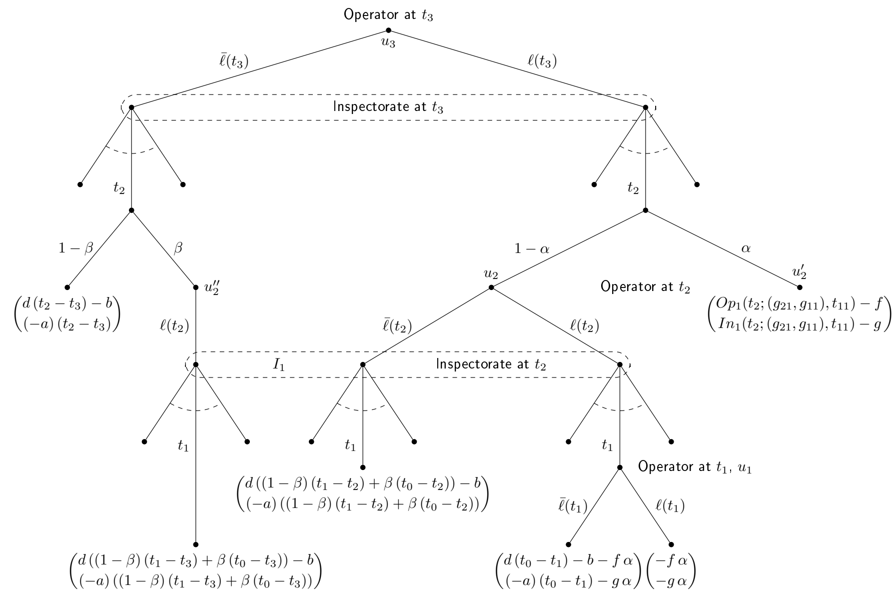
In Figure 12.4, the reduced extensive form of the inspection game in Figure 12.3 is presented: All chance moves after the second interim inspection at t_1 are eliminated and the payoffs are replaced by their (expected) payoffs. Furthermore, the proper subgame after the clarified false alarm at t_2 is replaced by the payoffs $Op_1(t_2; (g_{21}, g_{11}), t_{11}) - f$ and $In_1(t_2; (g_{21}, g_{11}), t_{11}) - g$ as given by (12.2) and (12.3).

Using Figure 12.4, the (expected) payoff to the Operator is, for any $\mathbf{g} \in G_2 \times G_1$ and any $\mathbf{t} \in \mathcal{T}_2 \times \mathcal{T}_1$, given by

$$\begin{aligned} Op_2(t_3; \mathbf{g}, \mathbf{t}) := {} & (1 - g_3) \Big[d\left((1-\beta)\,(t_2 - t_3) + \beta\,(1-\beta)\,(t_1 - t_3) + \beta^2\,(t_0 - t_3) \right) - b \Big] \\ & + g_3 \Big[(1-\alpha) \Big((1 - g_2(t_2)) \Big[d\left((1-\beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2) \right) - b \Big] \\ & \quad + g_2(t_2) \Big[(1 - g_1(t_1))\,(d\,(t_0 - t_1) - b - f\,\alpha) + g_1(t_1)\,(-f\,\alpha) \Big] \Big) \\ & \quad + \alpha \Big(Op_1(t_2; (g_{21}, g_{11}), t_{11}) - f \Big) \Big] \,, \end{aligned} \tag{12.19}$$

[2] Note that because of the constant shifts of the payoffs, see Figure 12.3, the equilibrium probability at node u_1 coincides with the one at node u_1'. Therefore, we only introduce a single probability $g_1(t_1)$.

Figure 12.4 Reduced extensive form of the inspection game of Figure 12.3.



where $Op_1(t_2;(g_{21},g_{11}),t_{11})$ is defined by (12.2), and that of the Inspectorate by

$$\begin{aligned} In_2(t_3;\mathbf{g},\mathbf{t}) := {} & (1-g_3)\,(-a)\,((1-\beta)\,(t_2-t_3)+\beta\,(1-\beta)\,(t_1-t_3)+\beta^2\,(t_0-t_3)) \\ & + g_3\Big[(1-\alpha)\Big((1-g_2(t_2))\,(-a)\,((1-\beta)\,(t_1-t_2)+\beta\,(t_0-t_2)) \\ & \quad + g_2(t_2)\Big[(1-g_1(t_1))\,((-a)\,(t_0-t_1)-g\,\alpha)+g_1(t_1)\,(-g\,\alpha)\Big]\Big) \\ & + \alpha\Big(In_1(t_2;(g_{21},g_{11}),t_{11})-g\Big)\Big]\,, \end{aligned} \tag{12.20}$$

where $In_1(t_2;(g_{21},g_{11}),t_{11})$ is given by (12.3). Again, the Inspectorate's payoff parameter g should not be confused with the Operator's probabilities g_3, g_2, g_1, g_{21} and g_{11}.

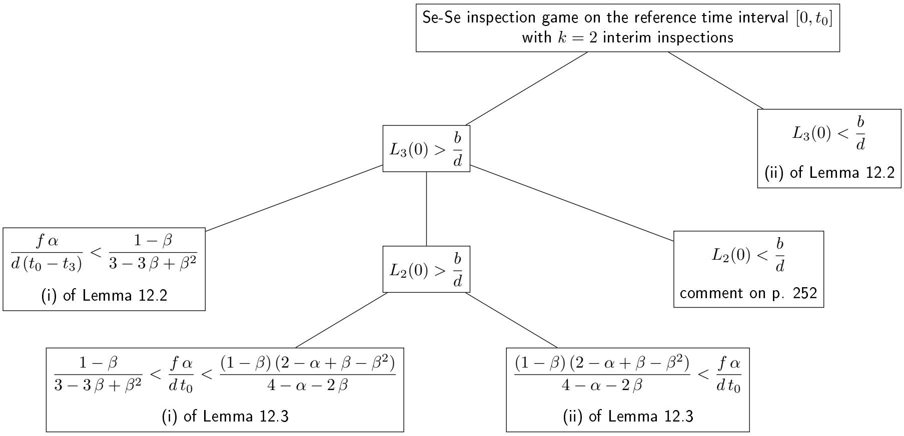
Note that in contrast to the Se-No inspection game treated in Section 10.3 in which the Inspectorate chooses both time points (t_2, t_1) at t_3, here – and due to the sequential nature of its behaviour – only t_2 is chosen at t_3, and at time point t_2 – depending whether a false alarm is raised or not – it chooses t_1 or t_{11}. This is reflected in (12.19) and (12.20).

Figure 12.5 gives an overview of the different cases to be treated in the game theoretical analysis.

The game theoretical solution of this inspection game, see Avenhaus and Canty (2005), is presented in

Lemma 12.2. *Given the Se-Se inspection game on the reference time interval* $[t_3, t_0]$ *with*

Figure 12.5 Distinction of cases of the game theoretical solution of the Se-Se inspection game on the reference time interval $[0, t_0]$ with $k = 2$ interim inspections. $L_2(t_2)$ and $L_3(t_3)$ are defined in (12.5) and (12.21), respectively.



$k = 2$ interim inspections, errors of the first and second kind, and an unbiased test procedure. The sets of behavioural resp. pure strategies are given by (12.17) and (12.18), and the payoffs to both players by (12.19) and (12.20).

Define, using (12.4), A_3 and B_3 by

$$A_3 = \frac{A_2}{1 + (1-\beta)\,A_2} = \frac{1}{3 - 2\,\beta} \qquad \text{and} \qquad B_3 = \frac{B_2 + (1-\beta)\,A_2}{1 + (1-\beta)\,A_2} = \frac{3\,(1-\beta)}{3 - 2\,\beta}\,,$$

and furthermore,

$$L_3(t_3) := A_3\,(t_0 - t_3) - \frac{f}{d}\,\alpha\,(B_3 - 2)\,. \tag{12.21}$$

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op_2^*(t_3) := Op_2(t_3; \mathbf{g}^*, \mathbf{t}^*)$ and $In_2^*(t_3) := In_2(t_3; \mathbf{g}^*, \mathbf{t}^*)$:

(i) For

$$L_3(t_3) > \frac{b}{d} \tag{12.22}$$

the Operator behaves illegally. For[3]

$$\frac{f\,\alpha}{d\,(t_0 - t_3)} < \frac{1-\beta}{3 - 3\,\beta + \beta^2} \tag{12.23}$$

an equilibrium strategy of the Operator is given by

$$\begin{aligned} &g_3^* = 1 - A_3\,,\; g_2^*(t_2) = 1 - \frac{2\,(1-\alpha) - \beta}{2\,(1-\alpha)}\,A_2\,,\; g_1^*(t_1) = 0\,,\; t_3 < t_2, t_1 < t_0\,, \\ &g_{21}^* = 1 - A_2\,,\; g_{11}^*(t_{11}) = 0\,, \quad t_2^* < t_{11} < t_0\,, \end{aligned} \tag{12.24}$$

[3]Note that in Lemma 12.1 there is no equivalent to (12.23).

where t_2^* is given by (12.25).

An equilibrium strategy of the Inspectorate is given by

$$
\begin{aligned}
t_2^* - t_3 &= (1-\beta)\, A_3\,(t_0 - t_3) - \frac{f}{d}\,\alpha\,((1-\beta)\, B_3 + \beta)\,, \\
t_1^* - t_2^* &= (1-\beta)\, A_2\,(t_0 - t_2^*) - \frac{f}{d}\,\alpha\,((1-\beta)\, B_2 + \beta) \qquad \textit{and} \\
t_{11}^* &= t_1^*\,.
\end{aligned} \tag{12.25}
$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$
\begin{aligned}
Op_2^*(t_3) &= d\, A_3\,(t_0 - t_3) - f\,\alpha\, B_3 - b \qquad \textit{and} \\
In_2^*(t_3) &= -a\, A_3\,(t_0 - t_3) - g\,\alpha\, B_3\,.
\end{aligned} \tag{12.26}
$$

(ii) For

$$L_3(t_3) < \frac{b}{d}$$

the Operator behaves legally, i.e., $g_3^* = g_2^*(t_2) = g_1^*(t_1) = 1$ for all $t_3 < t_2, t_1 < t_0$ and $g_{21}^* = g_{11}^*(t_{11}) = 1$ for all $t_2^* < t_{11} < t_0$, where t_2^* fulfils (12.27). The Inspectorate's set of equilibrium strategies is given by

$$
\begin{aligned}
\frac{b}{d} - \frac{2\,f}{d}\,\alpha &\geq (1-\beta)\,(t_2^* - t_3) + \beta\,(1-\beta)\,(t_1^* - t_3) + \beta^2\,(t_0 - t_3) \\
\frac{b}{d} - \frac{f}{d}\,\alpha &\geq (1-\beta)\,(t_1^* - t_2^*) + \beta\,(t_0 - t_2^*) \\
\frac{b}{d} &\geq t_0 - t_1^* \\
\frac{b}{d} - \frac{f}{d}\,\alpha &\geq (1-\beta)\,(t_{11}^* - t_2^*) + \beta\,(t_0 - t_2^*) \\
\frac{b}{d} &\geq t_0 - t_{11}^*\,.
\end{aligned} \tag{12.27}
$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_2^*(t_3) = -2\,f\,\alpha \qquad \textit{and} \qquad In_2^*(t_3) = -2\,g\,\alpha\,.$$

Proof. We have to prove that, in analogy to (19.5), the Nash equilibrium conditions

$$Op_2^*(t_3) \geq Op_2(t_3; \mathbf{g}, \mathbf{t}^*) \qquad \text{and} \qquad In_2^*(t_3) \geq In_2(t_3; \mathbf{g}^*, \mathbf{t}) \tag{12.28}$$

are fulfilled for any $\mathbf{g} = (g_3, g_2, g_1, g_{21}, g_{11}) \in G_2 \times G_1$ and any $\mathbf{t} = (t_2, t_1, t_{11}) \in \mathcal{T}_2 \times \mathcal{T}_1$.

Ad (i): We first see that $\mathbf{g}^* \in G_2 \times G_1$. Because $A_3\,(3 - 3\,\beta + \beta^2) = (1-\beta)\, B_3 + \beta$, (12.23) is, using (12.25), equivalent to $t_2^* - t_3 > 0$. With the same argument as on p. 238 we have $t_1^* - t_2^* > 0$ resp. $t_{11}^* - t_2^* > 0$. Thus, $\mathbf{t}^* \in \mathcal{T}_2 \times \mathcal{T}_1$.

Now we show that in case of legal behaviour at t_3 ($\ell(t_3)$) and a false alarm, the Operator will behave illegally in the subsequent proper subgame: Using (12.5), (12.21) and (12.25), we get

$$\begin{aligned} L_2(t_2^*) &= \frac{1}{2-\beta}\left(t_0 - t_2^* + \frac{f}{d}\,\alpha\right) = \frac{1}{2-\beta}\left(t_0 - t_3 - (t_2^* - t_3) + \frac{f}{d}\,\alpha\right) \\ &= \frac{1}{2-\beta}\left(t_0 - t_3 - \frac{1-\beta}{3-2\,\beta}\,(t_0 - t_3) + \frac{f}{d}\,\alpha\,\frac{3-3\,\beta+\beta^2}{3-2\,\beta} + \frac{f}{d}\,\alpha\right) \\ &= \frac{1}{3-2\,\beta}\left(t_0 - t_3 + \frac{f}{d}\,\alpha\,(3-\beta)\right) = L_3(t_3)\,. \end{aligned}$$

Thus, (12.22) implies $L_2(t_2^*) > b/d$, i.e., case (i) of Lemma 12.1 is satisfied.

Like on p. 238, we consider the Operator's strategies to start the illegal activity immediately at t_3, to start it at t_2, to start it at t_1 or to behave legally throughout the game, are equivalent to $\{g_3 = 0, g_2(t_2^*) \in [0,1], g_1(t_1^*) \in [0,1]\}$, $\{g_3 = 1, g_2(t_2^*) = 0, g_1(t_1^*) \in [0,1]\}$, $\{g_3 = 1, g_2(t_2^*) = 1, g_1(t_1^*) = 0\}$ and $\{g_3 = 1, g_2(t_2^*) = 1, g_1(t_1^*) = 1\}$. Using (12.19) and the same argument as on p. 409, the Operator's Nash equilibrium condition (12.28) is equivalent to the four inequalities

$$Op_2^*(t_3) \geq Op_2(t_3; (0, g_2, g_1), \mathbf{t}^*)\,, \qquad Op_2^*(t_3) \geq Op_2(t_3; (1, 0, g_1), \mathbf{t}^*)\,,$$

$$Op_2^*(t_3) \geq Op_2(t_3; (1, 1, 0), \mathbf{t}^*)\,, \qquad Op_2^*(t_3) \geq Op_2(t_3; (1, 1, 1), \mathbf{t}^*)\,,$$

i.e., equivalent to

$$\begin{aligned} Op_2^*(t_3) &\geq d\,(1-\beta)\,(t_2^* - t_3) + d\,\beta\,((1-\beta)\,(t_1^* - t_3) + \beta\,(t_0 - t_3)) - b \\ Op_2^*(t_3) &\geq (1-\alpha)\,(d\,(1-\beta)\,(t_1^* - t_2^*) + d\,\beta\,(t_0 - t_2^*) - b) + \alpha\left(Op_1^*(t_2^*) - f\right) \\ Op_2^*(t_3) &\geq (1-\alpha)\,(-f\,\alpha + d\,(t_0 - t_1^*) - b) + \alpha\left(Op_1^*(t_2^*) - f\right) \\ Op_2^*(t_3) &\geq (1-\alpha)\,(-f\,\alpha) + \alpha\left(Op_1^*(t_2^*) - f\right), \end{aligned} \tag{12.29}$$

keeping in mind that the Operator will behave illegally in the proper subgame starting at u_2'. Now, using $Op_1^*(t_2)$ from (12.9), (12.25) and $Op_2^*(t_3)$ from (12.26), a cumbersome calculation shows that

$$\begin{aligned} Op_2^*(t_3) &= d\,(1-\beta)\,(t_2^* - t_3) + d\,\beta\,((1-\beta)\,(t_1^* - t_3) + \beta\,(t_0 - t_3)) - b \\ Op_2^*(t_3) &= (1-\alpha)\,(d\,(1-\beta)\,(t_1^* - t_2^*) + d\,\beta\,(t_0 - t_2^*) - b) + \alpha\,(Op_1^*(t_2^*) - f) \\ Op_2^*(t_3) &= (1-\alpha)\,(-f\,\alpha + d\,(t_0 - t_1^*) - b) + \alpha\,(Op_1^*(t_2^*) - f)\,, \end{aligned} \tag{12.30}$$

i.e., the first three inequalities in (12.29) are fulfilled as equality. The last inequality of (12.29) follows from (12.22). In sum, the Operator's Nash equilibrium condition (12.28) is fulfilled.

To prove the Inspectorate's Nash equilibrium condition we consider the coefficients of t_2 and t_1 in $In_2(t_3; \mathbf{g}^*, \mathbf{t})$. Because, as mentioned above, case (i) of Lemma 12.1 is satisfied,

$In_1(t_2; (g^*_{21}, g^*_{11}), t_{11})$ evaluates, using (12.15), to $In^*_1(t_2)$ for all $t_2 < t_{11} < t_0$. Thus, (12.9) and (12.20) yield for the coefficients of t_2 and t_1

$$\text{for } t_2: \quad a\Big(-(1-g^*_3)(1-\beta) + g^*_3(1-\alpha)(1-g^*_2(t_2)) + g^*_3\,\alpha\,A_2\Big)$$

$$\text{for } t_1: \quad a\Big(-(1-g^*_3)\,\beta\,(1-\beta) - g^*_3(1-\alpha)(1-g^*_2(t_2))(1-\beta) + g^*_3(1-\alpha)\,g^*_2(t_2)\Big)$$

which both evaluate to zero because of (12.24). Therefore, we get after some lengthy calculation, using (12.24) and (12.26), that $In_2(t_3; \mathbf{g}^*, \mathbf{t}) = In^*_2(t_3)$ for any $(t_2, t_1, t_{11}) \in \mathcal{T}_2 \times \mathcal{T}_1$, i.e., the Inspectorate's Nash equilibrium condition (12.28) is fulfilled as equality.

Ad (ii): Let $(t^*_2, t^*_1, t^*_{11}) \in \mathcal{T}_2 \times \mathcal{T}_1$ be *any* equilibrium strategy of the Inspectorate in case of legal behaviour of the Operator. Note that the components t^*_2 and t^*_1 are different from (12.25); see the comment on p. 248. We apply the backward induction principle directly to the reduced extensive form of the game in Figure 12.4. At node u_1 the Operator will have legally if $d\,(t_0 - t^*_1) - b - f\,\alpha \le -f\,\alpha$ which is equivalent to the third inequality in (12.27). At node u_2 he behaves legally if

$$d\,((1-\beta)\,(t^*_1 - t^*_2) + d\,\beta\,(t_0 - t^*_2)) - b \le -f\,\alpha\,,$$

which is equivalent to the second inequality in (12.27). What happens at node u'_2, i.e., a false alarm is raised? The second and third inequality in (12.27) imply

$$\frac{b}{d} - \frac{f}{d}\,\alpha \ge (1-\beta)\,(t^*_1 - t_0 + t_0 - t^*_2) + \beta\,(t_0 - t^*_2) = -(1-\beta)\,(t_0 - t^*_1) + t_0 - t^*_2$$

$$\ge -(1-\beta)\,\frac{b}{d} + t_0 - t^*_2\,,$$

which yield, using (12.4) and (12.5),

$$L_2(t^*_2) = A_2\,(t_0 - t^*_2) - \frac{f}{d}\,\alpha\,(B_2 - 1) \le A_2\,\left(\frac{b}{d} - \frac{f}{d}\,\alpha + (1-\beta)\,\frac{b}{d}\right) - \frac{f}{d}\,\alpha\,(B_2 - 1) = \frac{b}{d}\,.$$

Thus, the Operator will – because of (12.10) – behave legally in the proper subgame with the equilibrium strategy t^*_{11} given by the fourth and fifth inequality in (12.27) (use (12.11) and write t^*_{11} instead of t^*_1). Thus, if the Operator behaves legally at t_3 ($\ell(t_3)$) then his payoff is $(1-\alpha)\,(-f\,\alpha) + \alpha\,(-f\,\alpha - f) = -2\,f\,\alpha$. Thus, he will behave legally in the entire game if

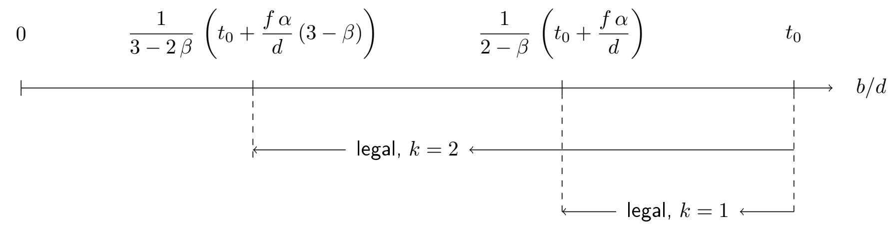
$$d\,((1-\beta)\,(t_2 - t_3) + \beta\,(1-\beta)\,(t_1 - t_3) + \beta^2\,(t_0 - t_3)) - b \le -2\,f\,\alpha\,,$$

which is equivalent to the first inequality of (12.27).

For the Inspectorate we get $In_2(t_3; \mathbf{g}^*, \mathbf{t}) = -2\,f\,\alpha$ for any $(t_2, t_1, t_{11}) \in \mathcal{T}_2 \times \mathcal{T}_1$, i.e., its Nash equilibrium condition is fulfilled as equality, which completes the proof. □

Let us comment the results of Lemma 12.2: First, we compare the regions of b/d-values in the Se-Se inspection game with $k = 1$ and $k = 2$ interim inspection(s) each played over the full reference time interval $[0, t_0]$, which lead to legal and illegal behaviour. Given (12.23), condition (12.22) for $t_3 = 0$ gives a smaller bound for b/d than (12.6) for $t_2 = 0$, so that we have the situation shown in Figure 12.6. In other words, for a smaller inspection effort ($k = 1$) the ratio of sanctions to gains b/d for the Operator has to be larger to induce him to legal behaviour than for a larger inspection effort ($k = 2$).

Figure 12.6 Representation of (12.6) and (12.22) for $t_3 = t_2 = 0$ provided that (12.23) holds.



Second, according to (12.25) the Inspectorate's equilibrium strategy at its information set I_1 is the same as its equilibrium strategy in the Se-Se inspection game with $k = 1$ interim inspection, see (12.8), pointing to a straightforward generalization to any number of interim inspections. On the other hand we see from (12.24) that this is not the case for the Operator at his decision node u_2. This phenomenon is a consequence of the game's information structure.

Third, the equilibrium of the inspection game considered here is very close to that of the Se-No inspection game considered in Section 10.3 and given by Lemma 10.3. The equilibrium strategy of the Inspectorate as well as the equilibrium payoffs to both players are the same, whereas the component $g_2^*(t_2)$ of the Operator's equilibrium strategy in case of illegal behaviour are different; see (10.42) and (12.24). One may explain this result as follows: For the Inspectorate there is only one advantage in the Se-Se inspection game as compared to the Se-No inspection game which exists only if both types of errors are possible: Whereas in both variants without first kind errors, but eventually second kind errors, the Inspectorate does not know after the first interim inspection without detection of the illegal activity whether or not it took place, after a false alarm and its clarification it does know that there was no illegal activity. In the Se-Se inspection game therefore the Inspectorate can use this information for the planning of the second interim inspection, whereas this is not possible in the Se-No inspection game. The Operator, on his side, reacts to this difference by an appropriately modified equilibrium strategy such that the advantage of the Inspectorate is neutralized. A weak point of this argument is that without both error types we also have the situation that after an interim inspection the Inspectorate knows whether or not an illegal activity took place, but in both variants, as well as in the variant without errors of the first kind, the equilibrium strategies of both players are the same. Maybe these games are too simple to contain as subtle differences as described above.

Fourth, again the Inspectorate's equilibrium strategy in Lemma 12.2 is deterministic and the Inspectorate may announce the equilibrium interim inspection time points, if it wishes so. Or it might randomize as described on p. 192 without having any advantage in doing so.

Finally, as illustrated for the Se-No inspection game on p. 203, the Inspectorate's equilibrium inspection time points (t_2^*, t_1^*, t_{11}^*) given by (12.25) is not a robust equilibrium strategy, because the Inspectorate has to check whether (12.23) is fulfilled in order to assure that $t_2^* - t_3 > 0$; see the proof of Lemma 12.2. For $\alpha = 0$, however, (12.23) vanishes, and (t_2^*, t_1^*, t_{11}^*) given by (12.25) is a robust equilibrium strategy; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

After these comments we consider now some special solutions which arise when condition

(12.23) in Lemma 12.2 is not fulfilled. Since these solutions will not be required for the generalization to any number k of interim inspections in Section 12.3, we will set $t_3 = 0$ for convenience. Also, let us repeat our introductory remark: Even though these solutions cover an unrealistic area of large values of α, and even though these solutions are of no practical value, they are presented here in order to demonstrate the intricacies of such models.

While the Operator's strategy set is same as the one given in (12.17), the Inspectorate's strategy set needs to be modified: We allow not only that the two interim inspections are performed at the same time point, but also at time point $t_3 = 0$, i.e., at the initial PIV. Thus, we have in contrast to $\mathcal{T}_1$ and $\mathcal{T}_2$:

$$\widetilde{\mathcal{T}}_1 = \{t_{11} \in \mathbb{R} : 0 \le t_{11} < t_0\} \qquad \text{and} \qquad \widetilde{\mathcal{T}}_2 = \{(t_2, t_1) \in \mathbb{R}^2 : 0 \le t_2 \le t_1 < t_0\}\,,$$

and the Inspectorate's set of pure strategies is given by

$$\widetilde{\mathcal{T}}_2 \times \widetilde{\mathcal{T}}_1\,, \tag{12.31}$$

and an element of this set is denoted by $\mathbf{t} := (t_2, t_1, t_{11})$.

The game theoretical solution of this inspection game, see Avenhaus and Canty (2005), is presented in

Lemma 12.3. *Given the Se-Se inspection game on the reference time interval $[0, t_0]$ with $k = 2$ interim inspections, errors of the first and second kind, and an unbiased test procedure. The sets of behavioural resp. pure strategies are given by (12.17) and (12.31), and the payoffs to both players by (12.19) and (12.20) for $t_3 = 0$.*

If, using (12.5) and (12.22),

$$L_3(0) > \frac{b}{d} \qquad \text{and} \qquad L_2(0) > \frac{b}{d} \tag{12.32}$$

then the Operator behaves illegally and a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op_2^*(0) := Op_2(0; \mathbf{g}^*, \mathbf{t}^*)$ and $In_2^*(0) := In_2(0; \mathbf{g}^*, \mathbf{t}^*)$:

(i) For

$$\frac{1-\beta}{3 - 3\,\beta + \beta^2} < \frac{f\,\alpha}{d\,t_0} < \frac{(1-\beta)\,(2 - \alpha + \beta - \beta^2)}{4 - \alpha - 2\,\beta} \tag{12.33}$$

an equilibrium strategy of the Operator is given by

$$\begin{aligned} &g_3^* = 1 - \frac{1-\alpha}{1 - \alpha + \beta - \beta^2}\,, \quad g_2^*(t_2) = 1\,, \quad g_1^*(t_1) = 0\,, \quad t_3 < t_2, t_1 < t_0\,, \\ &g_{21}^* = 1 - A_2\,, \quad g_{11}^*(t_{11}) = 0\,, \qquad 0 < t_{11} < t_0\,. \end{aligned} \tag{12.34}$$

An equilibrium strategy of the Inspectorate is given by

$$\begin{aligned} &t_2^* = 0\,, \qquad t_1^* = \frac{Op_2^*(0) + b - d\,\beta^2\,t_0}{d\,\beta\,(1-\beta)} \qquad \text{and} \\ &t_{11}^* = (1-\beta)\,A_2\,t_0 - \frac{f}{d}\,\alpha\,((1-\beta)\,B_2 + \beta)\,. \end{aligned} \tag{12.35}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_2^*(0) = \beta\, A_2 \left(\frac{d\, t_0\, (2-\alpha-\beta) - f\, \alpha\, (1-\beta)\, (4-\alpha-2\,\beta)}{1-\alpha+\beta-\beta^2} \right) - b \quad \text{and}$$
$$In_2^*(0) = -\beta\, A_2 \left(\frac{a\, t_0\, (2-\alpha-\beta) + g\, \alpha\, (1-\beta)\, (4-\alpha-2\,\beta)}{1-\alpha+\beta-\beta^2} \right) . \tag{12.36}$$

(ii) For

$$\frac{(1-\beta)\,(2-\alpha+\beta-\beta^2)}{4-\alpha-2\,\beta} < \frac{f\,\alpha}{d\,t_0} \tag{12.37}$$

an equilibrium strategy of the Operator is given by

$$g_3^* = 0\,, \quad g_2^*(t_2) = 1\,, \quad g_1^*(t_1) = 0\,, \quad t_3 < t_2, t_1 < t_0\,,$$
$$g_{21}^* = 1 - A_2\,, \quad g_{11}^*(t_{11}) = 0\,, \quad 0 < t_{11} < t_0\,.$$

An equilibrium strategy of the Inspectorate is given by

$$t_2^* = t_1^* = 0 \qquad \text{and} \qquad t_{11}^* = (1-\beta)\, A_2\, t_0 - \frac{f}{d}\, \alpha\, ((1-\beta)\, B_2 + \beta)\,. \tag{12.38}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_2^*(0) = \beta^2\, d\, t_0\, - b \qquad \text{and} \qquad In_2^*(0) = -\beta^2\, a\, t_0\,. \tag{12.39}$$

Proof. Because $t_2^* = 0$, see (12.35) and (12.38), we get by (12.29) for the Operator's Nash equilibrium condition

$$Op_2^*(0) \geq d\,\beta\,((1-\beta)\, t_1^* + \beta\, t_0) - b \tag{12.40}$$

$$Op_2^*(0) \geq (1-\alpha)\,(d\,(1-\beta)\, t_1^* + d\,\beta\, t_0 - b) + \alpha \left(Op_1^*(0) - f \right) \tag{12.41}$$

$$Op_2^*(0) \geq (1-\alpha)\,(-f\,\alpha + d\,(t_0 - t_1^*) - b) + \alpha \left(Op_1^*(0) - f \right) \tag{12.42}$$

$$Op_2^*(0) \geq (1-\alpha)\,(-f\,\alpha) + \alpha \left(Op_1^*(0) - f \right), \tag{12.43}$$

where, because of (12.32), we have $Op_1^*(0) = d\, A_2\, t_0 - f\, \alpha\, B_2 - b$; see (12.9).

Ad (i): Using (12.35) and (12.36), it can be see that (12.40) and (12.42) are fulfilled as equality, whereas (12.41) follows from the left-hand inequality of (12.33). Again, the left-hand inequality of (12.33) is equivalent to

$$\frac{f\,\alpha}{d\,t_0}\,(4-\alpha-2\,\beta) > (1-\alpha+\beta-\beta^2)\,(2-\beta)\, L_2(0)\, \frac{1}{t_0} + \alpha - 2\,\beta + \beta^2\,,$$

which implies, because of (12.32), that

$$\frac{f\,\alpha}{d\,t_0}\,(4-\alpha-2\,\beta) > (1-\alpha+\beta-\beta^2)\,(2-\beta)\, \frac{b}{d\,t_0} + \alpha - 2\,\beta + \beta^2\,,$$

and, which implies that – after some algebra – (12.43) is fulfilled. Thus, the Operator's Nash equilibrium condition is fulfilled.

Because of the right hand side of (12.32), the Inspectorate's payoff (12.20) can be, using (12.9) and (12.34), written as

$$\begin{aligned} In_2(0;\mathbf{g}^*,(t_2,t_1,t_{11})) = (1-g_3^*)\,(-a)\,((1-\beta)\,t_2+\beta\,(1-\beta)\,t_1+\beta^2\,t_0) \\ + g_3^*\Big[(1-\alpha)\Big(((-a)\,(t_0-t_1)-g\,\alpha)\Big) \\ +\ \alpha\Big((-a)\,A_2\,(t_0-t_2)-g\,\alpha\,B_2-g\Big)\Big]\,. \end{aligned} \tag{12.44}$$

The coefficient of t_1 in (12.44) is given by

$$(-a)\Big((1-g_3^*)\,\beta\,(1-\beta)-g_3^*\,(1-\alpha)\Big)\,,$$

which evaluates to zero due to (12.34). Also, the Inspectorate's payoff (12.44) is maximized for $t_2^*=0$. Thus, using (12.36), we have $In_2(0;\mathbf{g}^*,(0,t_1,t_{11}))=In_2^*(0)$ and the Inspectorate's Nash equilibrium condition (12.28) is fulfilled. Finally, $t_1^*>0$ can be seen to be equivalent to the right-hand inequality of (12.33).

Ad (ii): Inequality (12.40) is by (12.39) fulfilled as equality, while (12.42) follows from (12.37). Because

$$\begin{aligned} &\frac{(1-\beta)\,(2-\alpha+\beta-\beta^2)}{4-\alpha-2\,\beta}-\frac{(1-\beta)\,(\alpha\,(1-\beta)+\beta\,(2-\beta))}{2-\beta+\alpha\,(1-\beta)} \\ &=\frac{(1-\alpha)\,(2-3\,\beta+\beta^2)^2}{(2-\beta+\alpha\,(1-\beta))\,(4-\alpha-2\,\beta)}>0\,, \end{aligned}$$

we obtain from (12.37) that

$$\frac{(1-\beta)\,(\alpha\,(1-\beta)+\beta\,(2-\beta))}{2-\beta+\alpha\,(1-\beta)}<\frac{f\,\alpha}{d\,t_0}\,,$$

which implies that (12.41) is valid. Because

$$\frac{(1-\beta)\,(2-\alpha+\beta-\beta^2)}{4-\alpha-2\,\beta}-\frac{1-\beta^2\,(2-\beta)}{3-2\,\beta}=\frac{(1-\alpha)\,(2-\beta)\,(1-\beta)^2}{(3-2\,\beta)\,(4-\alpha-2\,\beta)}>0\,,$$

(12.37) implies

$$\frac{1-\beta^2\,(2-\beta)}{3-2\,\beta}<\frac{f\,\alpha}{d\,t_0}\,,$$

and we get

$$\frac{f\,\alpha}{d\,t_0}\,(4-\alpha-2\,\beta)>(2-\beta)\,\left((1-\alpha)\,L_2(0)\,\frac{1}{t_0}-\beta^2\right)+\alpha\,.$$

Due to the right inequality of (12.32) this inequality simplifies to

$$\frac{f\,\alpha}{d\,t_0}\,(4-\alpha-2\,\beta)>(2-\beta)\,\left((1-\alpha)\,\frac{b}{d\,t_0}-\beta^2\right)+\alpha\,,$$

which implies that (12.43) is valid.

Finally, because $-a < 0$, $t_2^* = t_1^* = 0$ are best replies to the Operator's equilibrium strategy $g_3^* = 0$. □

We finalize this section with two comments on the results of Lemma 12.3: First, the equilibrium payoffs (12.36) are quite complicated and were obtained by brute force using vertex enumeration of the convex polyhedra associated with an equivalent bimatrix game programmed by M. Canty on the computer-algebra system Mathematica®; see Canty (2003).

Second, in Figure 12.5 the case $L_2(0) < b/d$ is mentioned. In this case the Operator behaves legally in the proper subgame which arises after a false alarm, but not necessarily in the rest of the game tree. The game theoretical solutions are not presented here, because they are of less interest from application view point; see the argumentation at the beginning of the next section.

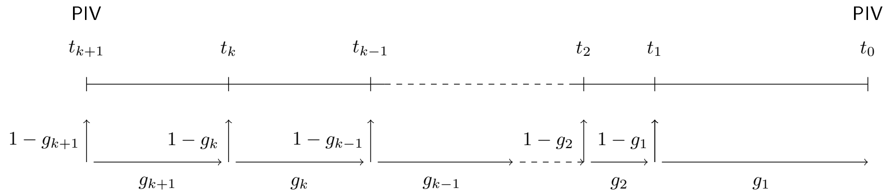
12.3 Any number of interim inspections; errors of the first and second kind

The special Nash equilibria of Lemma 12.3 are of questionable practical value, since placing *interim* inspections at the beginning of the reference time interval is a contradiction in terms. In solution (ii) for example, the comparatively large false alarm costs to the Operator, see condition (12.37), compel him to violate immediately in order to avoid false alarms altogether, and the Inspectorate must react by also inspecting immediately. Although justifiable from the theoretical point of view, this is not likely to be an acceptable inspection strategy in real situations. By reducing the number of inspections by one the chance of a false alarm is reduced, leading to solution (12.8) with an "unsaturated" interim inspection. In the sequel, we take the point of view that the number of interim inspections should always be chosen such that, given the Operator's utilities d, f and error probabilities α, β, the equilibrium interim inspection time points are positive. For unbiased test procedures the number of interim inspections satisfying this requirement will never be less than one; see Lemma 12.1. We shall therefore generalize only the unsaturated equilibria, i.e., the Nash equilibria with $t_k^* - t_{k+1} > 0$, see Lemmata 12.1 and 12.2, to an arbitrary number of interim inspections. Condition (12.47) in Theorem 12.1 below guarantees that $t_k^* - t_{k+1} > 0$.

The time line of the interim inspections and probabilities for starting or postponing the illegal activity is represented in Figure 12.7. It is a generalization of that in Figure 10.3 because here the Operator does not necessarily have to behave illegally.

As on p. 193, the Operator starts the illegal activity at t_{k+1} with probability $1 - g_{k+1}$ or he postpones its start with probability g_{k+1}, in the latter case he starts it at t_k with probability $1 - g_k(t_k)$ which depends on t_k or he postpones its start again with probability $g_k(t_k)$. If the Operator postpones the start of the illegal activity until time point t_n, $n = 1, \ldots, k$, he starts it there with probability $1 - g_n(t_n)$ and postpones its start again with probability $g_n(t_n)$. It is important to note that the probability $g_n(t_n)$ refers only to the case that at all time points $t_k, \ldots, t_n$ *no* false alarm is raised. If during some interim inspection a false alarm is raised, then the Se-Se inspection game starts again as a proper subgame with the appropriate smaller number of interim inspections. Again, and as justified on p. 193, we assume that g_n depends only on t_n and not on the whole history $t_k, \ldots, t_n$, i.e., $g_n = g_n(t_n)$. Also the proof

Figure 12.7 Time line of the interim inspections and probabilities for starting or postponing the illegal activity for the Se-Se inspection game with k interim inspections. For reasons of clarity we write g_n instead of $g_n(t_n)$, $n = 2, \dots, k$.



of Theorem 12.1 shows that even a dependence of g_n on the whole history $t_k, \dots, t_n$, i.e., $g_n = g_n(t_k, \dots, t_n)$ would not change the Operator's Nash equilibrium strategy.

Again, as an extension of (10.21), we define the Operator's behavioural strategy set to be

$$G_k := \{\mathbf{g} := (g_{k+1}, g_k, \dots, g_2, g_1) : g_{k+1} \in [0,1]\,, \quad g_n : (t_{k+1}, t_0) \to [0,1]\,, \quad n = 1, 2, \dots, k\}\,. \tag{12.45}$$

The Inspectorate's strategy set is given by (10.20).

The payoffs to the two players are generalizations of those for the case of $k = 2$ interim inspections and can be deduced from Figure 12.4.

The game theoretical solution of this inspection game, see Avenhaus and Canty (2005), is presented in

Theorem 12.1. *Given the Se-Se inspection game on the reference time interval $[t_{k+1}, t_0]$ with k interim inspections, errors of the first and second kind, and an unbiased test procedure. The sets of behavioural resp. pure strategies are given by (12.45) and (10.20), and the payoffs to the two players can be deduced from Figure 12.4.*

Define for all $n = 1, 2, \dots$ the constants A_n and B_n by

$$A_n = \frac{1}{1 + (n-1)(1-\beta)} \qquad \text{and} \qquad B_n = \frac{n}{2}(1 - A_n)\,, \tag{12.46}$$

and furthermore,

$$L_{k+1}(t_{k+1}) := A_{k+1}(t_0 - t_{k+1}) - \frac{f}{d}\,\alpha\,(B_{k+1} - k)\,.$$

Assume that for $k > 1$[4]

$$\frac{f\,\alpha}{d\,(t_0 - t_{k+1})} < \frac{A_{k+1}}{B_{k+1} + \dfrac{\beta}{1-\beta}}\,. \tag{12.47}$$

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op_k^*(t_{k+1})$ and $In_k^*(t_{k+1})$:

[4] Note that for $k = 1$ condition (12.47) is satisfied due to assumptions (8.2) and $\alpha + \beta < 1$, and therefore, it does not occur in Lemma 12.1.

(i) For

$$L_{k+1}(t_{k+1}) > \frac{b}{d} \tag{12.48}$$

the Operator behaves illegally and an equilibrium strategy is given by

$$g^*_{k+1} = 1 - A_{k+1} \qquad \text{and} \qquad g^*_n(t_n) = 1 - \frac{n\,(1-\alpha) - (n-1)\,\beta}{n\,(1-\alpha)}\,A_n \tag{12.49}$$

for all $t_{k+1} < t_n < t_0$, $n = 1, \ldots, k$, *which implies* $g^*_1(t_1) = 0$.
An equilibrium strategy of the Inspectorate is given by

$$t^*_n - t^*_{n+1} = (1-\beta)\,A_{n+1}\,(t_0 - t^*_{n+1}) - \frac{f\,\alpha}{d}\,\Big((1-\beta)\,B_{n+1} + \beta\Big) \tag{12.50}$$

for $n = 1, \ldots, k$ *and* $t^*_{k+1} = 0$, *which fulfils* $t^*_k - t_{k+1} > 0$.
The equilibrium payoffs to the Operator and to the Inspectorate are

$$\begin{aligned} Op^*_k(t_{k+1}) &= d\,A_{k+1}\,(t_0 - t_{k+1}) - f\,\alpha\,B_{k+1} - b \\ In^*_k(t_{k+1}) &= -a\,A_{k+1}\,(t_0 - t_{k+1}) - g\,\alpha\,B_{k+1}\,. \end{aligned} \tag{12.51}$$

(ii) For

$$L_{k+1}(t_{k+1}) < \frac{b}{d} \tag{12.52}$$

the Operator behaves legally, i.e., $g^*_{k+1} = g^*_k(t_k) = \ldots = g^*_1(t_1) = 1$ *for all* $t_{k+1} < t_n < t_0$, $n = 1, \ldots, k$, *and the Inspectorate's set of equilibrium strategies is given by*

$$\frac{b}{d} - \frac{n\,f}{d}\,\alpha \geq (1-\beta)\,\sum_{m=1}^{n}\,\beta^{n-m}\,(t^*_m - t^*_{n+1}) + \beta^n\,(t_0 - t^*_{n+1})\,, \tag{12.53}$$

with $\sum_{m=1}^{0} := 0$ *and* $t^*_{k+1} := t_{k+1}$ *for* $n = 0, \ldots, k$.
The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{k+1}(t_{k+1}) = -k\,f\,\alpha \qquad \text{and} \qquad In^*_{k+1}(t_{k+1}) = -k\,g\,\alpha\,. \tag{12.54}$$

Proof. In analogy to the proof of Lemma 12.2, the Operator's Nash equilibrium condition is equivalent to the following $k+2$ inequalities:

$$\begin{aligned} Op^*_k(t_{k+1}) &\geq d\,\Big[(1-\beta)\,(t^*_k - t_{k+1}) + \beta\,\Big[(1-\beta)\,(t^*_{k-1} - t_{k+1}) \\ &\qquad + \beta\,[(1-\beta)\,(t^*_{k-2} - t_{k+1}) + \ldots + \beta\,(t_0 - t_{k+1})]\ldots\Big]\Big] - b \\ Op^*_k(t_{k+1}) &\geq -f\,\alpha + d\,\Big[(1-\beta)\,(t^*_{k-1} - t^*_k) + \beta\,\Big[(1-\beta)\,(t^*_{k-2} - t^*_k) \\ &\qquad + \beta\,[(1-\beta)\,(t^*_{k-3} - t^*_k) + \ldots + \beta\,(t_0 - t^*_k)]\ \ldots\Big]\Big] - b \\ &\ \ \vdots \\ Op^*_k(t_{k+1}) &\geq -k\,f\,\alpha + d\,(t_0 - t^*_1) - b \\ Op^*_k(t_{k+1}) &\geq -k\,f\,\alpha\,. \end{aligned} \tag{12.55}$$

Ad (i): Under condition (12.48) it can be shown that the Operator will behave illegally in any proper subgame. We show by induction that all other inequalities in (12.55) are fulfilled as equalities. For $k = 2$ we have proved this in Lemma 12.2; see (12.30). Now assume that for $k-1$ interim inspections the first k inequalities corresponding to (12.55) hold as equalities. Then (12.55) can be written as follows:

$$
\begin{aligned}
Op_k^*(t_{k+1}) &\geq \beta\, Op_{k-1}^*(t_k^*) + d\,(t_k^* - t_{k+1}) - (1-\beta)\, b \\
Op_k^*(t_{k+1}) &\geq -f\,\alpha + Op_{k-1}^*(t_k^*) \\
Op_k^*(t_{k+1}) &\geq -2\,f\,\alpha + Op_{k-2}^*(t_{k-1}^*) \\
&\vdots \\
Op_k^*(t_{k+1}) &\geq -(k-1)\,f\,\alpha + d\,(t_0 - t_1^*) - b\,,
\end{aligned}
\tag{12.56}
$$

where by (12.51) $Op_{n-1}^*(t_n^*)$ is given by

$$Op_{n-1}^*(t_n^*) = d\,A_n\,(t_0 - t_n^*) - f\,\alpha\,B_n - b \tag{12.57}$$

for $n = 2, \ldots, k+1$ and t_{n+1}^* is implicitly given by (12.50) for $n = 2, \ldots, k$ with $t_{k+1}^* := t_{k+1}$. Using (12.57) we get by (12.50) for all $n = 2, \ldots, k+1$

$$
\begin{aligned}
&-f\,\alpha + Op_{n-2}^*(t_{n-1}^*) \\
&= -f\,\alpha + d\,A_{n-1}\,(t_0 - t_{n-1}^*) - f\,\alpha\,B_{n-1} - b \\
&= d\,A_{n-1}\,(t_0 - t_n^*) - d\,A_{n-1}\,(t_{n-1}^* - t_n^*) - f\,\alpha\,(B_{n-1}+1) - b \\
&= A_{n-1}\,(t_0 - t_n^*) - d\,A_{n-1}\,\left((1-\beta)\,A_n\,(t_0 - t_n^*) - \frac{f\,\alpha}{d}\Big((1-\beta)\,B_n + \beta\Big)\right) \\
&\qquad - f\,\alpha\,(B_{n-1}+1) - b \\
&= d\,A_{n-1}\,(1 - (1-\beta)\,A_n)\,(t_0 - t_n^*) \\
&\qquad - f\,\alpha\Big((-A_{n-1})\,((1-\beta)\,B_n + \beta) + B_{n-1} + 1\Big) - b\,.
\end{aligned}
\tag{12.58}
$$

Because A_n and B_n as given by (12.46) satisfy for all $n = 2, 3, \ldots$ the recursive relations

$$A_n = A_{n-1}\,(1 - (1-\beta)\,A_{n-1}) \qquad \text{and} \qquad B_n = \frac{B_{n-1} + 1 - \beta\,A_{n-1}}{1 + (1-\beta)\,A_{n-1}}\,,$$

the coefficient of $d\,(t_0 - t_n^*)$ reduces to A_n and that of $-f\,\alpha$ to B_n. Thus, we have by (12.58), using (12.57) for $n = 2, \ldots, k+1$,

$$-f\,\alpha + Op_{n-2}^*(t_{n-1}^*) = d\,A_n\,(t_0 - t_n^*) - f\,\alpha\,B_n - b = Op_{n-1}^*(t_n^*)\,,$$

which implies

$$Op_k^*(t_{k+1}^*) = -(k+1-n)\,f\,\alpha + Op_{n-1}(t_n^*)\,, \qquad n = 3, \ldots, k\,. \tag{12.59}$$

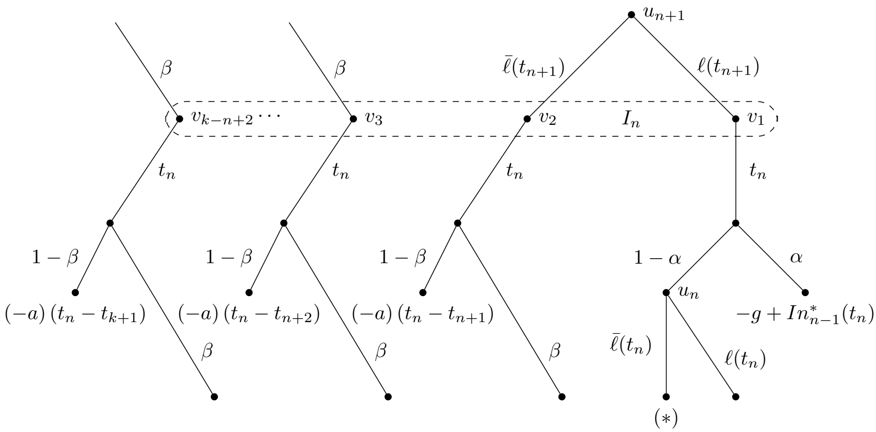
Therefore, the second to the last inequality in (12.56) is fulfilled as equality. It remains to show that the first inequality in (12.56) also holds as equality. This claim is, using (12.59) for $n = k$, equivalent to

$$Op^*_{k-1}(t^*_k) = \frac{1}{1-\beta}\,(d\,(t^*_k - t_{k+1}) + f\,\alpha) - b\,,$$

the right hand side of which simplifies with (12.50) to $Op^*_{k-1}(t^*_k)$ given by (12.51). The last inequality of (12.55) is fulfilled because of (12.48) and (12.51). Finally, condition (12.47) is equivalent to $t^*_k - t_{k+1} > 0$, where $t^*_k - t_{k+1}$ is given by (12.50) with $t^*_{k+1} := t_{k+1}$.

To show the Nash condition is satisfied for the Inspectorate, consider the Inspectorate's information set I_n and the edges leading to and from it as shown in Figure 12.8.

Figure 12.8 The Inspectorate's information set I_n and its payoffs involving t_n.



In Figure 12.8, the terminal nodes are labelled with the Inspectorate's payoff. The symbol $(*)$ denotes the continuation in which the Inspectorate's payoff is a function of t_n. It corresponds to the payoff

$$-a\,[(1-\beta)\,(t_{n-1} - t_n) + \beta\,[(1-\beta)\,(t_{n-2} - t_n) + \beta\,[\ldots + \beta\,(t_0 - t_n)]\;\ldots]]\,,$$

where the number of square bracket pairs is equal to n and where the combined coefficient of t_n is simply $+a$, because

$$\begin{aligned}&(1-\beta) + \beta\,(1-\beta) + \beta^2\,(1-\beta) + \ldots + \beta^{n-1}\,(1-\beta) + \beta^n \\ &= (1-\beta)\,\frac{1-\beta^n}{1-\beta} + \beta^n = 1\,.\end{aligned} \tag{12.60}$$

For $k = 2$ interim inspections it was proven in Lemma 12.2 that the equilibrium strategy of the Operator makes the Inspectorate indifferent to its choice of t_2 and t_1 and that its equilibrium payoff is $In^*_1(t_2) = -a\,A_2\,(t_0 - t_2) - g\,\alpha\,B_2$. We shall therefore assume inductively that for $k-1$ interim inspections the Inspectorate is indifferent with respect to its choice of t_n, $n = 1, \ldots, k-1$, and that its equilibrium payoff in any proper subgame beginning at time t_n is

$$In^*_{n-1}(t_n) = -a\, A_n\,(t_0 - t_n) - g\,\alpha\, B_n\,, \qquad n = 2, \ldots, k-1\,. \tag{12.61}$$

Now we determine the probabilities $\rho_{k+1}(v_m)$ that the decision point v_m, $m = 1, \ldots, k-n+2$, in I_n is reached during the course of the game. The point v_{k-n+2} is reached if and only if the Operator behaves illegally at time point t_{k+1} and is not detected at the $(k-n)$ time points $t_k, \ldots, t_{n+1}$. Thus, we have

$$\rho_{k+1}(v_{k-n+2}) = (1 - g^*_{k+1})\,\beta^{k-n} = A_{k+1}\,\beta^{k-n}\,. \tag{12.62}$$

The point v_m, $m = 2, \ldots, k-n+1$ is reached if and only if the Operator

- behaves legally at the time points $t_{k+1}, \ldots, t_{n+m}$, and $(k-(n+m)+2)$ non-false alarms occur at the time points $t_k, \ldots, t_{n+m}, t_{n+m-1}$;
- starts the illegal activity at the time point t_{n+m-1};
- is not detected at the $(m-2)$ time points $t_{n+m-2}, \ldots, t_{n+1}$.

Therefore, we obtain for all $m = 2, \ldots, k-n+1$

$$\begin{aligned} &\rho_{k+1}(v_m) \\ &= g^*_{k+1}\, g^*_k(t_k)\ \ldots\ g^*_{m+n}(t_{m+n})\,(1 - g^*_{m+n-1}(t_{m+n-1}))\,(1-\alpha)^{k-m-n+2}\,\beta^{m-2}\,. \end{aligned} \tag{12.63}$$

The point v_1 is reached if and only if the Operator behaves legally at the time points $t_{k+1}, \ldots,$ t_{n+1}, and $k-(n+1)+1$ non-false alarms occur at the time points $t_k, \ldots, t_{n+1}$, which leads to

$$\rho_{k+1}(v_1) = g^*_{k+1}\, g^*_k(t_k)\ \ldots\ g^*_{n+1}(t_{n+1})\,(1-\alpha)^{k-n}\,. \tag{12.64}$$

Thus, the coefficient of t_n in the Inspectorate's payoff is, using Figure 12.8, (12.60) and (12.61), given by

$$-a\,(1-\beta) \sum_{m=2}^{k-n+2} \rho_{k+1}(v_m) + a\,[(1-\alpha)\,(1 - g^*_n(t_n)) + \alpha\, A_n]\,\rho_{k+1}(v_1)\,.$$

If this coefficient vanishes, then the Inspectorate is indifferent as to its choice of t_n. We therefore wish to demonstrate that

$$[(1-\alpha)\,(1 - g^*_n(t_n)) + \alpha\, A_n]\,\rho_{k+1}(v_1) = (1-\beta) \sum_{m=2}^{k-n+2} \rho_{k+1}(v_m)\,. \tag{12.65}$$

According to the induction assumption, we have

$$[(1-\alpha)\,(1 - g^*_n(t_n)) + \alpha\, A_n]\,\rho_k(v_1) = (1-\beta) \sum_{m=2}^{k-n+1} \rho_k(v_m)\,. \tag{12.66}$$

Rewriting (12.62) – (12.64), we obtain, using (12.49),

$$\begin{aligned}\rho_{k+1}(v_{k-n+2}) &= A_{k+1}\,\beta^{k-n}\\ \rho_{k+1}(v_{k-n+1}) &= (1-A_{k+1})\,(1-g_k^*(t_k))\,(1-\alpha)\,\beta^{k-n-1}\\ \rho_{k+1}(v_m) &= \frac{1-A_{k+1}}{1-A_k}\,g_k^*(t_k)\,(1-\alpha)\,\rho_k(v_m)\,, \qquad m=1,\ldots,k-n\,.\end{aligned} \tag{12.67}$$

Thus, (12.65) is equivalent to

$$[(1-\alpha)\,(1-g_n^*(t_n))+\alpha\,A_n]\;\frac{1-A_{k+1}}{1-A_k}\,g_k^*(t_k)\,(1-\alpha)\,\rho_k(v_1)=(1-\beta)\sum_{m=2}^{k-n+2}\rho_{k+1}(v_m)\,,$$

or to

$$\begin{aligned}&[(1-\alpha)\,(1-g_n^*(t_n))+\alpha\,A_n]\,\frac{1-A_{k+1}}{1-A_k}\,g_k^*(t_k)\,(1-\alpha)\,\rho_k(v_1)\\ &=(1-\beta)\left[A_{k+1}\,\beta^{k-n}+(1-A_{k+1})\,(1-g_k^*(t_k))\,(1-\alpha)\,\beta^{k-n-1}\right.\\ &\qquad\left.+\frac{1-A_{k+1}}{1-A_k}\,g_k^*(t_k)\,(1-\alpha)\,\left(\sum_{m=2}^{k-n+1}\rho_k(v_m)-\rho_k(v_{k-n+1})\right)\right].\end{aligned}$$

With the induction assumption (12.66) this becomes, using (12.67),

$$\begin{aligned}0=A_{k+1}\,\beta+(1-A_{k+1})\,(1-g_k^*(t_k))\,(1-\alpha)&\\ -\frac{1-A_{k+1}}{1-A_k}\,g_k^*(t_k)\,(1-\alpha)\,\rho_k(v_{k-n+1})\,\beta^{-(k-n-1)}&\,,\end{aligned}$$

which is fulfilled by (12.46) and (12.49). Thus the Inspectorate is indifferent to its choice of t_n, $n=1,\ldots,k$.

Finally we determine the Inspectorate's equilibrium payoff. Since the Inspectorate is indifferent as to the choice of t_n, $n=1,\ldots,k$, we choose the time points $t_0-\epsilon<t_k<t_{k-1}<\ldots<t_1<t_0$ for arbitrarily small ϵ. Then, apart from terms involving ϵ, the Inspectorate's payoff $In_k^*(t_{k+1})$ is given by

$$\begin{aligned}In_{k+1}&=-a\,A_{k+1}+(1-A_{k+1})\left[\alpha\,(-g+In_k^*(t_k))+\right.\\ &\left.+(1-\alpha)\,g_k^*(t_k)\,[\alpha\,(-g+In_{k-1}^*(t_{k-1}))+\ldots+(1-\alpha)\,g_2^*(t_2)\,(-g\,\alpha)]\,\cdots\right].\end{aligned} \tag{12.68}$$

It follows from Lemma 12.2 that $In_3=In_3^*(0)$. Therefore, we assume inductively that

$$In_k=In_k^*(0)\,,$$

or, equivalently, that

$$\frac{In_k^*(0)+a\,A_k}{1-A_k}=\alpha\,(-g+In_{k-1}^*(t_{k-1}))+$$

$$+ (1-\alpha)\, g^*_{k-2}(t_{k-2}) \Big[\alpha\,(-g + In^*_{k-1}(t_{k-1})) + \ldots + (1-\alpha)\, g^*_2(t_2)\,(-g\,\alpha) \cdots\Big] .$$

Substituting this into (12.68), we have

$$In_{k+1} = -a\,A_{k+1} + (1-A_{k+1}) \left[\alpha\,(-g + In^*_k(t_k)) + \frac{(1-\alpha)\, g^*_k(t_k)}{1-A_k}\,(In^*_k(0) + a\,A_k\,)\right] ,$$

which, with (12.44) and $t_k \to t_0$, is equivalent to

$$In_{k+1} = -a\,A_{k+1} - g\,\alpha\,(1-A_{k+1}) \left[(1-\alpha\,B_k) + \frac{(1-\alpha)\, g^*_k(t_k)}{1-A_k}\,B_k\right] = In^*_{k+1}(0) ,$$

where the last equality follows from (12.46) and (12.49).

Ad (ii): Turning to the legal equilibrium, we see immediately that (12.53) and (12.54) satisfy the Nash equilibrium conditions (12.55). □

Let us comment the results of Theorem 12.1: First, although the general form (12.50) of the Inspectorate's equilibrium strategy could be guessed from Lemmata 12.1 and 12.2, this was not so for the Operator's equilibrium strategy (12.49). Also, there exists no simple form for the equilibrium expected time point S of the start of the illegal activity of the Operator which would correspond to those given in Chapters 4 and 10. As mentioned on p. 248, the equilibrium inspection time points given by (12.50) constitute a robust equilibrium strategy only in the case of $\alpha = 0$; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

Second, assume $\alpha = 0$. From (12.49) we get

$$g^*_{k+1} = \frac{k\,(1-\beta)}{1+k\,(1-\beta)} \qquad \text{and} \qquad g^*_n(t_n) = \frac{n-1}{n} , \quad t_{k+1} < t_n < t_0 \,, n = 2, \ldots, k .$$

This is the same result as that for the continuous time Se-No inspection game in Theorem 10.1 and, if in addition $\beta = 0$, as that for the discrete time Se-No inspection game in Theorem 4.1. Like in the Se-No inspection game treated in Section 10.1, $1 - g^*_2(t_2), \ldots, 1 - g^*_k(t_k)$ as given by (12.49) form only in case of $\alpha = 0$ a harmonic progression; see also Table 4.1 on p. 72 for an overview of inspection games with this property. As mentioned on p. 153, for $\alpha = 0$ and illegal behaviour of the Operator the game is strategically equivalent to a zero-sum game with the expected detection time as payoff to the Operator; see p. 398. In fact, from (8.1) and (12.51) we get for $\alpha = 0$ the optimal expected detection time

$$\frac{t_0 - t_{k+1}}{1 + k\,(1-\beta)} ,$$

which is the same as that for the Se-No inspection game given by (10.27). We will come back to this point in Chapter 13. Also, due to this equivalence the equilibrium strategies of both players do not depend on the payoff parameters a, b and d. Also, (12.50) and (22.3) yield

$$t^*_n - t^*_{n+1} = \frac{1-\beta}{1+n\,(1-\beta)}\,(t_0 - t^*_{n+1}) \qquad \text{for} \qquad n = 1, \ldots, k ,$$

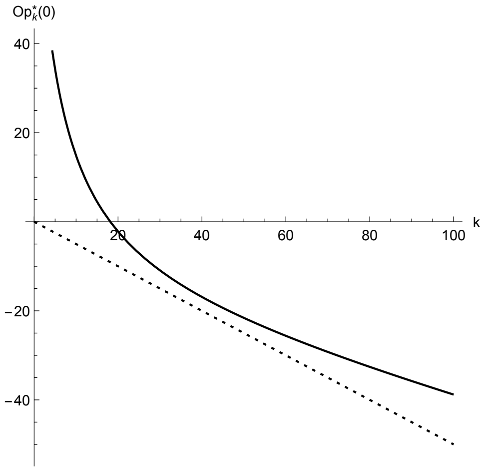
which is the same as in the Se-No inspection game; see (10.26). From (22.3) we get

$$t^*_n - t^*_{n+1} = \frac{1-\beta}{1+n\,(1-\beta)}\,(t_0 - t^*_{n+1}) = \frac{1-\beta}{1+k\,(1-\beta)}\,(t_0 - t^*_{k+1}) , \tag{12.69}$$

i.e., the differences between the optimal interim inspection time points are the same; see also (10.35). This does no longer hold for $\alpha > 0$.

Finally, we discuss whether the Operator can be induced to legal behaviour. We first show that this is not always possible even if the number k of interim inspections is increased. Consider for example $t_0 = 10, t_{k+1} = 0, d = 11, b = 20, f = 10, \alpha = 0.05$ and $\beta = 0.8$. Then the left hand inequality in (8.2), and (9.49) are fulfilled. Figure 12.9 plots the equilibrium payoffs to the Operator (12.51) and $-k\,f\,\alpha$ according to (12.54). We see that the equilibrium payoff (12.51) is always larger than $-k\,f\,\alpha$. Thus, the Operator cannot be induced to legal behaviour, and the Inspectorate should choose the value of k which maximises its equilibrium payoff $In_k^*(0)$ in (12.51). This is possible because $In_k^*(0)$ is increasing for all $k \leq k_{max}$, and decreasing for all $k \geq k_{max}$, where k_{max} is the positive solution of the equation $2\,a = g\,\alpha\,(1 + 2\,k + k^2\,(1-\beta))$.

Figure 12.9 Equilibrium payoffs to the Operator for the parameters $t_0 = 10, t_{k+1} = 0, d = 11, b = 20, f = 10, \alpha = 0.05$ and $\beta = 0.8$. Solid curve: (12.51), dashed curve: $-k\,f\,\alpha$.



As mentioned on p. 252, we are only interested in the unsaturated Nash equilibria, i.e., we require $t_k^* - t_{k+1} > 0$, which is, using (12.47) with $t_{k+1} = 0$, equivalent to

$$H(k+1) := A_{k+1}\,t_0 - \frac{f\,\alpha}{d}\left(B_{k+1} + \frac{\beta}{1-\beta}\right) > 0\,.$$

Because A_k resp. B_k is a monotone decreasing resp. increasing sequence in k, we have $H(k) > H(k+1)$. Furthermore, we have $\lim_{k\to\infty} H(k+1) \to -\infty$. Therefore, there exists an upper limit k_0 given by $H(k_0) = 0$ such that (12.47) is valid for all $k \leq k_0$.

On the other hand, the Operator will behave legally for all k with $-k\,f\,\alpha > Op_k^*(0)$, where $Op_k^*(0)$ is given by (12.51). Because of associated inspection costs, the practitioner will nevertheless be interested in the smallest number k_1 fulfilling $-k_1\,f\,\alpha > Op_{k_1}^*(0)$. If $k_1 > k_0$ then the Operator cannot be induced to legal behaviour and the Inspectorate should choose the value of k which maximises its equilibrium payoff; see above. If $k_1 < k_0$ then he can be induced to legal behaviour and k_1 interim inspections should be performed. Numerical calculations indicate $k_1 < k_0$ for reasonable values of the parameters α, β, d, b and f.

12.4 Choice of the false alarm probability

Like in Sections 9.4 and 10.3 we have considered in this chapter so far the value of the false alarm probability α a parameter of the model, but – like in Section 9.5 – we ask now which value should be chosen by the Inspectorate. It will become clear that the structure of finding the optimal value of α is the same as in Section 9.5, the only difference is the analytical form of the equilibrium payoffs to both players in case of illegal behaviour of the Operator. For the same reasons as those given in Section 9.5 we limit our consideration to just $k = 1$ interim inspection and assume that (9.69) is again fulfilled.

Using (12.6) and (12.9) as well as (12.10) and (12.12), the equilibrium payoff to the Operator is given by

$$Op_1^*(\alpha) := \begin{cases} Op_1^*(t_2) & \text{for Operator's illegal behaviour} \\ -f\,\alpha & \text{for Operator's legal behaviour} \end{cases}$$

$$= \begin{cases} d\,(t_0 - t_2)\,\dfrac{1}{2-\beta(\alpha)} - f\,\alpha\,\dfrac{1-\beta(\alpha)}{2-\beta(\alpha)} - b & \text{for} \quad L_2(t_2) > b/d \\ -f\,\alpha & \text{for} \quad L_2(t_2) < b/d \end{cases} . \tag{12.70}$$

Define for any $\alpha \in [0, 1]$

$$F(\alpha) := d\,(t_0 - t_2)\,\frac{1}{2-\beta(\alpha)} - f\,\alpha\,\frac{1-\beta(\alpha)}{2-\beta(\alpha)} - b\,. \tag{12.71}$$

Then $F(\alpha)$ is equal to $Op_1^*(t_2)$, see (12.70), if and only if $L_2(t_2) > b/d$, i.e., only for those $\alpha \in [0, 1]$ for which, using (12.6), we have

$$\frac{1}{2-\beta(\alpha)}\left(t_0 - t_2 + \frac{f\,\alpha}{d}\right) > \frac{b}{d}\,.$$

Using (9.69), (8.2) and (12.71) we obtain

$$F(0) = d\,(t_0 - t_2) - b > 0 \qquad \text{and} \qquad F(1) = \frac{1}{2}\,(d\,(t_0 - t_2) - f) - b\,.$$

To prove that $F(\alpha)$ is a monotone decreasing function on $[0, 1]$, we proceed as in Section 9.5 and define

$$\tilde{F}(\alpha, \beta) := d\,(t_0 - t_2)\,\frac{1}{2-\beta} - f\,\alpha\,\frac{1-\beta}{2-\beta} - b\,,$$

which implies $F(\alpha) = \tilde{F}(\alpha, \beta(\alpha))$. Assuming that $\beta(\alpha)$ is a differentiable function on $(0, 1)$ and applying the chain rule from calculus, yields for any $\alpha \in (0, 1)$

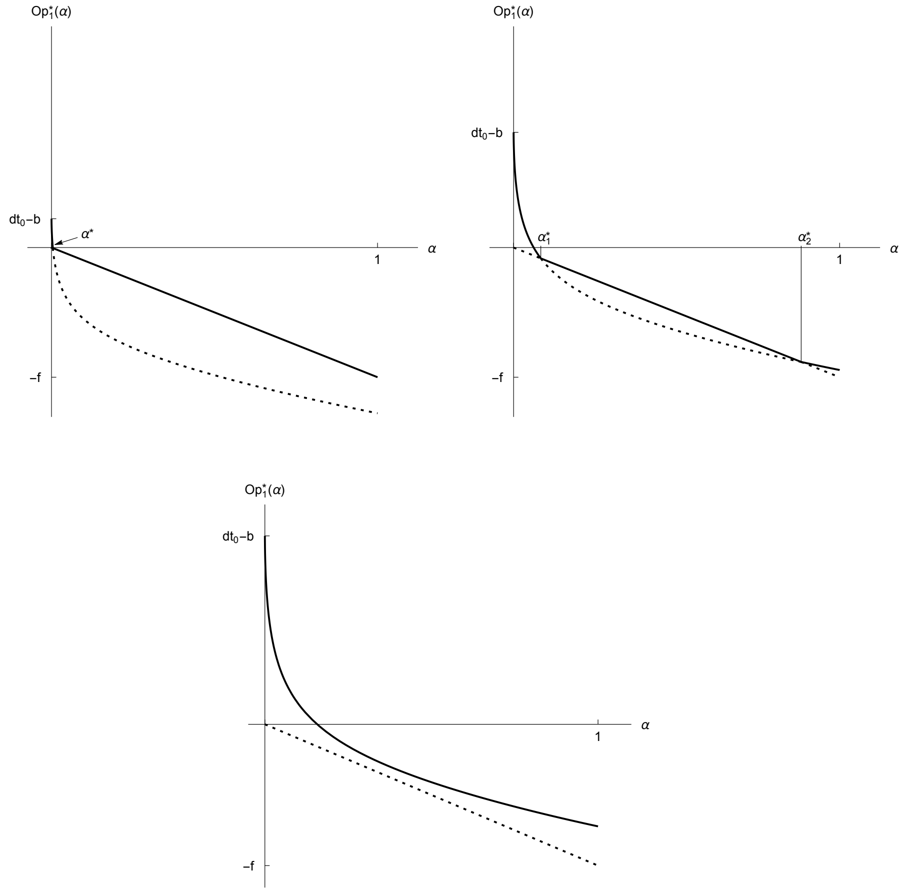
$$\frac{d}{d\alpha}\,F(\alpha) = \left(\frac{\partial}{\partial\alpha}\,\tilde{F}(\alpha,\beta), \frac{\partial}{\partial\beta}\,\tilde{F}(\alpha,\beta)\right)\Big|_{\alpha=\alpha,\beta=\beta(\alpha)} \begin{pmatrix} 1 \\ \beta'(\alpha) \end{pmatrix}$$

$$= \left(-f\,\frac{1-\beta}{2-\beta}, \frac{1}{(2-\beta)^2}\,(d\,(t_0 - t_2) + f\,\alpha)\right)\Big|_{\alpha=\alpha,\beta=\beta(\alpha)} \begin{pmatrix} 1 \\ \beta'(\alpha) \end{pmatrix},$$

which is less than zero, because of $\beta'(\alpha) < 0$.

Figure 12.10 represents $F(\alpha)$ and $-f\,\alpha$ as well as the resulting $Op_1^*(\alpha)$ using (9.74) with $(\mu_1 - \mu_0)/\sigma = 1.5$. Depending on the regions of definition, see (12.70), $F(\alpha)$ and $-f\,\alpha$ are solid or dashed, and $Op_1^*(\alpha)$ is solid for any $\alpha \in [0,1]$. To see the same effects as in Figure 9.3, we choose here $t_0 = 1, t_2 = 0, b = 8, f = 4.5$; the three graphs correspond again to $d = 9$ (left top), $d = 12$ (right top) and $d = 14$ (bottom). Note that these parameters fulfil (8.2).

Figure 12.10 The equilibrium payoff (12.70) to the Operator for $t_0 = 1, t_2 = 0, b = 8, f = 4.5$ and $d = 9$ (top left), $d = 12$ (top right) and $d = 14$ (bottom).



Because $F(\alpha)$ is a monotone decreasing function on $[0,1]$, we distinguish as in Section 9.5 the cases (i) and (ii) from (9.75) with the special cases (9.76) and (9.77). Again, because $F(\alpha) > -f\,\alpha$ for any $\alpha \in [0,1]$, in case (ii) and no intersection point, the Operator will behave illegally for all values of α (bottom graph).

In order to determine the optimal value of α, we apply again the Inspectorate Leadership Principle, see Sections 7.4 and 9.5: According to (12.9) and (12.12), the Inspectorate's equilibrium payoff is given by

$$In_1^*(\alpha) := \begin{cases} In_1^*(t_2) & \text{for Operator's illegal behaviour} \\ -g\,\alpha & \text{for Operator's legal behaviour} \end{cases},$$

$$= \begin{cases} G(\alpha) & \text{for} \quad L_2(t_2) > b/d \\ -g\,\alpha & \text{for} \quad L_2(t_2) < b/d \end{cases}, \tag{12.72}$$

where $G(\alpha)$ is, using (12.4), for any $\alpha \in [0,1]$ defined by

$$G(\alpha) := -a\,(t_0 - t_2)\,\frac{1}{2-\beta(\alpha)} - g\,\alpha\,\frac{1-\beta(\alpha)}{2-\beta(\alpha)}\,. \tag{12.73}$$

$G(\alpha)$ coincides with $In_1^*(t_2)$, see (12.72), if and only if $L_2(t_2) > b/d$. Using (9.69), (8.2) and (12.73) we get

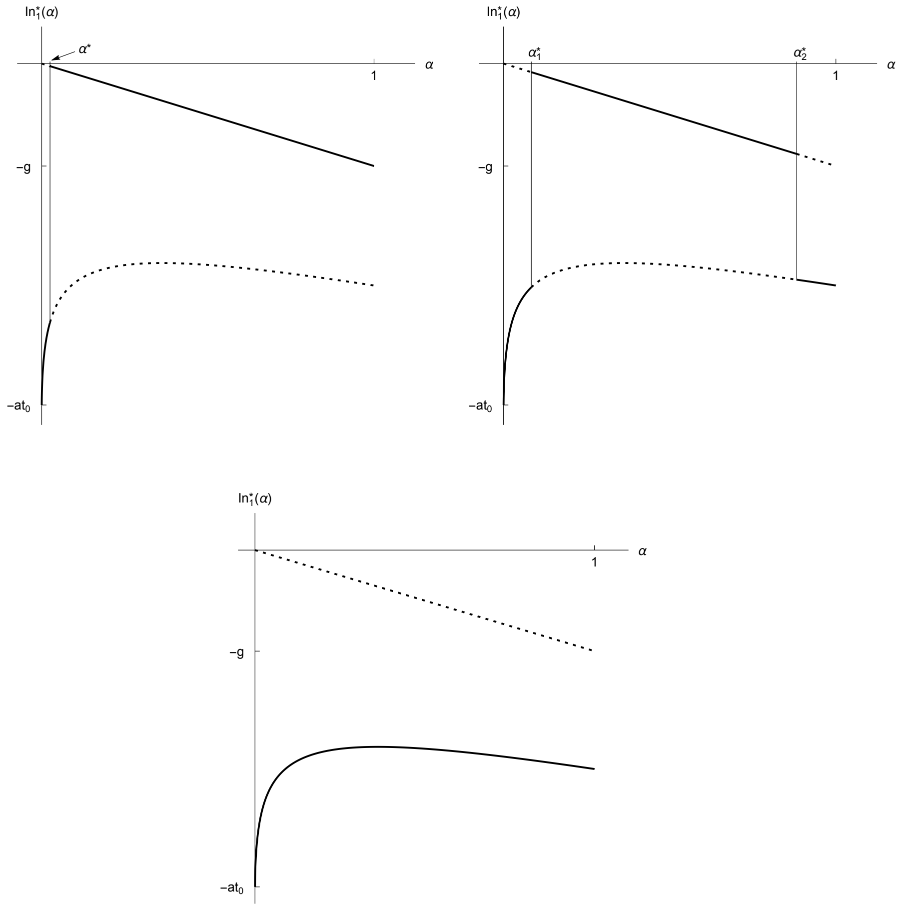
$$G(0) = -a\,(t_0 - t_2) < 0 \qquad \text{and} \qquad G(1) = -\frac{1}{2}\,(a\,(t_0 - t_2) + g) < -g\,.$$

In Figure 12.11, the solid curve represents $In_1^*(\alpha)$ for the sets of parameters used in Figure 12.10 and $a = 10$ and $g = 3$, which fulfil (8.2). Note that for plotting reasons, α^* in the top left graph is slightly shifted to the right.

We see that in case of $d = 9$ and $d = 12$ (top row), for $\alpha = \alpha^*$ resp. $\alpha = \alpha_1^*$ and legal behaviour of the Operator the Inspectorate's payoff is maximized which, as outlined in Section 9.4, is the optimal choice of both players in which the Operator is deterred from behaving illegally.

For the sake of completeness let us mention that – as in Section 9.4 – in case (ii) and no intersection point in (9.75), i.e., the bottom graph in Figure 12.11, the application of the Inspector Leadership Principle does not result in the deterrence of the Operator.

Figure 12.11 The equilibrium payoff (12.72) to the Inspectorate for the sets of parameters used in Figure 12.10 and $a = 10$ and $g = 3$.



Chapter 13

Comparison of models in Part II and between models in Parts I and II

Different as the inspection models presented in Part I are, their treatment as well as their game theoretical solutions showed considerable similarities. This is expressed best in Chapter 6 where for admittedly special cases surprising relations between optimal strategies of different models and a convincing order of the optimal payoffs was found; see (6.42) and Figure 6.7.

To some degree this holds also for the models of this Part II and also, if we compare the models and their game theoretical solutions of Parts I and II. The latter comparison, however, is not so easy for several reasons, even though Part II differs from Part I only by the fact that time is considered continuous. First, most of the inspection models in Part II take errors of the second and some of them even errors of the first kind into account which means in the latter case that payoff parameters for both players have to be introduced leading to non-zero-sum games. Second, not all variants listed in Table 2.1 have been taken into account: There is no published literature for the time continuous No-Se inspection game. Instead and due to requirements from the side of practitioners, inspection models are presented which deal with several facilities; see Chapter 11. Just recall that in Part I the No-Se inspection game was only solved in case of $N = 4$ possible time points for $k = 2$ interim inspections; see Lemma 4.1. Third, the analytical techniques for finding optimal resp. equilibrium strategies are different for the No-No inspection game and the Se-No resp. Se-Se inspection games of Part II.

First, let us start by comparing the No-No inspection game of Chapter 9 with the Se-No and Se-Se inspection games of Chapters 10 to 12. It turned out that there is a remarkable difference between these two types of inspection games. In the first one, the optimal interim inspection time points are characterized by a probability distribution. In the second one the optimal resp. equilibrium interim inspection time points are pure strategies, i.e., they may be announced, but they may also be randomized which, however, does not change the equilibrium payoffs to both players; see, e.g., p. 189 and the explanations at the end of the next paragraph.

What does this mean for practitioners? The unpredictability of interim inspections is appealing, as they would seem to place the potential violator in a permanent state of uncertainty and thus serve to deter illegal activity. In the context of routine verification under the Nuclear Weapons Non-Proliferation Treaty, Sanborn (2001) contrasts the intuitive attractiveness of (unannounced) random inspections with the substantial practical difficulties of implementing them and with the burden to the Operator in trying to accommodate them. Significantly, as mentioned above, in the models presented in Chapters 10 to 12 the Inspectorate has an optimal resp. equilibrium strategy which is a pure strategy, although it can play any mixed strategy which leads to expected optimal resp. equilibrium interim inspection time points being the same as the deterministic optimal resp. equilibrium inspection time points given by its pure strategy. Thus there is no need to randomize and the inspection schedule can be common knowledge.

Second, let us compare the game theoretical solutions of the Se-No and the Se-Se inspection games. In Part I we saw that – under conditions given there – the No-No and the No-Se inspection games on the one hand, and the Se-No and the Se-Se inspection games on the other led – after an appropriate transformation of Inspectorate's strategies – to the same optimal strategies of both players and the same optimal payoffs to the Operator. Since in Part II the No-Se inspection game is not considered, we can only compare the Se-No and the Se-Se inspection games. Taking into account the fact that in the Se-No inspection game payoff parameters are not considered whereas in the Se-Se inspection game they are, Theorems 10.1 and 12.1 and (12.69) show that for *one* facility and for $\beta > 0$ and $\alpha = 0$ optimal strategies of both players and the optimal payoffs to the Operator are the same.

But also for several facilities it can be shown – at least for the special case of $k = 2$ interim inspections in $N = 2$ facilities and no error of the first kind – that the optimal strategies of the Operator and the optimal expected detection times are the same for both inspection games and furthermore, that the optimal strategies of the Inspectorate are equivalent in the sense that they can uniquely be transformed into each other. Let us sketch this here: In the Se-Se inspection game the Inspectorate chooses at the beginning of the reference time interval in which facility the first interim inspection will take place. Let $h_3(1)$ resp. $h_3(2) = 1 - h_3(1)$ denote the probabilities that it chooses facility 1 resp. 2 for the first interim inspection and let $t_2(1)$ resp. $t_2(2)$ denote the resp. time points. If i, $i = 1, 2$, denotes the facility at which the first interim inspection is performed, then the Inspectorate chooses at time point $t_2(i)$ facility 1 with probability $h_2(1; i)$ resp. facility 2 with probability $h_2(2; i) = 1 - h_2(1; i)$ for the second interim inspection at the time points $t_1(1; 1), t_1(1; 2), t_1(2; 1)$ and $t_1(2; 2)$, depending in which facility the first inspection took place. The Operator decides like described in Section 11.2 for $N = 2$ facilities.

As in Section 11.2, let $q_{(i_2,i_1)}$ denotes the probabilities that the first interim inspection is performed in facility i_2, and the second one in facility i_1. Putting

$$\begin{aligned} h_3(1)\, h_2(1;1) &=: q_{(1,1)}\,, \qquad & h_3(1)\, h_2(2;1) &=: q_{(1,2)}\,, \\ h_3(2)\, h_2(1;2) &=: q_{(2,1)}\,, \qquad & h_3(2)\, h_2(2;2) &=: q_{(2,2)}\,, \end{aligned} \tag{13.1}$$

then it can be shown that the expected detection time for the Se-Se inspection game, which is not derived here, is the same as that for the Se-No inspection game as given by (11.24) for $N = 2$ facilities. Consequently, (11.25) and (11.26) constitutes for $N = 2$ facilities an optimal strategy of the Operator in both games. Using the inverse transformation of (13.1), i.e.,

$$h_3(1) = q_{(1,1)} + q_{(1,2)}\,, \qquad\qquad h_3(2) = q_{(2,1)} + q_{(2,2)}\,,$$

$$h_2(1;1) = \frac{q_{(1,1)}}{h_3(1)} = \frac{q_{(1,1)}}{q_{(1,1)} + q_{(1,2)}}\,, \qquad\qquad h_2(1;2) = \frac{q_{(1,2)}}{h_3(1)} = \frac{q_{(1,2)}}{q_{(1,1)} + q_{(1,2)}}\,,$$

$$h_2(2;1) = \frac{q_{(2,1)}}{h_3(2)} = \frac{q_{(2,1)}}{q_{(2,1)} + q_{(2,2)}}\,, \qquad\qquad h_2(2;2) = \frac{q_{(2,2)}}{h_3(2)} = \frac{q_{(2,2)}}{q_{(2,1)} + q_{(2,2)}}\,,$$

given the appropriate ratios exist, the Inspectorate's optimal strategy (11.27) and (11.28) for $N = 2$ facilities in the Se-No inspection game can be used to obtain an optimal strategy of the Inspectorate in the Se-Se inspection game. Obviously, the optimal expected detection times coincide. We dare to suppose that for any number of facilities and interim inspections such a transformation can be shown.

Third, let us compare the Operator's equilibrium payoffs of the No-No and the Se-Se inspection game. In Chapter 6 we saw for the special cases of $N = 3$ possible time points for $k = 2$ interim inspections and $\beta > 0$ that the Se-No and Se-Se inspection game led to larger or equal optimal expected detection times than the No-No and No-Se inspection game; see (6.42) and Figure 6.7. In Part II we can only compare the No-No and the Se-Se inspection game for $k = 1$ interim inspection and $\alpha > 0$ and $\beta > 0$. For the No-No inspection game we get for the Operator's equilibrium payoff, using (9.57) and (9.59), that

$$d\,t_0\,\frac{1}{e^{1-\beta}} + f\,\frac{\alpha}{1-\beta}\left(\beta - \frac{1}{e^{1-\beta}}\right) - b \gtrless -f\,\alpha$$

if and only if

$$\frac{b}{d} \lessgtr t_0 - \left(t_0 - \frac{\alpha}{1-\beta}\,\frac{f}{d}\right)\left(1 - \frac{1}{e^{1-\beta}}\right)\,,$$

whereas for the Se-Se inspection game we obtain by (12.9) and (12.12) for $t_2 = 0$ (in order to make both games comparable)

$$d\,t_0\,\frac{1}{2-\beta} - f\,\alpha\,\frac{1-\beta}{2-\beta} - b \gtrless -f\,\alpha$$

if and only if

$$\frac{b}{d} \lessgtr \frac{1}{2-\beta}\left(t_0 + \frac{f}{d}\,\alpha\right)\,.$$

As shown in the proof of Corollary 9.1, we know that

$$\text{if} \qquad t_0 - t^* > \frac{b}{d} \qquad \text{then} \qquad \frac{1}{2-\beta}\left(t_0 + \frac{f}{d}\,\alpha\right) > \frac{b}{d}\,.$$

Thus, in case of illegal behaviour the difference between these two equilibrium payoffs is, using (9.57) and (12.9), given by

$$d\,t_0\left(\frac{1}{e^{1-\beta}} - \frac{1}{2-\beta}\right)\left(1 - \frac{\alpha}{1-\beta}\,\frac{f}{d\,t_0}\right) < 0\,,$$

because of $\alpha + \beta < 1$ and $f < d\,t_0$; see (9.49), and (8.2) for $k = 1$ interim inspection. This means, not surprisingly, that it is a disadvantage for the Operator to play non-sequentially. Consistently with this, the limit for b/d to induce the Operator to legal behaviour is lower in the No-No inspection game.

This brings us to the equilibrium inspection strategy of the Inspectorate in case of the Operator's legal behaviour. For the No-No inspection game with $k = 1$ interim inspection, the pure equilibrium inspection time point t_1^* is according to (9.64) given by

$$t_0 - \frac{b}{d} \leq t_1^* \leq \frac{1}{1-\beta}\left(\frac{b}{d} - t_0\,\beta - \frac{f}{d}\,\alpha\right)$$

which is the same as that for the Se-Se inspection game given by (12.11) for $t_2 = 0$. Note that the condition for the existence of t_1^* coincide in both game; see (9.63) and (12.10).

The fact that the Inspectorate's set of equilibrium strategies in case of legal behaviour of the Operator coincide in the No-No and Se-Se inspection game with $k = 1$ interim inspection is at first sight so surprising, that we consider now the case of $k = 2$ interim inspections which has not been treated for the No-No inspection game with $\alpha > 0$ and $\beta > 0$. Suppose there exists a pure equilibrium strategy (t_2^*, t_1^*) of the Inspectorate. Using a generalization of (9.50), the Operator's Nash equilibrium conditions in the No-No inspection game are

$$-2\,f\,\alpha \geq \begin{cases} d\Big[(1-\beta)\,(t_2^* - s) + \\ \qquad + \beta\,(1-\beta)\,(t_1^* - s) + \ \beta^2\,(t_0 - s)\Big] - b & : \quad 0 \leq s < t_2^* < t_1^* < t_0 \\ d\Big[(1-\beta)\,(t_1^* - s) + \beta\,(t_0 - s)\Big] - \alpha\,f - b & : \quad 0 < t_2^* \leq s < t_1^* < t_0 \\ d\,(t_0 - s) - 2\,\alpha\,f - b & : \quad 0 < t_2^* < t_1^* \leq s < t_0 \end{cases},$$

which are equivalent to

$$\frac{b}{d} \geq \begin{cases} (1-\beta)\,(t_2^* - s) + \\ \qquad + \beta\,(1-\beta)\,(t_1^* - s) + \ \beta^2\,(t_0 - s) + 2\,\alpha\,\dfrac{f}{d} & : \quad 0 \leq s < t_2^* < t_1^* < t_0 \\ (1-\beta)\,(t_1^* - s) + \beta\,(t_0 - s) + \alpha\,\dfrac{f}{d} & : \quad 0 < t_2^* \leq s < t_1^* < t_0 \\ t_0 - s & : \quad 0 < t_2^* < t_1^* \leq s < t_0 \end{cases},$$

or, after taking into account those s which maximize the right hand sides of these inequalities,

$$\frac{b}{d} - \frac{2\,f}{d}\,\alpha \geq (1-\beta)\,t_2^* + \beta\,(1-\beta)\,t_1^* + \beta^2\,t_0$$

$$\frac{b}{d} - \frac{f}{d}\,\alpha \geq (1-\beta)\,(t_1^* - t_2^*) + \beta\,(t_0 - t_2^*)$$

$$\frac{b}{d} \geq t_0 - t_1^*\,,$$

which are the first three inequalities in (12.27) for the Se-Se inspection game if we take $t_3 = 0$. The Inspectorate's Nash equilibrium conditions are again fulfilled as equality.

In fact it can be seen nearly immediately that this equivalence holds for any number k of interim inspections. Suppose again the existence of a pure Inspectorate's equilibrium strategy $(t_k^*, \ldots, t_1^*)$. Then a generalization of (9.50) yields for the Operator's Nash conditions in the No-No inspection game

$$-\,k\,f\,\alpha$$

$$\geq \begin{cases} d\Big[(1-\beta)\,(t_k^*-s)+ \\ \quad +\beta\,(1-\beta)\,(t_{k-1}^*-s)+\ldots+\beta^k\,(t_0-s)\Big]-b & : \quad 0\leq s<t_k^* \\ d\Big[(1-\beta)\,(t_{k-1}^*-s)+ \\ \quad +\beta\,(1-\beta)\,(t_{k-2}^*-s)+\ldots+\beta^{k-1}\,(t_0-s)\Big]-\alpha\,f-b & : \quad t_k^*\leq s<t_{k-1}^* \\ \vdots & : \qquad \vdots \\ d\,(t_0-s)-k\,\alpha\,f-b & : \quad t_1^*\leq s<t_0 \end{cases}.$$

If we manipulate these inequalities in the same way as those for $k=2$ interim inspections and take those s which maximize their right hand sides, then we get the inequalities

$$\frac{b}{d}-\frac{k\,f}{d}\,\alpha\geq(1-\beta)\,t_k^*+\beta\,(1-\beta)\,t_{k-1}^*+\ldots+\beta^k\,t_0$$

$$\frac{b}{d}-\frac{(k-1)\,f}{d}\,\alpha\geq(1-\beta)\,(t_{k-1}^*-t_k^*)+\beta\,(1-\beta)\,(t_{k-2}^*-t_k^*)+\ldots+\beta^{k-1}\,(t_0-t_k^*)$$

$$\vdots$$

$$\frac{b}{d}\geq t_0-t_1^*\,.$$

These inequalities, however, are equivalent to those for the Se-Se inspection game; see (12.53) with $t_{k+1}=0$. Again, the Inspectorate's Nash equilibrium conditions are fulfilled as equality. Thus we have: If the Operator behaves legally and the Inspectorate has a pure equilibrium strategy $(t_k^*,\ldots,t_1^*)$, then the Inspectorate's set of pure equilibrium strategies coincides in both the No-No and the Se-Se inspection game.

Finally, let us come back to the system quantities, in our case primarily time points and intervals, which as mentioned in Section 1.4, are physical quantities and thus, may serve as important yardsticks for practitioners. Before, however, let us remember what we pointed out in detail on pp. 139, 153 and 259. The inspection games considered in Part II are either zero sum games, namely those in Sections 9.1 – 9.3, 10.1, and Chapter 11, or they are, for $\alpha=0$, strategically equivalent to zero sum games, namely those in Sections 9.4 and 10.3 and Chapter 12, all of them with the expected detection time as payoff to the Operator. This means that in the first case a priori only physical quantities are used, whereas in the second one these quantities can be deduced from the solutions as it has already been shown in Chapter 12.

In Tables 13.1 and 13.2 we have collected the most important system quantities of some of those models described and analysed in Parts I and II. In order to present an overview in which these system quantities can be compared directly, we have omitted many important results, e.g., in case of discrete time inspection games special results for $k>1$ interim inspections, or in case of continuous time inspection games results for $\alpha>0$. Note that time points, e.g., $E(S)$ and t^*, are always related to the beginning of the reference time interval so that they can also be considered as time intervals like Op_1^*. Only then an expression like $\mathbb{E}_{P^*}(S)=t^*-Op_1^*$ in Table 13.1 makes sense.

In Table 13.1 we present system quantities for the No-No inspection games with discrete and continuous time. Since, according to Theorem 3.1, there exists a solution for the discrete time case only for $k=1$ interim inspection and any number N of possible time points, for the purpose of comparison we present the corresponding results for the continuous time case also only for $k = 1$ interim inspection. Remember that the discrete and the continuous time No-Se inspection games are not generalized to any number k of interim inspections, but that for $k = 1$ interim inspection there is no difference between the No-No and the No-Se inspection games.

Table 13.1 System quantities for the No-No inspection games for $k = 1$ interim inspection and $\alpha = \beta = 0$.

	Discrete time	Continuous time
Length of the reference time interval	$N+1$	t_0
Cut-off value/ time point	$n^* = \min\left\{n : \sum_{j=1}^{n} \frac{1}{N-j+1} \geq 1\right\}$ N large: $n^* \approx (N+1)\left(1-\frac{1}{e}\right)$	$t^* = t_0\left(1-\frac{1}{e}\right)$
Optimal expected detection time	$Op^*_{N,1} = \sum_{j=1}^{n^*} \frac{N-n^*+1}{N-j+1}$ N large: $Op^*_{N,1} \approx \begin{cases} N+1-n^* \\ \frac{N+1}{e} \end{cases}$	$Op^*_1 = t_0 - t^* = \frac{t_0}{e}$ [1]
Optimal expected time point for the start of the illegal activity	$\mathbb{E}_{\mathbf{p}^*}(S) = n^* - Op^*_{N,1}$	$\mathbb{E}_{P^*}(S) = t^* - Op^*_1$
Optimal expected interim inspection time point	$\mathbb{E}_{\mathbf{q}^*}(T_1) = Op^*_{N,1}$	$\mathbb{E}_{Q^*}(T_1) = Op^*_1$ [1]

[1] Note that also for $k > 1$ interim inspections the optimal expected detection time is equal to the time from the cut-off value to the end of the reference time interval t_0, and is also equal to the time from the beginning of the reference time interval to the optimal expected time point of the first interim inspection.

We see striking similarities between the system quantities of both inspection games if we remember that the length of the reference time interval is $N + 1$ for the discrete time and t_0 for the continuous time case. Whereas this is not so surprising for large N, it is by no means trivial for the optimal expected time point for the start of the illegal activity and the optimal expected interim inspection time point; see the last two rows.

In Table 13.2 we present system quantities for the Se-No inspections games, now for any number k of interim inspections, but again only for the case $\alpha = \beta = 0$, even though for the continuous time Se-No inspections game Theorem 10.1 provides a solution for $\beta > 0$. Also remember that for the discrete and continuous time case and $\alpha = \beta = 0$, the Se-Se inspection games are equivalent to the Se-No inspection games in the sense that Inspectorate's optimal strategies can be transformed into each other. Note that the length of the reference time interval in the continuous time case is $t_0 - t_{k+1}$ due to the different procedures in both parts; in fact, in the Theorems we always could choose $t_{k+1} = 0$.

Table 13.2 System quantities for the Se-No inspection games for any number k of interim inspections, one facility and $\alpha = \beta = 0$.

	Discrete time	Continuous time
Length of the reference time interval	$N + 1$	$t_0 - t_{k+1}$
Cut-off value/ time point	Does not exist	
Optimal expected detection time	$Op^*_{N,k} = \dfrac{N+1}{k+1}$	$Op^*_k = \dfrac{t_0 - t_{k+1}}{k+1}$
Optimal expected time point for the start of the illegal activity	$\mathbb{E}_{(\mathbf{g}^*,\mathbf{q}^*)}(S) = \dfrac{k}{2}\, Op^*_{N,k}$	$\mathbb{E}_{(\mathbf{g}^*,\mathbf{t}^*)}(S) = \dfrac{k}{2}\, Op^*_k + t_{k+1}$
Optimal expected interim inspection time points	$\mathbb{E}_{\mathbf{q}^*}(T_n) = (k - n + 1)\, Op^*_{N,k}$ $n = 1, \ldots, k$	$t^*_n - t_{k+1} = (k - n + 1)\, Op^*_k$ [1] $n = 1, \ldots, k$
Optimal expected time point of the first interim inspection	$\mathbb{E}_{\mathbf{q}^*}(T_k) = Op^*_{N,k}$	$t^*_k - t_{k+1} = Op^*_k$ [1]
Optimal expected time point of the last interim inspection	$\mathbb{E}_{\mathbf{q}^*}(T_1) = k\, Op^*_{N,k}$ $= N + 1 - Op^*_{N,k}$	$t^*_1 = k\, Op^*_k = t_0 - Op^*_k$ [1]

[1] Note that because t^*_n, $n = 1, \ldots, k$, is not a random variable, they do not represent expectations but only the optimal time point(s) of the interim inspection(s).

Here, the similarities between the discrete and continuous time Se-No inspections games are even more pronounced maybe due to the fact that here a cut-off value/time point does not exist: If we identify $N + 1$ and $t_0 - t_{k+1}$, then all system quantities have the same analytical forms. But there are also similarities within the discrete and within the continuous time inspections games and between all models: The optimal expected time point of the first interim inspection (for $k = 1$ the only one) is just the optimal expected detection time.

Also, common to all these models is that the optimal expected detection times are related to the ending parts of the reference time interval: Either they are equal to the time between the last possibility for an interim inspection, n^* respectively t^*, and the end of the reference time interval, or equal to the time between the expected time point of the last interim inspection and, again, the end of the reference time interval. This holds even for the continuous time Se-No inspection game with $\beta > 0$; see (10.27).

Part III
Critical Time

In Part III inspection games are considered which are characterized by an inspection philosophy called critical time: The illegal behaviour of the Operator must be detected within that time, otherwise he is presumed to have achieved his illegal objectives. In practical applications the critical time may have quite different meanings. We mentioned on p. 4 that the critical time concept has its origin in the conversion time introduced by the IAEA: The Inspectorate has "won" the game if the illegal activity is detected within the critical time, otherwise it has "lost" it. In this and related applications, where a fixed critical time is considered, it can be argued that the Inspectorate performs its inspections only at integer multiples of the critical time. In other words, time is considered to be discrete.

In other applications, like the control of waterways or straits by Customs, the critical time may mean a night during which smugglers have a chance to cross the water without being detected even though the illegal activity itself, i.e., the crossing, may need only one hour. In fact, here there is a given number of nights in which Smugglers may try to cross the water with their boats, and customs officers try to catch them during some of these nights. It does not interest either party at which night and at which part/time of a night Smugglers are caught or not. The point is that time is not an issue in this case, but the mathematical models are basically the same for all conflict situations considered in Part III.

An important consequence of this new philosophy is that, as indicated by the wording "achieved his illegal objectives", from the very beginning idealized payoffs, in technical terms utilities, – in contrast to the detection time in Parts I and II which has a physical meaning – have to be introduced which describe the gains or losses of both players in case of detected or undetected illegal behaviour, or legal behaviour of the Operator. More, however, will be said on this important issue in the course and in particular in the final Chapter 18 of Part III.

Part III is structured as follows: We consider after the general assumptions in Chapter 14 the No-No and the Se-Se inspection games in Chapters 15 and 16. There will be, however, no No-Se nor Se-No inspection games since they have not yet been analysed. In addition, because of the large literature and due to some peculiarities, in Chapter 17 strait control models and models with multiple illegal activities are treated.

In the final Chapter 18 the models of Part III are compared and research gaps are identified.

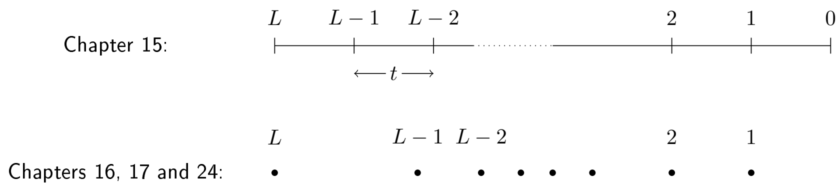
Chapter 14

General assumptions

This Part III differs from Parts I and II basically by the inspection philosophy. This implies further reaching consequences than one would imagine at the first sight. Therefore the full set of assumptions for the inspection models to be described and analysed subsequently is presented here. But, as already mentioned in the beginning of Part II, this also serves the purpose that Part III can be read independently.

Like in Chapters 2 and 8, we consider in Chapter 15 a simple inspected object, for example a production line, or a nuclear or chemical facility which is subject to inspections in the framework of agreed rules, formal agreements or an international treaty, and a reference time interval (a week, a month, or a calender year) consisting of L complete critical time periods of length t; see Figure 14.1.

Figure 14.1 Top: A reference time interval consisting of L complete critical time periods of length t. Below: L inspection opportunities with not necessarily equidistant time points.



In contrast to Parts I and II, in the inspection games in Part III fixed inspection activities at the beginning and at the end of the reference time interval (PIVs) *do not* exist. This is also the reason why the inspections are no longer called interim inspections, because the word "interim" indicates that there are some fixed inspection activities at the beginning and at the end of the reference time interval.

In addition, it is assumed that by agreement k inspections are strategically placed at integer multiples of the critical time, which is defined as the time period within which an illegal activity has to be detected. Thus, if t is the critical time and if the reference time interval consists of L complete critical time periods, the game starts at L and has the inspection opportunities $(L-1)\,t, \ldots, 1\,t, 0$, which are abbreviated by $L-1, \ldots, 1, 0$ and which are called steps; see

Figure 14.1. At an inspection a preceding failure or illegal activity will eventually be detected with some probability lower or equal than one. Also, associated with each inspection which is not preceded by a failure or an illegal activity a false alarm may be raised which is assumed to be clarified with certainty.

The Operator may behave illegally at the beginning of the reference time interval or at multiple integers of the critical time except for the end of the reference time interval, i.e., he may behave illegally at steps $L, \ldots, 1$. Again, we label the potential inspection opportunities in backward order, i.e., according to the number of remaining inspection opportunities.

The objective of the Operator is – in case he behaves illegally – to place the illegal activity such that it is not detected within the critical time, and the objective of the Inspectorate is to perform its inspections such that the illegal activity is detected within the critical time. Due to this new inspection philosophy utility functions have to be used from the very beginning of the analysis which describe the gains and losses of both the Operator and the Inspectorate for all outcomes of the inspection game; see (14.1) and (14.2).

The situation in Chapters 16 and 17 is in some aspects different to that of Chapter 15. While in Chapter 16 it can still be assumed that inspections are performed within the framework of agreed rules, formal agreements or an international treaty, this assumption can no longer be made for the inspection games in Chapter 17 between Customs and Smuggler. As already mentioned on p. 18 there exist no agreed rules, formal agreements or international treaties in these conflict situations. Nevertheless, we assume that the following assumptions hold also for these inspection games.

Again, it is assumed that in Chapter 16 by agreement and in Chapter 17 by Smugglers' long term observation of Customs' activities, see p. 18, k inspections are strategically placed at steps $L, \ldots, 1$, and that the Operator may behave illegally at steps $L, \ldots, 1$; see Figure 14.1. What has been said above about the detection of the illegal activity at an inspection resp. false alarms holds here as well. Note that in Figure 14.1 we intentionally do not use equidistant time points, because the steps can be any event which does not have to happen the same time after the previous step. Absolute time in the sense of detection time does matter in the inspection games considered here but only the fact of timeliness.

A note on wording: To be consistent with Chapter 17, in Chapter 16 we use the word "control(s)" although we refer to "inspection(s)". In Chapter 17, the steps are also called nights, due to the different applications for which the models were originally developed, and instead of Operator and Inspectorate the players are called Smuggler and Customs.

Let us summarize the assumptions in the following list, which are structured like those for Parts I and II. Note, however, that while in Parts I and II eleven assumptions are listed, here only ten are needed: Because PIVs do not exist in the inspection problems analyzed in Part III, assumption (iii) of Chapters 2 and 8 is omitted. For the sake of brevity, we formulate the following list only using the terms Operator, Inspectorate, steps and inspections.

(i) There are two players: the Operator and the Inspectorate.

(ii) The Inspectorate performs k inspections at steps $L-1, \ldots, 0$ in Chapter 15, and at steps $L, \ldots, 1$ in Chapters 16, 17 and 24.

(iii) The Operator may behave illegally at most once at the steps $L, \ldots, 1$. In Section 17.2 this assumption is extended.

(iv) During an inspection the Inspectorate may commit errors of the first and second kind with probabilities α and β. These error probabilities are the same for all k inspections.

(v) The number k of inspections is known to the Operator.[1]

(vi) The Operator decides – if he behaves illegally at all – at the beginning of the reference time interval at which step to behave illegally, or he only decides whether to behave illegally immediately at step L or to postpone it; in the latter case he decides whether to behave illegally at step $L-1$ or to postpone it again; and so on.

The Inspectorate decides at the beginning of the reference time interval at which steps to perform the inspection(s), or it decides only whether to perform an inspection at step L or not; in the latter case it decides whether to perform an inspection at step $L-1$, and so on, keeping in mind that it has to fully use k inspections.

(vii) Both players decide independently of each other, i.e., no bindings agreements are made.

(viii) The payoffs to the two players (Operator, Inspectorate) are given by

$$
\begin{array}{rl}
(d,-c) & \text{for an untimely inspection or} \\
 & \text{a timely inspection and no detection of the illegal behaviour} \\
(-b,-a) & \text{for a timely inspection and detection of the illegal behaviour} \\
(-f,-g) & \text{for legal behaviour and a false alarm} \\
(0,0) & \text{for legal behaviour and no false alarm}\,,
\end{array}
\tag{14.1}
$$

where the parameters satisfy the conditions

$$0 < f < min(b,d) \qquad \text{and} \qquad 0 < g < a < c\,. \tag{14.2}$$

(ix) An inspection does not consume time. For the model in Chapter 15 and in case of the coincidence of the start of the illegal behaviour and an inspection, the illegal behaviour may only be detected at the occasion of the next inspection.

(x) The condition when an inspection game ends are specified separately in the following chapters.

Let us comment assumption (viii): The Operator's most desirable outcome is to behave illegally without being detected, i.e., his maximum payoff is $d > 0$. Next he prefers to behave legally and that no false alarm occurs, which gives him the second-best payoff 0. His third-best outcome is to have to accept a false accusation leading to the payoff $-f$ and least of all does he want to be caught leading to the payoff $-b$. This ordering implies $-b < -f < 0 < d$ which is equivalent to the left hand inequality of (14.2) assuming that the false alarm costs f are less than the gain d in case of an undetected illegal behaviour, i.e., $f < d$.

The best, i.e., most desirable, outcome for the Inspectorate is legal behaviour of the Operator and no false alarm leading to the payoff 0. Thus, the Inspectorate's highest priority is deterrence; see Section 7.3. The second-best is to have raised a false alarm leading to the payoff $-g$. Its third-best outcome is to have detected the illegal behaviour leading to the payoff $-a$, and finally

[1] The possibility that the *expected* number of inspections is fixed and known to the Operator is addressed in Chapter 24; see also the comment on p. 18.

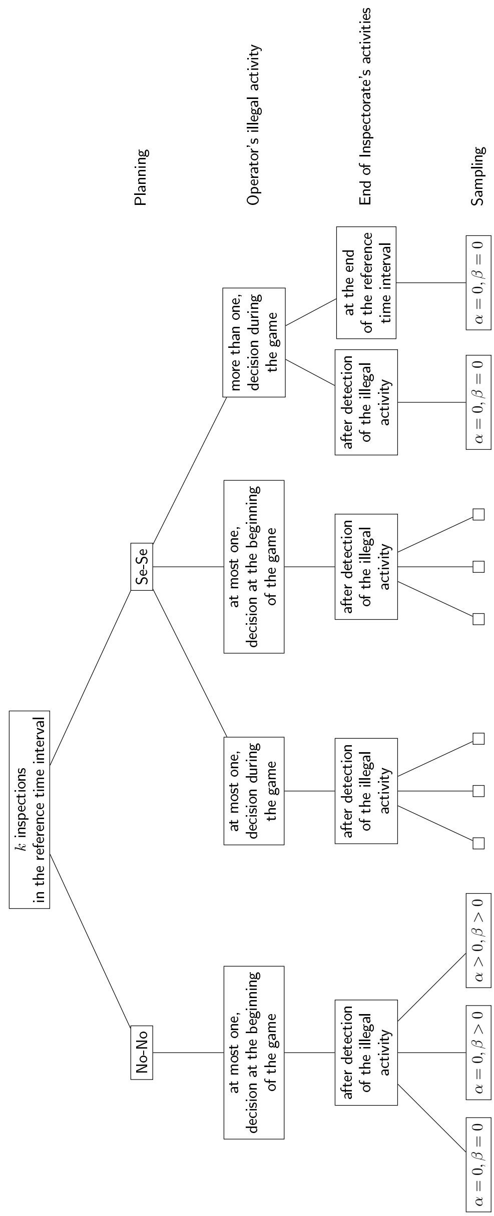
not to have detected the illegal behaviour, which gives, because it is the worst outcome, the minimum payoff $-c$. Therefore we get the ordering $-c < -a < -g < 0$ which is equivalent to the right hand inequality of (14.2).

Note that the payoff $(0, 0)$ in case of legal behaviour and no false alarm implies that inspection costs are not part of the Inspectorate's payoff, but rather imposed by the external parameter k; see also p. 136. Also note that we could have normalized a second payoff parameter for each player, e.g., to one. For the sake of clarity and for convenience in dimensional considerations, however, we have preferred not to do so. Note that the payoff $-c$ to the Inspectorate did not occur in Parts I and II, see (7.5), (7.26) and (8.1), since there it was assumed that an illegal activity is always, eventually only at the end of the reference time interval, detected by the Inspectorate and furthermore, timeliness in the sense of this part was not an issue. Also note that, as indicated in (14.1), the Operator gains d and the Inspectorate looses c not only in case of an untimely inspection, but also in case of a timely inspection and no detection – with probability β – of the illegal activity. Finally note that a and d are absolute payoffs, i.e., not proportional to time like in Parts I and II.

There is an important correspondence to what has been said on p. 153 in the context of the playing for time inspection games: If false alarms can be excluded and the Operator decides at the beginning of the game whether to behave legally or not, then the game is strategically equivalent to a zero-sum game with the timely detection probability as payoff to the Inspectorate; see also p. 391.

Finally, more assumptions have to be made which are neither covered by the classification scheme given by Figure 1.1 nor by the list presented above. In order to give an idea of this kind of assumptions, in Figure 14.2 the branch "critical time and discrete time" of Figure 1.1 is extended by two levels: "Operator's illegal activity" and "End of Inspectorate's activities". One realizes that not all possibilities of this classification scheme are shown, in fact only those are taken into account for which game theoretical models exist and are described and analysed in the following three chapters. Note that for the sake of simplicity and better overview in Figure 14.2 the wording reference time interval and inspections is maintained throughout, i.e., also for those models for which the wording steps and controls is used.

Figure 14.2 Classification of critical time inspection games. Note that in case of at most one illegal activity – which is common knowledge – the payoffs to both players are the same if the Inspectorate's activities end after the detection of an illegal activity or only at the end of the reference time; see Section 15.4 for an exception.



Chapter 15

No-No inspection game: Canty-Rothenstein-Avenhaus model

Like in Parts I and II we consider first the No-No inspection game which has been analysed by Canty et al. (2001) and which we call Canty-Rothenstein-Avenhaus inspection model. This model can also be used for the description of the conflict between Smugglers and Customs which is the subject of Chapter 17. Quite a different simple application will be discussed on p. 302. In the following we present this game theoretical model and its solution along the lines of the paper; we will, however, add some essential new findings and also some explanations. More than that, we will return to this model in Chapters 16, 17 and 24.

In this chapter, assumption (vi) of Chapter 14 is specified as follows:

(vi') The Operator decides at the beginning of the reference time interval, i.e., at the beginning of period L, when to behave illegally or whether to behave legally throughout the game.

The Inspectorate decides at the beginning of the reference time interval when to perform its k inspections.

Assumptions (iv) and (x) will be specified in the following sections, while the remaining assumptions of Chapter 14 hold throughout this chapter.

Assumption (viii) of Chapter 14 defines the payoffs to the two players (Operator, Inspectorate). Note that in Canty et al. (2001) no relation between d and f is specified, i.e., the false alarm costs f are allowed to be higher than the gain d in case of an undetected illegal activity, i.e., in case of an untimely inspection or a timely inspection and no detection of the illegal behaviour. We will come back to this point on pp. 293 and 304. Associating with each inspection an error second kind with probability β and an error first kind with probability α, the payoffs to the two players (Operator, Inspectorate) are

$$\begin{aligned} (-b\,(1-\beta)+d\,\beta, -a\,(1-\beta)-c\,\beta) &\qquad \text{for timely inspection and illegal behaviour} \\ (-f\,\alpha, -g\,\alpha) &\qquad \text{for legal behaviour.} \end{aligned} \tag{15.1}$$

If we introduce the quantities

$$A := (b+d)\,(1-\beta) > 0 \qquad \text{and} \qquad B := (c-a)\,(1-\beta) > 0\,, \tag{15.2}$$

then (15.1) simplifies to

$$\begin{array}{rl} (d-A, B-c) & \text{for timely inspection and illegal behaviour} \\ (-f\,\alpha, -g\,\alpha) & \text{for legal behaviour} \end{array} \qquad (15.3)$$

Furthermore, we assume that the Inspectorate's decision whether or not to call an alarm is based on an *unbiased test procedure*, i.e., if $\alpha+\beta<1$, which implies, using (14.2) and (15.2),

$$A - f\,\alpha > 2\,f\,(1-\beta) - f\,\alpha = 2\,f\,\alpha - f\,\alpha > 0\,, \qquad (15.4)$$

a relation, that is frequently used throughout this chapter.

In Sections 15.1 – 15.3 the No-No inspection game is analysed under the assumption that only inspections which are performed before an illegal activity may incur false alarm costs. These sections go along the lines of Canty et al. (2001). In Section 15.4 we demonstrate the sensitivity of the game theoretical solution of the No-No inspection game if now inspections which are performed before and after an illegal activity may incur false alarm costs. Section 15.5 deals with the optimal value of the false alarm probability α.

Note that in Chapter 24 a Se-No critical time inspection game with an *expected* number of inspections in one facility is analysed, see also p. 18, which has surprising relations to the No-No inspection game of Sections 15.1 – 15.3.

15.1 Two periods and one inspection; errors of the first and second kind

The inspection game analysed in this section and Sections 15.2 and 15.3 is based on the following specifications:

(iv') During an inspection the Inspectorate may commit errors of the first and second kind with probabilities α and β. These error probabilities are the same for all inspections. Only inspections which are performed before an illegal activity may incur false alarm costs. This assumption is soften on p. 307.

(x') The game ends either at the beginning of period L in case the Operator behaves legally throughout the game, or one period after the Operator behaves illegally.

Let us discuss these assumptions. Ad (iv'): Why, should one assume that only inspections which are performed before an illegal activity may incur false alarm costs? This can be justified as follows: Even though both players plan non-sequentially, the actual inspection game is played over time. If an illegal activity is detected timely, i.e., in case of a timely inspection and detection of the illegal behaviour, the game ends anyhow because of assumption (x'), and thus, false alarms and their costs after the illegal activity will not occur. If an illegal activity is not detected timely, i.e., in case of an untimely inspection or a timely inspection and no detection of the illegal behaviour, we assume that the Inspectorate will somehow find out about it and any inspections that may still be planned will no longer make sense, i.e., will not take place, because the Inspectorate has lost the game anyhow. This is expressed by the elimination of the false alarm costs. Formally: Suppose the Operator behaves illegally at the beginning of the period i, $i=L,\ldots,1$, i.e., at the beginning of $L-i+1$-th period, then at all inspections which are performed at $L-1,\ldots,i$ a false alarm may be raised which leads to false alarm costs for both players, while at all eventually planned inspections at $i-2,\ldots,0$ false alarm costs are put to zero, i.e., do not occur; see p. 307 for an inspection game with a modified assumption (iv'). Note that while in the critical time inspection games of Part III it has to be defined whether false alarms may occur after an undetected illegal activity, in the playing for time inspection games treated in Section 10.3 and Chapter 12 of Part II, this is obsolete, because once an illegal activity is started it is persistent until the interim inspection at which it is detected or the final PIV. Thus, false alarms at interim inspections performed after the start of the illegal activity do not occur by definition.

Ad (x'): Suppose the Operator behaves illegally at the beginning of the period i, $i=L,\ldots,1$, then the game ends at the beginning of period $i-1$ regardless whether the illegal behaviour is detected at $i-1$ or not. In the latter case, the Operator has successfully performed his illegal activity and thus, the game ends as well.

Let us begin by considering the inspection game with $L=2$ complete critical time intervals, i.e., two periods and $k=1$ inspection. The Operator's three pure strategies are to behave illegally at the beginning of the reference time interval, i.e., at 2, at the beginning of period 1, i.e., at 1, or to behave legally. Thus, his set of pure strategies is given by

$$I_2 = \{2, 1, \mathsf{le}\}\,.$$

The Inspectorate inspects at the end of period 2, i.e., at 1, or at the end of period 1, i.e., at 0. Hence its set of pure strategies is given by

$$J_{2,1} = \{1, 0\}\,.$$

There are two strategy combinations of both players in which the inspection is timely, namely $(2,1)$ and $(1,0)$, leading to the payoffs $d-A$ and $B-c$; see (15.3). For the strategy combination $(2,0)$ the inspection is untimely and the payoffs are d and $-c$. Note that in this case by assumption (iv') no false alarm is incurred at 0. In case of the strategy combination $(1,1)$, at 1 a false alarm may occur before the Operator behaves illegally at 1, thus, false alarm costs have to be included and the payoffs are given by $d-f\,\alpha$ and $-c-g\,\alpha$. In case of legal behaviour of the Operator the payoffs to both players are $-f\alpha$ and $-g\,\alpha$, independently of the Inspectorate's strategy. The normal form of this inspection game is given in Table 15.1.

Let p_i denote the probability for behaving illegally at i, $i=2,1$, or behaving legally p_{le}, and let q_j denote the probability of inspecting at j, $j=1,0$. Then, the set of mixed strategies of the Operator is given by

$$P_2 := \left\{\mathbf{p} = (p_2, p_1, p_{\mathsf{le}})^T \in [0,1]^3 : p_2 + p_1 + p_{\mathsf{le}} = 1\right\} \tag{15.5}$$

and that of the Inspectorate by

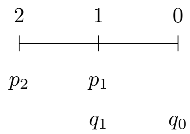
$$Q_{2,1} := \left\{\mathbf{q} := (q_1, q_0)^T \in [0,1]^2 : q_1 + q_0 = 1\right\}\,. \tag{15.6}$$

Then the (expected) payoff to the Operator is, for any $\mathbf{p}\in P_2$ and any $\mathbf{q}\in Q_{2,1}$, using (19.3), given by

$$Op_{2,1}(\mathbf{p},\mathbf{q}) = (p_2, p_1, p_{\mathsf{le}}) \begin{pmatrix} d-A & d \\ d-f\,\alpha & d-A \\ -f\,\alpha & -f\,\alpha \end{pmatrix} \begin{pmatrix} q_1 \\ q_0 \end{pmatrix}\,, \tag{15.7}$$

Table 15.1 Left: Normal form of the No-No inspection game with $L = 2$ periods and $k = 1$ inspection. Right: Reference time interval consisting of $L = 2$ complete critical time periods. The probabilities p_i, $i = 2, 1$, and q_j, $j = 1, 0$, are explained in the text.

	1	0
2	$B-c$ $d-A$	$-c$ d
1	$-c-g\,\alpha$ $d-f\,\alpha$	$B-c$ $d-A$
le	$-g\,\alpha$ $-f\,\alpha$	$-g\,\alpha$ $-f\,\alpha$



and, using (19.4), that to the Inspectorate by

$$In_{2,1}(\mathbf{p},\mathbf{q}) = (p_2, p_1 p_{\text{le}}) \begin{pmatrix} B-c & -c \\ -c-g\,\alpha & B-c \\ -g\,\alpha & -g\,\alpha \end{pmatrix} \begin{pmatrix} q_1 \\ q_0 \end{pmatrix} . \tag{15.8}$$

The game theoretical solution of this inspection game, see Canty et al. (2001), is presented in

Lemma 15.1. *Given the No-No inspection game with $L = 2$ periods and $k = 1$ inspection, errors of the first and second kind, and an unbiased test procedure. The sets of mixed strategies are given by (15.5) and (15.6), and the payoffs to both players by (15.7) and (15.8).*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{2,1} := Op_{2,1}(\mathbf{p}^*,\mathbf{q}^*)$ and $In^*_{2,1} := In_{2,1}(\mathbf{p}^*,\mathbf{q}^*)$:

(i) For

$$d > \frac{(A-f\,\alpha)^2}{2\,A-f\,\alpha} \tag{15.9}$$

the Operator behaves illegally and an equilibrium strategy is given by

$$p_2^* = \frac{B+g\,\alpha}{2\,B+g\,\alpha}, \qquad p_1^* = \frac{B}{2\,B+g\,\alpha} \qquad \text{and} \qquad p_{\text{le}}^* = 0\,. \tag{15.10}$$

An equilibrium strategy of the Inspectorate is given by

$$q_1^* = \frac{A}{2\,A-f\,\alpha} \qquad \text{and} \qquad q_0^* = \frac{A-f\,\alpha}{2\,A-f\,\alpha}\,. \tag{15.11}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{2,1} = d - \frac{A^2}{2\,A-f\,\alpha} \qquad \text{and} \qquad In^*_{2,1} = -c + \frac{B^2}{2\,B+g\,\alpha}\,. \tag{15.12}$$

(ii) For

$$d < \frac{(A - f\,\alpha)^2}{2\,A - f\,\alpha} \tag{15.13}$$

the Operator behaves legally, i.e., $p_2^* = p_1^* = 0$ *and* $p_{\text{le}}^* = 1$. *The Inspectorate's set of equilibrium strategies is given by*

$$\frac{d + f\,\alpha}{A} \leq q_1^* \leq 1 - \frac{d}{A - f\,\alpha} \qquad \textit{and} \qquad q_0^* = 1 - q_1^*\,. \tag{15.14}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_{2,1}^* = -f\,\alpha \qquad \textit{and} \qquad In_{2,1}^* = -g\,\alpha\,. \tag{15.15}$$

Proof. According to (19.6) we have to prove that $(\mathbf{p}^*, \mathbf{q}^*) \in P_2 \times Q_{2,1}$ fulfils

$$Op_{2,1}^* \geq Op_{2,1}(i, \mathbf{q}^*) \qquad \text{and} \qquad In_{2,1}^* \geq In_{2,1}(\mathbf{p}^*, j) \tag{15.16}$$

for all $i = 2, 1, \text{le}$ and for all $j = 1, 0$.

Ad (i): Because $-c < B - c$, $d - A < d - f\,\alpha$ due to (15.4), $-c - g\alpha < B - c$ and $d - A < d$, the payoffs are cyclic. Thus, the normal form game possesses a unique Nash equilibrium which is the solution of the two equations

$$Op_{2,1}(2, \mathbf{q}^*) = Op_{2,1}(1, \mathbf{q}^*) \qquad \text{and} \qquad In_{2,1}(\mathbf{p}^*, 1) = In_{2,1}(\mathbf{p}^*, 0)$$

with the solutions (15.10), (15.11) and (15.12). Condition (15.9) guarantees that

$$Op_{2,1}^* > -f\,\alpha = Op_{2,1}(\text{le}, \mathbf{q}^*)\,.$$

Thus, the Nash equilibrium conditions (15.16) are fulfilled.

Ad (ii): The Operator's Nash equilibrium condition can be, using (15.15) and (15.16), written as

$$-f\,\alpha \geq Op_{2,1}(1, \mathbf{q}^*) = q_1^*\,(d - A) + (1 - q_1^*)\,d \qquad \text{and}$$

$$-f\,\alpha \geq Op_{2,1}(2, \mathbf{q}^*) = q_1^*\,(d - f\,\alpha) + (1 - q_1^*)\,(d - A)\,,$$

which holds because of (15.14). Furthermore, we have for any $\mathbf{q} \in Q_{2,1}$ that $In_{2,1}(\text{le}, \mathbf{q}) = In_{2,1}^* = -g\,\alpha$, i.e., the Inspectorate's Nash equilibrium condition is also fulfilled. Inequality (15.13) assures the existence of q_1^* with $0 < q_1^* < 1$. □

Let us comment the results of Lemma 15.1: First, equality will be considered neither in (15.9) nor in (15.13) since the payoff parameters can be estimated only imprecisely in reality. For later purposes let us note the following equivalent formulation of condition (15.9): With

$$x := 1 - \frac{f\,\alpha}{A}$$

for which $0 < x < 1$ holds because of (15.4), we get, using (15.9), that

$$d > \frac{A\,x^2}{1 + x}$$

is equivalent to

$$x^2 < \left(1 + \frac{A}{d}\,(1-x)\right)^{-1} . \tag{15.17}$$

We will find a generalization of this condition in Theorem 15.1.

Second, in case of $\alpha = 0$, i.e., if attribute sampling is considered, case (i) leads to equilibrium strategies which are independent of the payoff parameters. In fact, this could have been guessed since in this case neither the first nor the second inspection opportunity can be preferred to the other.

Third, because condition (15.13) is equivalent to both

$$\frac{d + f\,\alpha}{A} < \frac{A}{2\,A - f\,\alpha} \qquad \text{and} \qquad \frac{A}{2\,A - f\,\alpha} < 1 - \frac{d}{A - f\,\alpha}\,,$$

the equilibrium strategy $(q_1^*, q_0^*)^T$ given by (15.11) fulfills (15.14). Thus, $(q_1^*, q_0^*)^T$ is a robust equilibrium strategy in the sense that given the payoff parameters d, b and f, and the probabilities α and β, the Inspectorate can just play $(q_1^*, q_0^*)^T$ according to (15.11) and does not need to check whether (15.9) or (15.13) is fulfilled. A corresponding statement for No-No inspection games with $k \geq 2$ inspections and $L > k$ periods only holds for $\alpha = 0$; see pp. 292 and 304, and Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

Fourth, it can be shown, for example with the LCP-formalism, see Avenhaus and Canty (1996) Chapter 9, that (15.10) and (15.11) together with the legal behaviour solution (15.14) are in fact the only Nash equilibria.

Finally, since the detection probability $1 - \beta$ is a function of the false alarm probability α and since the latter can be fixed by the Inspectorate, the question arises as to the "best" value of α. We will return to this question in Section 15.5.

15.2 Three periods and two inspections; errors of the first and second kind

Before turning to the general case, let us consider the inspection game with $L = 3$ periods and $k = 2$ inspections, because here two new features which do not yet occur in the case of $L = 3$ periods and $k = 1$ inspection can be studied: First, the Inspectorate's mixed strategies are transformed into strategies better suited the solve the inspection game, see (15.21), and second, in the game theoretical solution three cases instead of two need to be distinguished in Lemma 15.2. Also, the case of $L = 3$ periods and $k = 2$ inspections shows already many features of the general case which is analysed in Section 15.3.

The inspection game analysed in this section is based on the specifications (iv') and (x') on p. 282.

The Operator's four pure strategies are to behave illegally at the beginning of period i, $i = 3, 2, 1$, or to behave legally. Thus, his set of pure strategies is given by

$$I_3 = \{3, 2, 1, \mathsf{le}\}\,.$$

The Inspectorate performs its inspections at $(2,1), (2,0)$ or $(1,0)$. Thus, its set of pure strategies is given by

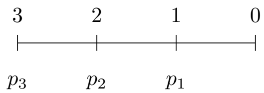
$$\tilde{J}_{3,2} := \{(2,1), (2,0), (1,0)\} \ .$$

The notation $\tilde{J}_{3,2}$ instead of $J_{3,2}$ is chosen, because for the general game (L, k) treated in Section 15.3 a different representation of the pure strategies is used.

In Table 15.2 the normal form of this inspection game is represented. The entries in the bimatrix have, using (15.3), exactly the same meaning as those in the bimatrix in Table 15.1, and need not be explained again.

Table 15.2 Left: Normal form of the No-No inspection game with $L = 3$ periods and $k = 2$ inspections. Right: Reference time interval consisting of $L = 3$ complete critical time periods. The probabilities p_i, $i = 3, 2, 1$, are explained in the text.

	$(2,1)$	$(2,0)$	$(1,0)$
3	$B-c$ $d-A$	$B-c$ $d-A$	$-c$ d
2	$B-c-g\,\alpha$ $d-A-f\,\alpha$	$-c-g\,\alpha$ $d-f\,\alpha$	$B-c$ $d-A$
1	$-c-2\,g\,\alpha$ $d-2\,f\,\alpha$	$B-c-g\,\alpha$ $d-A-f\,\alpha$	$B-c-g\,\alpha$ $d-A-f\,\alpha$
le	$-2\,g\,\alpha$ $-2\,f\,\alpha$	$-2\,g\,\alpha$ $-2\,f\,\alpha$	$-2\,g\,\alpha$ $-2\,f\,\alpha$



In analogy to the Section 15.1, let p_i denote the probability for behaving illegally at i, $i = 3, 2, 1$, or behaving legally p_{le}. Thus, the Operator's set of mixed strategies is given by

$$P_3 := \left\{ \mathbf{p} = (p_3, p_2, p_1, p_{\mathsf{le}})^T \in [0,1]^4 : p_3 + p_2 + p_1 + p_{\mathsf{le}} = 1 \right\} \ . \tag{15.18}$$

The mixed strategies of the Inspectorate are, for reasons to be understood in Section 15.3, denoted by $\tilde{q}_{(2,1)}, \tilde{q}_{(2,0)}$ and $\tilde{q}_{(1,0)}$. Thus, its set of mixed strategies is given by

$$\tilde{Q}_{3,2} := \left\{ \tilde{\mathbf{q}} = (\tilde{q}_{(2,1)}, \tilde{q}_{(2,0)}, \tilde{q}_{(1,0)})^T \in [0,1]^3 : \tilde{q}_{(2,1)} + \tilde{q}_{(2,0)} + \tilde{q}_{(1,0)} = 1 \right\} \ .$$

Then, using Table 15.2, the payoff to the Operator is for any of his pure strategies and any

$\tilde{\mathbf{q}} = (\tilde{q}_{(2,1)}, \tilde{q}_{(2,0)}, \tilde{q}_{(1,0)})^T \in \tilde{Q}_{3,2}$ given by

$$\begin{aligned} Op_{3,2}(3, \tilde{\mathbf{q}}) &= \tilde{q}_{(2,1)}\,(d - A) + \tilde{q}_{(2,0)}\,(d - A) + \tilde{q}_{(1,0)}\,d \\ Op_{3,2}(2, \tilde{\mathbf{q}}) &= \tilde{q}_{(2,1)}\,(d - A - f\,\alpha) + \tilde{q}_{(2,0)}\,(d - f\,\alpha) + \tilde{q}_{(1,0)}\,(d - A) \\ Op_{3,2}(1, \tilde{\mathbf{q}}) &= \tilde{q}_{(2,1)}\,(d - 2\,f\,\alpha) + \tilde{q}_{(2,0)}\,(d - A - f\,\alpha) + \tilde{q}_{(1,0)}\,(d - A - f\,\alpha) \\ Op_{3,2}(\mathsf{le}, \tilde{\mathbf{q}}) &= -2\,f\,\alpha\,. \end{aligned} \tag{15.19}$$

Let q_j, $j = 2, 1, 0$, be the probability that at j an inspection is performed. Then we have

$$q_2 = \tilde{q}_{(2,1)} + \tilde{q}_{(2,0)}\,, \qquad q_1 = \tilde{q}_{(2,1)} + \tilde{q}_{(1,0)} \qquad \text{and} \qquad q_0 = \tilde{q}_{(2,0)} + \tilde{q}_{(1,0)}\,, \tag{15.20}$$

and consequently

$$\sum_{j=0}^{2} q_j = 2\,.$$

Thus, we define the Inspectorate's strategy set (not a set of mixed strategies) by

$$Q_{3,2} := \left\{ \mathbf{q} = (q_2, q_1, q_0)^T \in [0,1]^3 : \sum_{j=0}^{2} q_j = 2 \right\}. \tag{15.21}$$

Note that here in case of $L = 3$ periods and $k = 2$ inspections there is a one-to-one correspondence between q_j, $j = 2, 1, 0$, in (15.21) and the elements of (15.20):

$$\begin{pmatrix} \tilde{q}_{(2,1)} \\ \tilde{q}_{(2,0)} \\ \tilde{q}_{(1,0)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} q_2 \\ q_1 \\ q_0 \end{pmatrix}. \tag{15.22}$$

Thus, for any $\tilde{\mathbf{q}} \in \tilde{Q}_{3,2}$ there exists a $\mathbf{q} \in Q_{3,2}$ such that (15.19) can be written as

$$\begin{aligned} Op_{3,2}(3, \tilde{\mathbf{q}}) &= d - A\,q_2 =: Op_{3,2}(3, \mathbf{q}) \\ Op_{3,2}(2, \tilde{\mathbf{q}}) &= d - (A\,q_1 + f\,\alpha\,q_2) =: Op_{3,2}(2, \mathbf{q}) \\ Op_{3,2}(1, \tilde{\mathbf{q}}) &= d - (A\,q_0 + f\,\alpha\,(q_1 + q_2)) =: Op_{3,2}(1, \mathbf{q}) \\ Op_{3,2}(\mathsf{le}, \tilde{\mathbf{q}}) &= -2\,f\,\alpha =: Op_{3,2}(\mathsf{le}, \mathbf{q}) \end{aligned} \tag{15.23}$$

or, in closed form, as

$$Op_{3,2}(i, \mathbf{q}) = d - A\,q_{i-1} - f\,\alpha \sum_{j=i}^{2} q_j\,, i = 3, 2, 1, \quad \text{and} \quad Op_{3,2}(\mathsf{le}, \mathbf{q}) = -2\,f\,\alpha\,, \tag{15.24}$$

where $\sum_{j=3}^{2} q_j := 0$. Hence, the (expected) payoff to the Operator is, for any $\mathbf{p} \in P_3$ and any $\mathbf{q} \in Q_{3,2}$, given by

$$\begin{aligned} Op_{3,2}(\mathbf{p}, \mathbf{q}) &= \sum_{i=1}^{3} p_i\,Op_{3,2}(i, \mathbf{q}) - p_{\mathsf{le}}\,2\,f\,\alpha \\ &= \sum_{i=1}^{3} p_i \left[d - A\,q_{i-1} - f\,\alpha \sum_{j=i}^{2} q_j \right] - p_{\mathsf{le}}\,2\,f\,\alpha\,. \end{aligned} \tag{15.25}$$

To obtain a closed form of the Inspectorate's payoff, we first get with Table 15.2 for any $\tilde{\mathbf{q}} = (\tilde{q}_{(2,1)}, \tilde{q}_{(2,0)}, \tilde{q}_{(1,0)})^T \in \tilde{Q}_{3,2}$

$$\begin{aligned}
In_{3,2}(3, \tilde{\mathbf{q}}) &= B\,(\tilde{q}_{(2,1)} + \tilde{q}_{(2,0)}) - c \\
In_{3,2}(2, \tilde{\mathbf{q}}) &= B\,(\tilde{q}_{(2,1)} + \tilde{q}_{(1,0)}) - g\,\alpha\,(\tilde{q}_{(2,1)} + \tilde{q}_{(2,0)}) - c \\
In_{3,2}(1, \tilde{\mathbf{q}}) &= -2\,g\,\alpha\,\tilde{q}_{(2,1)} + (B - g\,\alpha)\,(\tilde{q}_{(2,0)} + \tilde{q}_{(1,0)}) - c \\
In_{3,2}(\mathsf{le}, \tilde{\mathbf{q}}) &= -2\,g\,\alpha\,,
\end{aligned}$$

which implies, using (15.20) and (15.22),

$$\begin{aligned}
In_{3,2}(3, \tilde{\mathbf{q}}) &= B\,q_2 - c =: In_{3,2}(3, \mathbf{q}) \\
In_{3,2}(2, \tilde{\mathbf{q}}) &= B\,q_1 - g\,\alpha\,q_2 - c =: In_{3,2}(2, \mathbf{q}) \\
In_{3,2}(1, \tilde{\mathbf{q}}) &= -g\,\alpha\,(q_2 + q_1 - q_0) + (B - g\,\alpha)\,q_0 - c =: In_{3,2}(1, \mathbf{q}) \\
In_{3,2}(\mathsf{le}, \tilde{\mathbf{q}}) &= -2\,g\,\alpha =: In_{3,2}(\mathsf{le}, \mathbf{q})\,.
\end{aligned} \tag{15.26}$$

Thus, we get by (15.26) for $i = 3, 2, 1$

$$In_{3,2}(i, \mathbf{q}) = B\,q_{i-1} - c - g\,\alpha \sum_{j=i}^{2} q_j\,.$$

Therefore, the (expected) payoff to the Inspectorate is, for any $\mathbf{p} \in P_3$ and any $\mathbf{q} \in Q_{3,2}$, given by

$$\begin{aligned}
In_{3,2}(\mathbf{p}, \mathbf{q}) &= \sum_{i=1}^{3} p_i\,In_{3,2}(i, \mathbf{q}) - p_{\mathsf{le}}\,2\,g\,\alpha \\
&= \sum_{i=1}^{3} p_i \left[B\,q_{i-1} - c - g\,\alpha \sum_{j=i}^{2} q_j \right] - p_{\mathsf{le}}\,2\,g\,\alpha\,.
\end{aligned} \tag{15.27}$$

Define

$$x := 1 - \frac{f\,\alpha}{A} \quad (\in (0,1)) \qquad \text{and} \qquad y := 1 + \frac{g\,\alpha}{B} \quad (> 1)\,. \tag{15.28}$$

The game theoretical solution of this inspection game, which is structurally more complicated than the one of Lemma 15.1, is presented in

Lemma 15.2. *Given the No-No inspection game with $L = 3$ periods and $k = 2$ inspections, errors of the first and second kind, and an unbiased test procedure. The Operator's set of mixed strategies is given by (15.18), the Inspectorate's strategy set by (15.21), and the payoffs to both players by (15.25) and (15.27). Let x and y be defined by (15.28).*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{3,2} := Op_{3,2}(\mathbf{p}^*, \mathbf{q}^*)$ and $In^*_{3,2} := In_{3,2}(\mathbf{p}^*, \mathbf{q}^*)$:

(i) For

$$x^3 < \min\left[1 - 2\,(1-x), \left(1 + 2\,\frac{A}{d}\,(1-x)\right)^{-1}\right] \tag{15.29}$$

the Operator behaves illegally and an equilibrium strategy is given by

$$p_i^* = \frac{y-1}{y^3-1}\,y^{i-1}\,, \quad i = 3,2,1, \qquad \textit{and} \qquad p_{le}^* = 0\,. \tag{15.30}$$

An equilibrium strategy of the Inspectorate is given by

$$q_j^* = 2\,\frac{1-x}{1-x^3}\,x^{2-j}\,, \quad j = 2,1,0\,. \tag{15.31}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_{3,2}^* = d - 2\,A\,\frac{1-x}{1-x^3} \qquad \textit{and} \qquad In_{3,2}^* = -c + 2\,B\,\frac{y-1}{y^3-1}\,. \tag{15.32}$$

(ii) For

$$\left(1 + 2\,\frac{A}{d}\,(1-x)\right)^{-1} < 1 - 2\,(1-x) \quad \textit{and} \quad \left(1 + 2\,\frac{A}{d}\,(1-x)\right)^{-1} < x^3 \tag{15.33}$$

the Operator behaves legally, i.e., $p_3^* = p_2^* = p_1^* = 0$ *and* $p_{le}^* = 1$*. The Inspectorate's set of equilibrium strategies is given by*

$$q_2^* \geq \frac{d}{A} + 2\,(1-x) \tag{15.34}$$

$$q_1^* + (1-x)\,q_2^* \geq \frac{d}{A} + 2\,(1-x) \tag{15.35}$$

$$q_0^* + (1-x)\,(q_1^* + q_2^*) \geq \frac{d}{A} + 2\,(1-x) \tag{15.36}$$

$$q_2^* + q_1^* + q_0^* = 2\,. \tag{15.37}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_{3,2}^* = -2\,A\,(1-x) = -2\,f\,\alpha \qquad \textit{and} \qquad In_{3,2}^* = -2\,B\,(y-1) = -\,2\,g\,\alpha\,.$$

(iii) For

$$1 - 2\,(1-x) < \left(1 + 2\,\frac{A}{d}\,(1-x)\right)^{-1} \quad \textit{and} \quad 1 - 2\,(1-x) < x^3 \tag{15.38}$$

the Operator behaves illegally and an equilibrium strategy is given by

$$p_3^* = 1\,, \qquad p_2^* = p_3^* = 0 \qquad \textit{and} \qquad p_{le}^* = 0\,. \tag{15.39}$$

An equilibrium strategy of the Inspectorate is given by

$$q_2^* = 1\,, \qquad q_1^* = x \qquad \textit{and} \qquad q_0^* = 1 - x\,. \tag{15.40}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_{3,2}^* = d - A \qquad \textit{and} \qquad In_{3,2}^* = -c + B\,. \tag{15.41}$$

Proof. To prove the Nash equilibrium conditions, we have to show that the inequalities

$$Op^*_{3,2} \geq Op_{3,2}(i, \mathbf{q}^*) \,, \tag{15.42}$$

$$In^*_{3,2} \geq In_{3,2}(\mathbf{p}^*, \mathbf{q}) \tag{15.43}$$

are fulfilled for all $i = 3, 2, 1, \text{le}$ and any $\mathbf{q} \in Q_{3,2}$; see (19.8).

Ad (i): It is clear that p^*_i, $i = 3, 2, 1$, as well as q^*_j, $j = 2, 1, 0$, as given by (15.30) and (15.31) both sum up to 1. Because $y > 1$, we get from (15.30) that $0 < p^*_1 < p^*_2 < p^*_3 < 1$ and therefore, $\mathbf{p}^* \in P_3$. Using (15.31) and the fact that $x \in (0, 1)$, we obtain for the Inspectorate's equilibrium strategy $0 < q^*_0 < q^*_1 < q^*_2$. Thus, $q^*_2 < 1$ if an only if $x^3 < 1 - 2\,(1 - x)$, which is fulfilled due to (15.29), i.e., $\mathbf{q}^* \in Q_{3,2}$.

Because $f\,\alpha = A\,(1 - x)$, see (15.28), we get, using (15.24), (15.31) and (15.32),

$$Op_{3,2}(3, \mathbf{q}^*) = d - 2\,A\,\frac{1 - x}{1 - x^3} = Op^*_{3,2}$$

$$Op_{3,2}(2, \mathbf{q}^*) = d - 2\,\frac{1 - x}{1 - x^3}\,(A\,x + A\,(1 - x)) = Op^*_{3,2}$$

$$Op_{3,2}(1, \mathbf{q}^*) = d - 2\,\frac{1 - x}{1 - x^3}\,(A\,x^2 + A\,(1 - x)\,(1 + x)) = Op^*_{3,2} \,.$$

Because the inequality

$$x^3 < \left(1 + 2\,\frac{A}{d}\,(1 - x)\right)^{-1}$$

is equivalent to

$$-2\,f\,\alpha = -2\,A\,(1 - x) < d - 2\,A\,\frac{1 - x}{1 - x^3} \,,$$

we obtain, using (15.24),

$$Op_{3,2}(\text{le}, \mathbf{q}^*) = -2\,f\,\alpha < d - 2\,A\,\frac{1 - x}{1 - x^3} = Op^*_{3,2} \,.$$

Therefore, (15.42) is fulfilled as equality for $i = 3, 2, 1$ and as inequality for $i = \text{le}$.

To prove the Inspectorate's Nash equilibrium inequality we get, using (15.27), (15.30) and $g\,\alpha = (y - 1)\,B$,

$$\begin{aligned} In_{3,2}(\mathbf{p}^*, \mathbf{q}) &= \sum_{i=1}^{3} \frac{y - 1}{y^3 - 1}\,y^{i-1} \left[B\,q_{i-1} - c - g\,\alpha \sum_{j=i}^{2} q_j\right] \\ &= -c + B\left[\sum_{i=1}^{3} \frac{y - 1}{y^3 - 1}\,y^{i-1}\,q_{i-1} - (y - 1) \sum_{i=1}^{3} \frac{y - 1}{y^3 - 1}\,y^{i-1} \sum_{j=i}^{2} q_j\right] \\ &= -c + B\left[\sum_{i=1}^{3} \frac{y - 1}{y^3 - 1}\,y^{i-1}\,q_{i-1} - (y - 1)^2 \sum_{j=1}^{2} q_j \sum_{i=1}^{j} \frac{y^{i-1}}{y^3 - 1}\right] \end{aligned}$$

$$= -c + B \left[\sum_{i=1}^{3} \frac{y-1}{y^3-1}\, y^{i-1}\, q_{i-1} + (y-1) \sum_{j=1}^{2} q_j \, \frac{1-y^j}{y^3-1} \right]$$

$$= -c + B \left[\frac{y-1}{y^3-1}\, q_0 + \frac{y-1}{y^3-1} \sum_{j=1}^{2} q_j \right] = -c + 2\,B\, \frac{y-1}{y^3-1}$$

for any $\mathbf{q} \in Q_{3,2}$, i.e., (15.43) is fulfilled as equality.

Ad (ii): The left hand inequality in (15.33) is equivalent to $d/A+2\,(1-x) < 1$, which assures the existence of $q_2^* \in [0,1]$ fulfilling (15.34), and further implies the existence of q_1^* and q_0^* according to (15.35) and (15.36). Because (15.34) – (15.36) are equivalent to $-2\,f\,\alpha \geq Op_{3,2}(i, \mathbf{q}^*)$ for all $i = 3, 2, 1$, (15.42) is fulfilled. Using (15.27), we have $In_{3,2}(\text{le}, \mathbf{q}) = In_{3,2}^* = -2\,g\,\alpha$ for any $\mathbf{q} \in Q_{3,2}$, i.e., (15.43) is fulfilled as equality.

Ad (iii): From (15.23) and (15.40) we get

$$Op_{3,2}(3, \mathbf{q}^*) = Op_{3,2}(2, \mathbf{q}^*) = d - A$$

$$Op_{3,2}(1, \mathbf{q}^*) = d - (A\,(1-x) + f\,\alpha\,(1+x)) = d - A\,(2 - x - x^2)\,. \qquad (15.44)$$

Because the right hand inequality of (15.38) is equivalent to $1 < 2 - x - x^2$ we obtain $Op_{3,2}(1, \mathbf{q}^*) < d - A$ by (15.44). Finally, because the left hand inequality of (15.38) is equivalent to $-2\,f\,\alpha = -2\,A\,(1-x) < d - A$, we get $Op_{3,2}(\text{le}, \mathbf{q}^*) = -2\,f\,\alpha < d - A$, i.e., (15.42) is fulfilled. Using (15.27) and (15.39), we have $In_{3,2}(\mathbf{p}^*, \mathbf{q}) = B\,q_2 - c$, which is maximized for $q_2^* = 1$, i.e., (15.43) is valid. This completes the proof. □

Let us comment the results of Lemma 15.2: First, as mentioned at the beginning of this section, the solution given by Lemma 15.2 shows two new features as compared to that of Lemma 15.1: 1) there is a new type of solution, namely (iii), and 2) the Inspectorate's strategies are reformulated, because the equilibrium strategy can be better expressed the new way. Also, as shown in (15.22), there is a one-to-one-correspondence between the original strategies $\tilde{q}_{(2,1)}, \tilde{q}_{(2,0)}$ and $\tilde{q}_{(1,0)}$ and the transformed ones q_j, $2, 1, 0$. This is no longer true in the general inspection game with L periods and $L > k$ inspections to be considered in Section 15.3. On the other hand, we will see that the game theoretical solution of the general game has the same structure as that with $L = 3$ periods and $k = 2$ inspections.

Second, because $x > 0$, the case $\alpha = 0$ is not covered in Lemma 15.2, but can be obtained by considering $x \to 1$ and $y \to 1$. The result is a special case of Corollary 15.1.

Third, the regions of the solutions, i.e., conditions (15.29), (15.33) and (15.38), are represented in Figure 15.3.

Fourth, in the inspection game with $L = 2$ periods and $k = 1$ inspection we have shown that the equilibrium strategy $(q_1^*, q_0^*)^T$ given by (15.11) is a robust equilibrium strategy in the sense that playing $(q_1^*, q_0^*)^T$ is always an equilibrium strategy no matter whether (15.9) or (15.13) is fulfilled. A corresponding statement for the inspection game with $L = 3$ periods and $k = 2$ inspections does not hold for two reasons: First, the equilibrium strategy (15.31) yields $q_2^* > 1$ under the right hand inequality in (15.38), which is a contradiction. Second, (15.31) solves (15.34) – (15.37) if and only if in addition to (15.33) the inequality $x^3 < 1 - 2\,(1-x)$ is fulfilled, which assures $q_2^* < 1$. Thus, (15.31) is not a robust equilibrium strategy in the sense that it is an equilibrium strategy no matter whether (15.29), (15.33) or (15.38) is fulfilled. In

Corollary 15.1 it is shown that for $\alpha = 0$ the case (iii) in Lemma 15.2 vanishes, and that (15.31) is then indeed a robust equilibrium strategy; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

Finally, as mentioned on p. 281, Canty et al. (2001) do not specify a relation between d and f, but only $f < b$ is assumed. Thus, the case $1 - 2(1-x) = 1 - 2f\alpha/A < 0$ is considered that leads in case (iii) to an additional equilibrium strategy of the Inspectorate: $\mathbf{q}^* := (q_2^*, q_1^*, q_0^*)^T = (1,1,0)^T$ and to the same equilibrium payoffs (15.41): From (15.23) and (15.40) we get

$$Op_{3,2}(3,\mathbf{q}^*) = d - A\,, \qquad Op_{3,2}(2,\mathbf{q}^*) = d - A - f\,\alpha < d - A = Op_{3,2}(3,\mathbf{q}^*)$$

$$Op_{3,2}(1,\mathbf{q}^*) = d - 2\,f\,\alpha < d - A\,, \qquad Op_{3,2}(\mathsf{le},\mathbf{q}^*) = -2\,f\,\alpha < -A < d - A\,,$$

i.e., the Operator's inequalities in (15.42) are fulfilled. For the Inspectorate we get, using (15.27) and (15.39), as in the proof of the Lemma that $In_{3,2}(\mathbf{p}^*,\mathbf{q}) = B\,q_2 - c$, which is maximized for $q_2^* = 1$. The case $f > d$ seems to be of little practical relevance, because the payoff f in case of a false alarm is usually much smaller than the gain d for an undetected illegal behaviour; see also the comment on p. 304.

15.3 Any number of periods and inspections; errors of the first and second kind

Let us now turn to the general case of $L > k$ periods and k inspections. The inspection game analysed in this section is based again on the specifications (iv') and (x') on p. 282.

The Operator's pure strategies are to behave illegally at the beginning of period i, $i = L, \ldots, 1$, or to behave legally. Thus, his set of pure strategies is given by

$$I_L := \{L, L-1, \ldots, 2, 1, \mathsf{le}\}\,,$$

and the corresponding set of mixed strategies by

$$P_L := \left\{ \mathbf{p} = (p_L, \ldots, p_1, p_{\mathsf{le}})^T \in [0,1]^{L+1} : \sum_{i=1}^{L} p_i + p_{\mathsf{le}} = 1 \right\}. \tag{15.45}$$

Again, p_i denotes the probability of behaving illegally at i, $i = L, \ldots, 1$, or behaving legally with probability p_{le}. In contrast to Sections 15.1 and 15.2 we do not consider bimatrix games in this section. Therefore, it is of no importance whether the elements in (15.45) and below in (15.47) are seen to be row or column vectors. To be consistent with the notation in Sections 15.1 and 15.2, however, we keep the notation as column vectors.

A pure strategy of the Inspectorate is a k-tupel $(j_k, \ldots, j_1)$ with $L > j_k > \ldots > j_1 \geq 0$ where j_n means that the $k - n + 1$-th inspection is performed at j_n. A pure strategy $(j_k, \ldots, j_1)$ can be identified with an L-tupel $\mathbf{r} = (r_{L-1}, r_L, \ldots, r_0)^T \in \{0,1\}^L$ with $\sum_{j=0}^{L-1} r_j = k$. Since this representation is more useful in the context here we define the set of pure strategies $J_{L,k}$ of the Inspectorate by

$$J_{L,k} := \left\{ \mathbf{r} = (r_{L-1}, r_L, \ldots, r_0)^T \in \{0,1\}^L : \sum_{j=0}^{L-1} r_j = k \right\}, \tag{15.46}$$

containing $\binom{L}{k}$ elements. If $\tilde{q}_{\mathbf{r}}$ denotes the probability that the pure strategy $\mathbf{r} \in J_{L,k}$ is chosen, then a mixed strategy $\tilde{\mathbf{q}} = (\tilde{q}_{\mathbf{r}})^T_{\mathbf{r}\in J_{L,k}}$ with $\sum_{\mathbf{r}\in J_{L,k}} \tilde{q}_{\mathbf{r}} = 1$ of the Inspectorate is a probability distribution over its set of pure strategies, i.e., the Inspectorate's set of mixed strategies is given by

$$\tilde{Q}_{L,k} := \left\{ \tilde{\mathbf{q}} = (\tilde{q}_{\mathbf{r}})^T_{\mathbf{r}\in J_{L,k}} \in [0,1]^{\binom{L}{k}} : \sum_{\mathbf{r}\in J_{L,k}} \tilde{q}_{\mathbf{r}} = 1 \right\} . \tag{15.47}$$

Note that for solving the Canty-Rothenstein-Avenhaus inspection game a different strategy set of the Inspectorate is considered; see (15.51) and p. 304.

The payoff to the Operator resp. the Inspectorate is for any $\mathbf{r} \in J_{L,k}$ given by

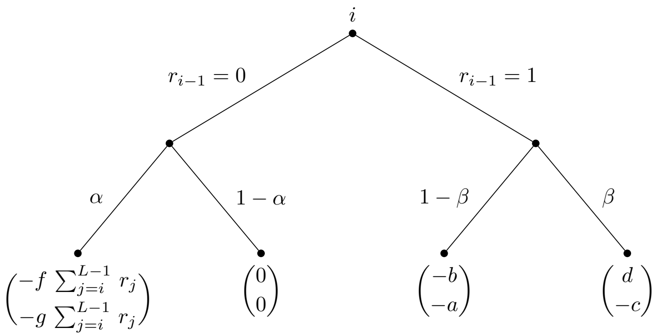
$$Op_{L,k}(i,\mathbf{r}) = \begin{cases} d - A\, r_{i-1} - f\,\alpha \sum_{j=i}^{L-1} r_j & \text{for} \quad i = L, L-1, \dots, 2, 1 \\ -k\, f\, \alpha & \text{for} \quad i = \text{le} \end{cases} \tag{15.48}$$

and

$$In_{L,k}(i,\mathbf{r}) = \begin{cases} B\, r_{i-1} - c - g\,\alpha \sum_{j=i}^{L-1} r_j & \text{for} \quad i = L, L-1, \dots, 2, 1 \\ -k\, g\, \alpha & \text{for} \quad i = \text{le} \end{cases} \tag{15.49}$$

where $\sum_{j=L}^{L-1} r_j := 0$. (15.48) can be justified as follows: Suppose the Operator behaves illegally at the beginning of period i. If the Inspectorate inspects at $i-1$ ($r_{i-1} = 1$), the Operator's payoff is $d\,\beta - b\,(1-\beta) = d - A$; see (15.3). If it does not inspect at $i-1$ ($r_{i-1} = 0$), the Operator's payoff d, because the next inspection is not timely. In addition, the payoff depends on the number of false alarms that the Inspectorate caused at $L-1, \dots, i$. Because $\sum_{j=i}^{L-1} r_j$ is the number of inspections performed until i, the expected false alarm costs are $f\,\alpha \sum_{j=i}^{L-1} r_j$. Inspections perform at $i-2, \dots, 0$ will not affect the Operator's payoff; see assumption (iv') and also Figure 15.1. A similar argumentation justifies (15.49).

Figure 15.1 Illustration of the payoffs (15.48) and (15.49) under the assumption that the Operator behaves illegally at the beginning of period i, $i = L, \dots, 1$.



In order to solve the Canty-Rothenstein-Avenhaus inspection game, a transformation into another game is necessary which was already demonstrated in Section 15.2. In analogy, let q_j,

$j = L-1, \ldots, 0$, be the probability that an inspection is performed at j, i.e., we have

$$q_j := \sum_{\mathbf{r} \in J_{L,k} : r_j = 1} \tilde{q}_{\mathbf{r}}, \qquad j = L-1, \ldots, 0.$$

Let $\mathbf{e}_n^T \in \{0,1\}^L$, $n = 1, \ldots, L$, be the unit vector with a 1 at the n-th component. Because $\mathbf{e}_{L-j}^T \mathbf{r} = 1$ if and only if $r_j = 1$, $j = L-1, \ldots, 0$, q_j can also be expressed as

$$q_j = \sum_{\mathbf{r} \in J_{L,k}} \mathbf{e}_{L-j}^T \mathbf{r}\, \tilde{q}_{\mathbf{r}}. \tag{15.50}$$

Using (15.50) we get

$$\sum_{j=0}^{L-1} q_j = \sum_{j=0}^{L-1} \left(\sum_{\mathbf{r} \in J_{L,k}} \mathbf{e}_{L-j}^T \mathbf{r}\, \tilde{q}_{\mathbf{r}} \right) = \sum_{\mathbf{r} \in J_{L,k}} \tilde{q}_{\mathbf{r}} \left(\sum_{j=0}^{L-1} \mathbf{e}_{L-j}^T \mathbf{r} \right) = \sum_{\mathbf{r} \in J_{L,k}} \tilde{q}_{\mathbf{r}}\, k = k$$

and define the Inspectorate's strategy set, which is, like $Q_{3,2}$, not a set of mixed strategies, by

$$Q_{L,k} := \left\{ \mathbf{q} = (q_{L-1}, \ldots, q_0)^T \in [0,1]^L : \sum_{j=0}^{L-1} q_j = k \right\}. \tag{15.51}$$

Only for $k = 1$ and $k = L-1$ inspection(s) we have the same numbers of $\tilde{q}_{\mathbf{r}}$ and q_j. For $1 < k < L-1$ there are more $\tilde{q}_{\mathbf{r}}$ than q_j thus, the q_j are uniquely determined by the $\tilde{q}_{\mathbf{r}}$, but not vice versa. We will come back to this point on p. 304 and in the context of the generalized Thomas-Nisgav inspection game; see Section 17.1.

The payoff to the Operator is, using (15.48) and (15.50), for all $i = L, L-1, \ldots, 1$ and any $\tilde{\mathbf{q}} = (\tilde{q}_{\mathbf{r}})_{\mathbf{r} \in J_{L,k}}^T \in \tilde{Q}_{L,k}$, given by

$$\begin{aligned} Op_{L,k}(i, \tilde{\mathbf{q}}) &= d - A \sum_{\mathbf{r} \in J_{L,k}} r_{i-1}\, \tilde{q}_{\mathbf{r}} - f\,\alpha \sum_{j=i}^{L-1} \sum_{\mathbf{r} \in J_{L,k}} r_j\, \tilde{q}_{\mathbf{r}} \\ &= d - A \sum_{\mathbf{r} \in J_{L,k}} \mathbf{e}_{L-(i-1)}^T \mathbf{r}\, \tilde{q}_{\mathbf{r}} - f\,\alpha \sum_{j=i}^{L-1} \sum_{\mathbf{r} \in J_{L,k}} \mathbf{e}_{L-j}^T \mathbf{r}\, \tilde{q}_{\mathbf{r}} \\ &= d - A\, q_{i-1} - f\,\alpha \sum_{j=i}^{L-1} q_j \\ &=: Op_{L,k}(i, \mathbf{q}). \end{aligned} \tag{15.52}$$

Hence, the (expected) payoff to the Operator is, for any $\mathbf{p} \in P_L$ and any $\mathbf{q} \in Q_{L,k}$, given by

$$\begin{aligned} Op_{L,k}(\mathbf{p}, \mathbf{q}) &= \sum_{i=1}^{L} p_i\, Op_{L,k}(i, \mathbf{q}) - p_{\mathsf{le}}\, k\, f\,\alpha \\ &= \sum_{i=1}^{L} p_i \left[d - A\, q_{i-1} - f\,\alpha \sum_{j=i}^{L-1} q_j \right] - p_{\mathsf{le}}\, k\, f\,\alpha. \end{aligned} \tag{15.53}$$

In analogy, the payoff to the Inspectorate is, using (15.49), for all $i = L, L-1, \ldots, 1$ and any $\mathbf{q} \in Q_{L,k}$, given by

$$In_{L,k}(i, \mathbf{q}) = B\, q_{i-1} - c - g\,\alpha \sum_{j=i}^{L-1} q_j\,,$$

and thus, the (expected) payoff to the Inspectorate is, for any $\mathbf{p} \in P_L$ and any $\mathbf{q} \in Q_{L,k}$, given by

$$In_{L,k}(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{L} p_i \left[B\, q_{i-1} - c - g\,\alpha \sum_{j=i}^{L-1} q_j \right] - p_{\text{le}}\, k\, g\, \alpha\,. \tag{15.54}$$

The game theoretical solution of this inspection game, see Canty et al. (2001), is presented in Theorem 15.1. Let us emphasize that according to our best knowledge this and the Se-Se playing for time inspection game treated in Chapter 12 is the only inspection game over time including errors of the first and second kind for which a game theoretical solution for the whole parameter space has been found.

Theorem 15.1. *Given the No-No inspection game with $L > k$ periods and k inspections, errors of the first and second kind, and an unbiased test procedure. The Operator's set of mixed strategies is given by (15.45), the Inspectorate's strategy set by (15.51), and the payoffs to both players by (15.53) and (15.54). Let x and y be defined by (15.28).*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{L,k} := Op_{L,k}(\mathbf{p}^*, \mathbf{q}^*)$ and $In^*_{L,k} := In_{L,k}(\mathbf{p}^*, \mathbf{q}^*)$:

(i) For

$$x^L < \min\left[1 - k\,(1-x), \left(1 + k\,\frac{A}{d}\,(1-x)\right)^{-1} \right] \tag{15.55}$$

the Operator behaves illegally and an equilibrium strategy is given by

$$p^*_i = \frac{y-1}{y^L - 1}\, y^{i-1}\,, \quad i = L, \ldots, 1 \qquad \text{and} \qquad p^*_{\text{le}} = 0\,. \tag{15.56}$$

An equilibrium strategy of the Inspectorate is given by

$$q^*_j = k\, \frac{1-x}{1-x^L}\, x^{L-j-1}\,, \qquad j = L-1, \ldots, 0\,. \tag{15.57}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{L,k} = d - k\,A\,\frac{1-x}{1-x^L} \qquad \text{and} \qquad In^*_{L,k} = -c + k\,B\,\frac{y-1}{y^L-1}\,. \tag{15.58}$$

(ii) For

$$\left(1 + k\,\frac{A}{d}\,(1-x)\right)^{-1} < 1 - k\,(1-x) \quad \text{and} \quad \left(1 + k\,\frac{A}{d}\,(1-x)\right)^{-1} < x^L \tag{15.59}$$

the Operator behaves legally, i.e., $p_L^* = \ldots = p_1^* = 0$ *and* $p_{le}^* = 1$. *The Inspectorate's set of equilibrium strategies is given by* ($\sum_{j=L}^{L-1} q_j^* := 0$)

$$-k\,f\,\alpha \geq d - A\,q_{i-1}^* - f\,\alpha \sum_{j=i}^{L-1} q_j^*\,, \quad i = L, \ldots, 1, \quad \text{and} \quad \sum_{j=0}^{L-1} q_j^* = k\,. \tag{15.60}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_{L,k}^* = -k\,f\,\alpha \qquad \text{and} \qquad In_{L,k}^* = -k\,g\,\alpha\,.$$

(iii) For

$$1 - k\,(1-x) < \left(1 + k\,\frac{A}{d}\,(1-x)\right)^{-1} \quad \text{and} \quad 1 - k\,(1-x) < x^L \tag{15.61}$$

let m *be the integer,* $0 < m < k$, *satisfying*

$$\begin{aligned} &x^{L-n} > 1 - (k-n)\,(1-x) > 0 \quad \text{for} \quad n = 0, 1, \ldots, m-1 \qquad \text{and} \\ &x^{L-m} \leq 1 - (k-m)\,(1-x)\,. \end{aligned} \tag{15.62}$$

The Operator behaves illegally and an equilibrium strategy is given by

$$p_L^* = 1\,, \qquad p_i^* = 0\,, \quad i = L-1, \ldots, 1, \qquad \text{and} \qquad p_{le}^* = 0\,. \tag{15.63}$$

An equilibrium strategy of the Inspectorate is given by

$$q_j^* = \begin{cases} x^{L-j-1} & \text{for } j = L-1, \ldots, m \\ k - m - x - \ldots - x^{L-m-1} & \text{for } j = m-1 \\ 1 & \text{for } j = m-2, \ldots, 0 \end{cases}\,. \tag{15.64}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_{L,k}^* = d - A \qquad \text{and} \qquad In_{L,k}^* = -c + B\,. \tag{15.65}$$

Proof: According to (19.8), the Nash equilibrium conditions are given by

$$Op_{L,k}^* \geq Op_{L,k}(i, \mathbf{q}^*) \tag{15.66}$$

$$In_{L,k}^* \geq In_{L,k}(\mathbf{p}^*, \mathbf{q})\,. \tag{15.67}$$

for all $i = L, \ldots, 1$, le and any $\mathbf{q} \in Q_{L,k}$.

Ad (i): Because of $y > 1$ we have by (15.56) that $0 < p_1^* < \ldots < p_L^* < 1$. Using the geometric series we see that $\mathbf{p}^* \in P_L$. Because $x \in (0,1)$ we have $0 < q_0^* < \ldots < q_{L-1}^*$. Then condition $x^L < 1 - k\,(1-x)$ in (15.55) is necessary for making sure that $q_{L-1}^* < 1$, which implies that $0 < q_j^* < 1$ for $j = L-2 \ldots, 0$. Again applying the geometric series gives us $\mathbf{q}^* \in Q_{L,k}$.

Using (15.28), (15.53) and (15.57), we get for all $i = L, L-1, \ldots, 1$

$$Op_{L,k}(i, \mathbf{q}^*) = d - A\,\left[q_{i-1}^* + (1-x)\,\sum_{j=i}^{L-1} q_j^*\right]$$

$$= d - A\left[k\,\frac{1-x}{1-x^L}\,x^{L-i} + (1-x)\,k\,\frac{1-x^{L-i}}{1-x^L}\right]$$

$$= d - A\,q^*_{L-1} = Op^*_{L,k}\,,$$

i.e., (15.66) is fulfilled as equality for all $i = L, L-1, \ldots, 1$. Using (15.55), the inequality

$$x^L < \left(1 + k\,\frac{A}{d}\,(1-x)\right)^{-1}$$

is equivalent to

$$d - A\,k\,\frac{1-x}{1-x^L} > -\,k\,(1-x)\,A = -\,k\,f\,\alpha\,.$$

Thus, we get $Op_{L,k}(\text{le}, \mathbf{q}^*) = -k\,f\,\alpha < d - A\,q^*_{L-1} = Op^*_{L,k}$, i.e., (15.66) is also fulfilled for $i = \text{le}$.

In order to show (15.67), note that (15.28) implies

$$\sum_{i=1}^{L} p^*_i\left[B\,q_{i-1} - g\,\alpha\sum_{j=i}^{L-1} q_j\right] = B\left[\sum_{i=1}^{L} p^*_i\,q_{i-1} - (y-1)\sum_{i=1}^{L} p^*_i\sum_{j=i}^{L-1} q_j\right], \qquad (15.68)$$

which yields, using (15.56),

$$(y-1)\sum_{i=1}^{L} p^*_i\sum_{j=i}^{L-1} q_j = (y-1)\sum_{j=1}^{L-1} q_j\sum_{i=1}^{j} p^*_i = (y-1)\sum_{j=1}^{L-1} q_j\,\frac{y^j-1}{y^L-1}$$

$$= \sum_{j=1}^{L-1} q_j\,(p^*_{j+1} - p^*_1)\,. \qquad (15.69)$$

Therefore, (15.68) simplifies by use of (15.69) to

$$B\left[\sum_{i=1}^{L} p^*_i\,q_{i-1} - \sum_{j=1}^{L-1} q_j\,(p^*_{j+1} - p^*_1)\right] = k\,p^*_1\,. \qquad (15.70)$$

Thus, (15.54) and (15.70) leads for any $\mathbf{q} \in Q_{L,k}$ to

$$In_{L,k}(\mathbf{p}^*, \mathbf{q}) = -c + k\,B\,p^*_1 = In^*_{L,k}(\mathbf{p}^*, \mathbf{q}^*)\,,$$

so that (15.67) is satisfied as equality.

Ad (ii): The left hand inequality in (15.59) assures that existence of $\mathbf{q}^*$ fulfilling (15.60). Furthermore, the inequalities (15.60) are equivalent to $-k\,f\,\alpha \geq Op_{L,k}(i, \mathbf{q}^*)$ for all i, $i = L, \ldots, 1$, and thus, the Operator's Nash equilibrium condition (15.66) is fulfilled. Using (15.54), we have $In_{L,k}(\text{le}, \mathbf{q}) = In^*_{L,k} = -k\,g\,\alpha$ for any $\mathbf{q} \in Q_{L,k}$, i.e., (15.67) is valid.

Ad (iii): The right hand inequality of (15.61) assures the existence of the integer m fulfilling (15.62). Using (15.51) and (15.64), q^*_{m-1} can be rewritten as

$$q^*_{m-1} = k - (m-1) - \frac{1-x^{L-m}}{1-x}\,. \qquad (15.71)$$

Because

$$x^{L-m} \leq 1 - (k-m)\,(1-x)$$

is equivalent to

$$k - (m-1) - \frac{1 - x^{L-m}}{1-x} = q^*_{m-1} \leq 1\,,$$

condition (15.62) guarantees that $q^*_{m-1} \leq 1$.

The payoff to the Operator, if he behaves illegally at the beginning of period i, $i = L\ldots, m+1$, is, using (15.28), (15.53) and (15.64), given by

$$\begin{aligned} Op_{L,k}(i, \mathbf{q}^*) &= d - A\,q^*_{i-1} - f\,\alpha \sum_{j=i}^{L-1} q^*_j = d - A\,x^{L-i} - f\,\alpha \sum_{j=i}^{L-1} x^{L-j-1} \\ &= d - A\,x^{L-i} - f\,\alpha\,\frac{1 - x^{L-i}}{1-x} \\ &= d - A\,\left[x^{L-i} + 1 - x^{L-i}\right] = d - A\,. \end{aligned}$$

If the Operator behaves illegally at the beginning of period $i = m$, we get for his payoff, again using (15.53) and (15.64),

$$\begin{aligned} Op_{L,k}(m, \mathbf{q}^*) &= d - A\,q^*_{m-1} - f\,\alpha \sum_{j=m}^{L-1} q^*_j \\ &= d - A\,\left[q^*_{m-1} + \frac{f\,\alpha}{A}\,(k - (m-1) - q^*_{m-1})\right] \end{aligned} \tag{15.72}$$

and prove that

$$q^*_{m-1} + \frac{f\,\alpha}{A}\,(k - (m-1) - q^*_{m-1}) > 1\,. \tag{15.73}$$

The left hand side of (15.73) simplifies by (15.71) to

$$\begin{aligned} & q^*_{m-1} + \frac{f\,\alpha}{A}\,(k - (m-1) - q^*_{m-1}) \\ &= k - (m-1) - \frac{1 - x^{L-m}}{1-x} + \frac{f\,\alpha}{A}\left(k - (m-1) - (k - (m-1)) + \frac{1 - x^{L-m}}{1-x}\right) \\ &= k - (m-1) - x\,\frac{1 - x^{L-m}}{1-x}\,. \end{aligned}$$

Using (15.62) for $n = m-1$, we see that

$$x^{L-(m-1)} > 1 - (k - (m-1))\,(1-x)$$

is equivalent to

$$k - (m-1) - x\,\frac{1 - x^{L-m}}{1-x} > 1\,.$$

Thus, (15.73) is fulfilled, and (15.72) yields $Op_{L,k}(m, \mathbf{q}^*) < d - A$. If the Operator behaves illegally at the beginning of period i, $i = m-1, \ldots, 1$, his payoff is, again using (15.53) and (15.64), given by

$$Op_{L,k}(i, \mathbf{q}^*) = d - A\, q^*_{i-1} - f\,\alpha \sum_{j=i}^{L-1} q^*_j = d - A - f\,\alpha\,(k-i) < d - A\,,$$

because $i \le m - 1 < k$. Because the left hand inequality of (15.61) is equivalent to $-k\,f\,\alpha < d - A$, we get $Op_{L,k}(\text{le}, \mathbf{q}^*) = -k\,f\,\alpha < d - A$, i.e., (15.66) is fulfilled. Using (15.54) and (15.63) we have $In_{L,k}(\mathbf{p}^*, \mathbf{q}) = B\, q_{L-1} - c$, which is maximized for $q^*_{L-1} = 1$. This completes the proof. □

Before we comment on the results of Theorem 15.1 on p. 303, we discuss the case $\alpha = 0$, i.e., the case $x = 1$, which is excluded in Theorem 15.1 because of $x \in (0, 1)$. Indeed, the conditions (15.55), (15.59), and (15.61) are meaningless in this case at first sight. We see, however, that

$$x^L \gtrless 1 - k\,(1 - x)$$

is equivalent to

$$k \gtrless 1 + x + \ldots + x^{L-1}\,, \tag{15.74}$$

and thus, for $x = 1$ equivalent to the condition $k \gtrless L$. Because of $k < L$, the inequality $x^L > 1 - k\,(1-x)$ in (15.74) does not hold for $x = 1$ and therefore, case (iii) of Theorem 15.1 vanishes. Furthermore,

$$x^L \gtrless \left(1 + k\,\frac{A}{d}\,(1 - x)\right)^{-1}$$

is equivalent to

$$x^L\, k\,\frac{A}{d} \gtrless 1 + x + \ldots + x^{L-1}\,,$$

therefore, for $x = 1$ we get

$$\frac{A}{d} \gtrless \frac{L}{k}\,.$$

Thus, one obtains the following Corollary 15.1, which describes an attribute sampling problem because of $\alpha = 0$. For later purposes we use the explicit forms of A and B as given by (15.2).

Corollary 15.1. *Given the No-No inspection game with $L > k$ periods and k inspections, errors of the first and second kind, and an unbiased test procedure analysed in Theorem 15.1. Then for $x \to 1$ and $y \to 1$ the Nash equilibrium in Theorem 15.1 reduces as follows:*

(i) For

$$\frac{k}{L} < \frac{1}{1-\beta}\,\frac{1}{1 + b/d} \tag{15.75}$$

the Operator behaves illegally and an equilibrium strategy is given by

$$p_i^* = \frac{1}{L}\,, \qquad i = L, \ldots, 1 \qquad \textit{and} \qquad p_{le}^* = 0\,. \tag{15.76}$$

An equilibrium strategy of the Inspectorate is given by

$$q_j^* = \frac{k}{L}\,, \qquad j = L-1, \ldots, 0\,. \tag{15.77}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op_{L,k}^* = d - (1-\beta)\,(b+d)\,\frac{k}{L} \qquad \textit{and} \qquad In_{L,k}^* = -c + (1-\beta)\,(c-a)\,\frac{k}{L}\,. \tag{15.78}$$

(ii) For

$$\frac{k}{L} > \frac{1}{1-\beta}\,\frac{1}{1+b/d} \tag{15.79}$$

the Operator behaves legally, i.e., $p_L^* = \ldots = p_1^* = 0$ *and* $p_{le}^* = 1$. *The Inspectorate's set of equilibrium strategies is given by*

$$0 \geq d - (b+d)\,(1-\beta)\,q_j^*\,, \quad j = L-1, \ldots, 0, \qquad \textit{and} \qquad \sum_{j=0}^{L-1} q_j^* = k\,, \tag{15.80}$$

where q_j^*, $j = L-1, \ldots, 0$, *given by (15.77) fulfils (15.80).*

The equilibrium payoffs to the Operator and to the Inspectorate are $Op_{L,k}^* = In_{L,k}^* = 0$.

Let us comment the results of Corollary 15.1: First, surprisingly enough, we will encounter the game theoretical results again in the next two chapters; see pp. 336, 368 and 391, and in Chapter 24. Note that one obtains the same results as in Corollary 15.1, if one uses immediately the payoffs (15.48) and (15.49) for $\alpha = 0$.

Second, because $\mathbf{q}^* = (q_{L-1}^*, \ldots, q_0^*)^T$ according to (15.77) constitutes a vector of probabilities no matter whether (15.75) or (15.79) is fulfilled, and because we have under condition (15.79)

$$0 > d - (1-\beta)\,(b+d)\,\frac{k}{L} = Op_{L,k}(i, \mathbf{q}^*) \qquad \text{for all} \qquad i = L, \ldots, 1\,,$$

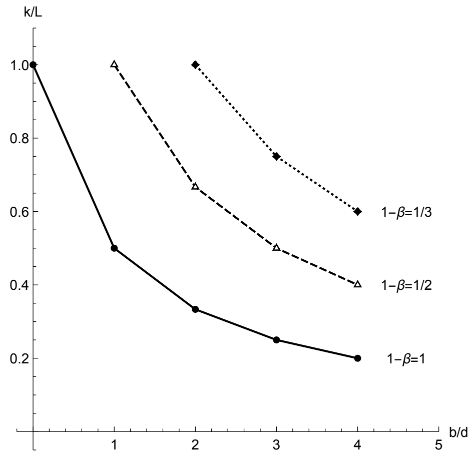
$\mathbf{q}^*$ is an element of (15.80), i.e., it is a robust equilibrium strategy in contrast to the case $\alpha > 0$ and $k \geq 2$ inspections; see p. 304 and Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

Third, the condition (15.79) for legal behaviour, i.e., $Op_{L,k}^* < 0$ in (15.78), can be interpreted as a minimum condition on the number of inspections required for deterrence. Obviously this inspection frequency has to be the larger, the smaller the ratio of sanctions to gains for the illegally behaving Operator. Note that because $k/L < 1$, a necessary condition for (15.79) is

$$1 > \frac{1}{1-\beta}\,\frac{1}{1+b/d}$$

which is equivalent to $-b\,(1-\beta) + d\,\beta < 0$, i.e., the expected payoff to the Operator for illegal behaviour in case of a timely inspection is smaller than zero.

Figure 15.2 Representation of (15.79) for three β-values.



Fourth, relation (15.79) is remarkable since it combines five model parameters in a simple and intuitive inequality. This simplicity is reflected in Figure 15.2 in which for a given value of $1-\beta$ the area above the $1-\beta$-curve represents legal behaviour of the Operator.

Fifth, let us consider a simple application. A facility with a valuable equipment or a museum with art treasures has to be protected by night watchmen against burglary. One inspection night shift lasts 12 hours. An illegal activity, e.g., removing and carrying away some expensive tool or art treasure lasts $1/4$ hour, about the same time, by design, as an inspection tour through the factory, thus, $L = 48$. For $1-\beta = 1$ and $b/d = 3$ we get $k/L = 1/4$ which means that $k = 12$ inspections have to be performed, i.e., on the average one inspection per hour.[1] Whoever would have thought to justify a night watchmen's rounds in this way? In practice, of course, inspection frequencies of this kind are determined on the basis of practicability and related common sense arguments. Nevertheless at least our results may help, given the model describes reality appropriately, to clarify underlying assumptions on the value of b/d and other model parameters.

Finally, the equilibrium strategies (15.76) and (15.77) do not depend on the payoff parameters a, b, c and d. Is this a surprising result? Denote for any $\mathbf{p} \in P_L$ and any $\mathbf{q} \in Q_{L,k}$ the probability that the illegal activity is timely detected by $w_{L,k}(\mathbf{p},\mathbf{q})$: If the Operator behaves illegally at the beginning of period i, $i = L, \ldots, 1$, then the Inspectorate needs to inspect at $i-1$ and must detect the illegal activity (with probability $1-\beta$). Thus, in the No-No inspection game the timely detection probability $w_{L,k}(\mathbf{p},\mathbf{q})$ is given by

$$w_{L,k}(\mathbf{p},\mathbf{q}) := (1-\beta) \sum_{i=1}^{L} p_i \, q_{i-1} \, . \tag{15.81}$$

[1] Conflict situations of this kind are presented professionally in some famous action movies with well-known actors, e.g., *Topkapi* (1964) with M. Mercouri, P. Ustinov and M. Schell, *How to Steel a Million* (1966) with A. Hepburn and P. O'Toole, and *Entrapment* (1999) with C. Zeta-Jones and S. Connery. In the second one manipulated false alarms are part of the burglars' strategy.

In case of illegal behaviour and no false alarms, i.e., $p_{\text{le}} = 0$ and $\alpha = 0$, the Operator's and Inspectorate's payoffs can be expressed as a function of the timely detection probability $w_{L,k}(\mathbf{p}, \mathbf{q})$: (14.1) in case of illegal behaviour implies

$$Op_{L,k}(\mathbf{p}, \mathbf{q}) = d\,(1 - w_{L,k}(\mathbf{p}, \mathbf{q})) - b\,w_{L,k}(\mathbf{p}, \mathbf{q}) \qquad \text{and}$$

$$In_{L,k}(\mathbf{p}, \mathbf{q}) = -c\,(1 - w_{L,k}(\mathbf{p}, \mathbf{q})) - a\,w_{L,k}(\mathbf{p}, \mathbf{q})\,,$$

and thus, using (14.2), $(\mathbf{p}^*, \mathbf{q}^*)$ is a Nash equilibrium if and only if the saddle point inequalities

$$w_{L,k}(\mathbf{p}^*, \mathbf{q}) \le w_{L,k}(\mathbf{p}^*, \mathbf{q}^*) \le w_{L,k}(\mathbf{p}, \mathbf{q}^*) \tag{15.82}$$

are fulfilled for any $\mathbf{p} \in P_L$ and any $\mathbf{q} \in Q_{L,k}$. Note that in (15.82) the Inspectorate is the maximizing player, while the Operator is the minimizing player, as expected due to the meaning of $w_{L,k}$. We will return to this important point on p. 391. Using the equilibrium strategies (15.76) and (15.77), we get by (15.81) for the timely detection probability in equilibrium

$$w_{L,k}(\mathbf{p}^*, \mathbf{q}^*) = \sum_{i=1}^{L} p_i^*\, q_{i-1}^*\,(1 - \beta) = (1 - \beta) \sum_{i=1}^{L} \frac{1}{L}\,\frac{k}{L} = (1 - \beta)\,\frac{k}{L}\,.$$

Let us now comment the results of Theorem 15.1: First, the regions of the solutions, i.e., conditions (15.55), (15.59) and (15.61), are represented in Figure 15.3 along the x^L-axis. Let us mention that

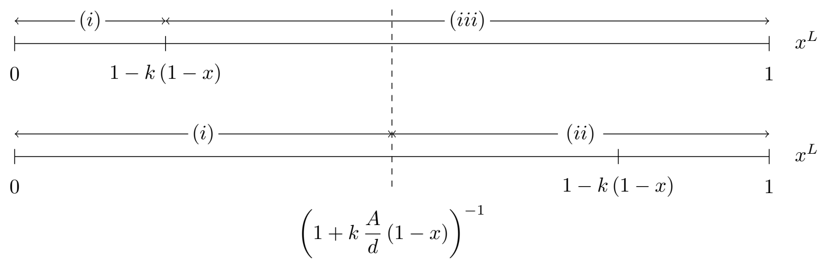
$$1 - k\,(1 - x) \gtreqless \left(1 + k\,\frac{A}{d}\,(1 - x)\right)^{-1}$$

is equivalent to

$$-k\,f\,\alpha \gtreqless d - A = d\,\beta - b\,(1 - \beta)\,,$$

the left hand side being the expected false alarm costs of the Operator in case of legal behaviour, and the right hand side his expected gain in case of a timely inspection.

Figure 15.3 Structure of the solutions of Theorem 15.1.



Second, note that for the case of $k = 1$ inspection the solution given by part (iii) of the Theorem does not exist because in this case $1 - k\,(1 - x) = x$ is larger than x^L for any $L > 1$ periods. Conditions (15.9) and (15.13) are, using (15.17), the same as (15.55) and (15.59) for $L = 2$ periods and $k = 1$ inspection.

Third, like in the inspection game with $L = 3$ periods and $k = 2$ inspections, the equilibrium strategy (15.57) is not a robust equilibrium strategy; see p. 292 and Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

Fourth, as mentioned on p. 293 for the game $L = 3$ periods and $k = 2$ inspections, the case $1 - k\,(1 - x) < 0$, which is excluded here because of (14.2) but treated in Canty et al. (2001), leads in case (iii) to the additional Inspectorate's equilibrium strategy:

$$(q^*_{L-1}, q^*_{L-2}, \ldots, q^*_0) = (\underbrace{1, 1 \ldots 1}_{k}, \underbrace{0, 0 \ldots 0}_{L-k}) \, .$$

The proof can be found in Canty et al. (2001).

Fifth, we mentioned on p. 295 that only for $k = 1$ and $k = L - 1$ inspection(s) we have the same numbers of $\tilde{q}_{\mathbf{r}}$ and q_j. In all other cases, i.e., $1 < k < L-1$, there are more $\tilde{q}_{\mathbf{r}}$ than q_j. To give an equilibrium strategy of the Inspectorate of the original No-No inspection game, i.e., with the strategy sets P_L and $\tilde{Q}_{L,k}$, see (15.47), the strategies $\tilde{\mathbf{q}}^*$ have to be somehow calculated utilizing $\mathbf{q}^*$ as given by Theorem 15.1. Because the $\tilde{\mathbf{q}}^*$ are not unique, the Inspectorate has the freedom to choose the strategy which fits best to its practice, e.g., mixing as few as possible pure strategies. A similar situation is discussed for the discrete time Se-No inspection game on p. 58.

Sixth, we discuss whether the Operator can be induced to legal behaviour; see p. 260 for similar considerations in the context of the continuous time Se-Se inspection game. For illustration we consider the parameters

$$d = b = 19 \, , \qquad f = 1 \, , \qquad \text{and} \qquad \alpha = \frac{4}{16} \, , \qquad \beta = \frac{1}{16} \, .$$

Then (14.2) and $\alpha + \beta < 1$ are fulfilled, and with (15.2) and (15.28) we get

$$A = (b + d)\,(1 - \beta) = 35.625 \qquad \text{and} \qquad x = \frac{283}{285} \approx 0.993 \, .$$

In Figure 15.4, the functions $1 - k\,(1 - x)$ and $(1 + k\,f\,\alpha/d)^{-1}$ as well as the three cases 1) x^{50} for $k = 1, \ldots, 50$ (top horizontal line), 2) x^{120} for $k = 1, \ldots, 120$ (middle horizontal line), and 3) x^{140} for $k = 1, \ldots, 140$ (bottom horizontal line), are depicted.

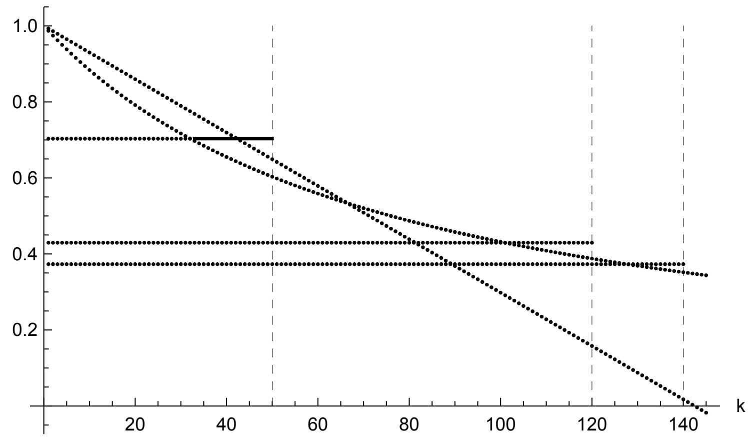
Let us comment on Figure 15.4: First, the case of $k = 1$ inspection leads in all three cases to (i) of Theorem 15.1. Second, for increasing k we observe a transition from (i) to (ii) in case 1), and a transition from (i) to (iii) for the cases 2) and 3). Thus, only in case 1) the Inspectorate has a deterrence strategy and it chooses the smallest k such that legal behaviour is induced, i.e., k^* with

$$\left(1 + k^* \, \frac{A}{d}\,(1 - x)\right)^{-1} < x^L < \left(1 + (k^* - 1)\, \frac{A}{d}\,(1 - x)\right)^{-1} \, .$$

In cases 2) and 3) the Inspectorate has no deterrence strategy and it should choose k such that its equilibrium payoff $In^*_{L,k}$ is maximized: Let k' be given by

$$1 - (k' + 1)\,(1 - x) < x^L < 1 - k'\,(1 - x) \, .$$

Figure 15.4 Illustration of the transition between the cases (i), (ii) and (iii) of Theorem 15.1. The solid part of the x^{50}-line describes legal behaviour of the Operator.



Then we obtain, using (15.58) and (15.65), the decision rule

$$\text{number of inspections is} \begin{cases} k' & \text{for} \quad k' > \dfrac{y^L - 1}{y - 1} \\ k' + 1 & \text{for} \quad k' < \dfrac{y^L - 1}{y - 1} \end{cases} .$$

Finally, let us consider a numerical example: We assume $L = 8$ periods and $k = 6$ inspections as well as

$$x = 0.95\,, \qquad d = 55\,, \qquad b = 47\,, \qquad f = 40 \qquad \text{and} \qquad \alpha = \frac{29}{1209}\,,$$

$$x = 0.90\,, \qquad d = 92\,, \qquad b = 88\,, \qquad f = 49 \qquad \text{and} \qquad \alpha = \frac{58}{387}\,,$$

$$x = 0.85\,, \qquad d = 17\,, \qquad b = 3\,, \qquad f = 2 \qquad \text{and} \qquad \alpha = \frac{87}{133}\,,$$

$$x = \frac{5}{6}\,, \qquad d = 85\,, \qquad b = 67 \qquad f = 17 \qquad \text{and} \qquad \alpha = \frac{22}{107}\,.$$

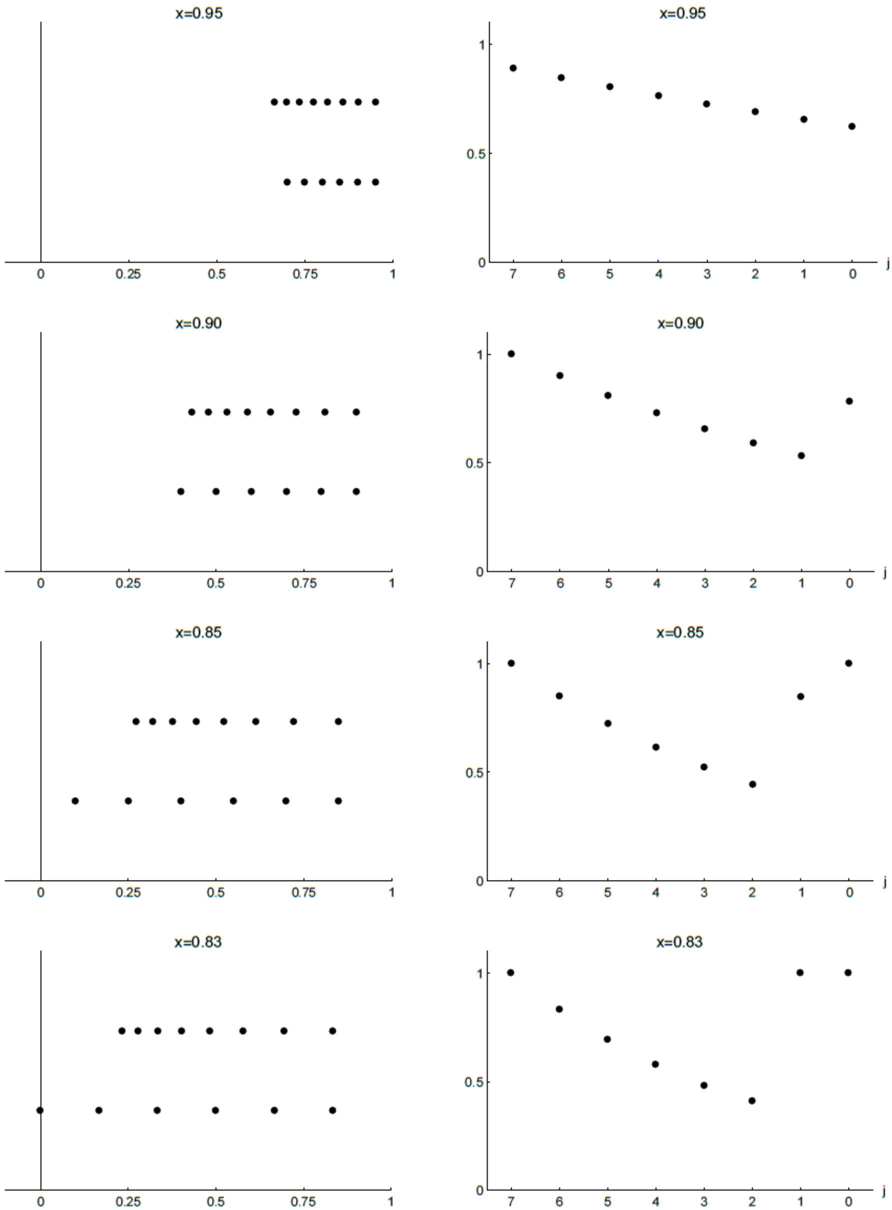
Then we have in all four cases that

$$1 - 6\,(1 - x) < \left(1 + 6\,\frac{A}{d}\,(1 - x)\right)^{-1}\,, \tag{15.83}$$

i.e., only case (i) and (iii) of the Theorem 15.1 can arise. In Figure 15.5, the Inspectorate's equilibrium strategies for the four examples are depicted from top to bottom. In the left-hand graphs the top rows of points show the sequence

$$x^8, x^7, \ldots, x\,,$$

Figure 15.5 Illustration of the Inspectorate's equilibrium strategy $\mathbf{q}^*$ for the inspection game with $L = 8$ periods and $k = 6$ inspections and various values of x. Left hand column: Upper sequence $x^8, x^7, \ldots, x$. Lower sequence $1-6(1-x), 1-5(1-x), \ldots, x$. Right hand column: $q_7^*, q_6^*, \ldots, q_0^*$.



and the bottom rows the sequence

$$1-6\,(1-x), 1-5\,(1-x), \ldots, 1-2\,(1-x), x$$

for the above given values of x. In the right hand plots the components $q_7^*, \ldots, q_0^*$ of the Inspectorate's equilibrium strategy are shown.

Let us consider the top pair of graphs: Because of $x^8 < 1-6\,(1-0.95)$, i.e., the leftmost point in the bottom row lies to the right of the leftmost point in the top row, and because of (15.83) condition (i) of Theorem 15.1 is valid, and the equilibrium strategy (15.57) is plotted on the right hand side. For the next three pairs of graphs we have $x^8 > 1-6\,(1-0.95)$ and because of (15.83) condition (15.61) is met. We get, using (15.62),

$$x = 0.9: \quad x^{8-0} > 1-(6-0)\,(1-x) \quad \text{and} \quad x^{8-1} \le 1-(6-1)\,(1-x), \text{ i.e., } m = 1$$

$$x = 0.85: \quad x^{8-1} > 1-(6-1)\,(1-x) \quad \text{and} \quad x^{8-2} \le 1-(6-2)\,(1-x), \text{ i.e., } m = 2$$

$$x = 5/6: \quad x^{8-2} > 1-(6-2)\,(1-x) \quad \text{and} \quad x^{8-3} \le 1-(6-3)\,(1-x), \text{ i.e., } m = 3\,.$$

Thus, we obtain in these cases by (15.64) the results that q_7^* is one; see the right hand plots of Figure 15.5.

15.4 Sensitivity considerations

In order to demonstrate the sensitivity of the game theoretical solution of the No-No inspection game on assumption (iv'), see p. 282, we assume in this section that also false alarm costs for inspections performed after the illegal activity are taken into account, i.e., the inspection game analysed in this section is based on the following specifications:

(iv") During an inspection the Inspectorate may commit errors of the first and second kind with probabilities α and β. These error probabilities are the same for all inspections. Inspections which are performed before and after an illegal activity may incur false alarm costs.

(x") The game ends either at the beginning of period L in case the Operator behaves legally throughout the game, or one period after the Operator behaves illegally, or after the last inspection.

Let us discuss these assumptions. Ad (iv"): Suppose the Operator behaves illegally at the beginning of the period i, $i = L, \ldots, 1$, then at all inspections which are performed at $L-1, \ldots, i$ and at $i-2, \ldots, 0$ a false alarm may be raised which leads to false alarm costs for both players. Ad (x"): In contrast to assumption (x') on p. 282 and its justification on p. 283, we have to include all inspections which are performed at the beginning of periods $i-2, \ldots, 0$ because false alarms may be raised.

In Figure 15.3 the payoff matrices for the No-No inspection game with $L = 2$ resp. $L = 3$ periods and $k = 1$ resp. $k = 2$ inspection(s) are presented under assumption (iv"). The entries in a box indicate differences to the payoff matrices in Tables 15.1 and 15.2. The entry $(2, 0)$ in the payoff matrix on top can be explained as follows: Because there is no inspection at 1, the illegal activity is not timely detected, i.e., payoff d to the Operator, and there may be a false alarm at 0, thus, the Operator's payoff is $d - f\,\alpha$. The entry $(2,(2,0))$ in the payoff matrix below can be explained as follows: The illegal activity is not timely detected, because there is no inspection at 1, but there may be a false alarm at 2 and one at 0. Therefore, we have $d - 2\,f\,\alpha$.

Table 15.3 Normal forms of the No-No inspection game under the assumption (iv") with $L = 2$ periods and $k = 1$ inspection (top), and with $L = 3$ periods and $k = 2$ inspections (below).

	1	0
2	$B-c$ $d-A$	$-c\boxed{-g\,\alpha}$ $d\boxed{-f\,\alpha}$
1	$-c-g\,\alpha$ $d-f\,\alpha$	$B-c$ $d-A$
le	$-g\,\alpha$ $-f\,\alpha$	$-g\,\alpha$ $-f\,\alpha$

	$(2,1)$	$(2,0)$	$(1,0)$
3	$B-c\boxed{-g\,\alpha}$ $d-A\boxed{-f\,\alpha}$	$B-c\boxed{-g\,\alpha}$ $d-A\boxed{-f\,\alpha}$	$-c\boxed{-2\,g\,\alpha}$ $d\boxed{-2\,f\,\alpha}$
2	$B-c-g\,\alpha$ $d-A-f\,\alpha$	$-c\boxed{-2\,g\,\alpha}$ $d\boxed{-2\,f\,\alpha}$	$B-c\boxed{-g\,\alpha}$ $d-A\boxed{-f\,\alpha}$
1	$-c-2\,g\,\alpha$ $d-2\,f\,\alpha$	$B-c-g\,\alpha$ $d-A-f\,\alpha$	$B-c-g\,\alpha$ $d-A-f\,\alpha$
le	$-2\,g\,\alpha$ $-2\,f\,\alpha$	$-2\,g\,\alpha$ $-2\,f\,\alpha$	$-2\,g\,\alpha$ $-2\,f\,\alpha$

The payoff matrices without the row "legal behaviour" are symmetric, and thus, a game theoretical solution for the case of illegal behaviour of the Operator, i.e., $p^*_{\text{le}} = 0$, can be given easily: In case of $L = 2$ periods and $k = 1$ inspection, and under the condition $d > (A - f\,\alpha)/2$, we have

$$p^*_2 = p^*_1 = q^*_1 = q^*_0 = \frac{1}{2} \qquad \text{and}$$

$$Op^*_{2,1} = d - \frac{1}{2}\,(A + f\,\alpha) \qquad \text{and} \qquad In^*_{2,1} = -c - \frac{1}{2}\,(B + g\,\alpha)\,,$$

which is very different from the equilibrium strategies and payoffs given in (i) of Lemma 15.1. We see that a change from modelling assumption (iv') to (iv") leads to a completely different game theoretical solution. For the inspection game with $L = 3$ periods and $k = 2$ inspections the game theoretical solution in case of illegal behaviour of the Operator, i.e., under the condition $d > 2\,(A - f\,\alpha)/3$, is given by

$$p^*_3 = p^*_2 = p^*_1 = \frac{1}{3} \qquad \text{and} \qquad q^*_{(2,1)} = q^*_{(2,0)} = q^*_{(1,0)} = \frac{1}{3} \qquad \text{and} \tag{15.84}$$

$$Op^*_{2,1} = d - \frac{2}{3}\,(A + 2\,f\,\alpha) \qquad \text{and} \qquad In^*_{2,1} = -c - \frac{2}{3}\,(B + 2\,g\,\alpha)\,.$$

In terms of the probabilities q_j, $j = 2, 1, 0$, that at j an inspection is performed, we get from (15.84) that $q^*_2 = q^*_1 = q^*_0 = 1/3$, which is also very different from the equilibrium strategies and payoffs given in (i) of Lemma 15.2.

A generalization to any number L of periods and any number k of inspections can be found easily. We consider again the Operator's set of mixed strategies (15.45) and the Inspectorate's strategy set (15.51). If the Operator behaves illegally at the beginning of period i, $i = L, \ldots, 1$, and the Inspectorate performs an inspection at $i - 1$, then the Operator's payoff is given by

$$d - A - (k-1)\,f\,\alpha\,, \tag{15.85}$$

because $d - A$ is the Operator's payoff for a detected illegal activity and the remaining $k - 1$ inspections all may lead to a false alarm. If, however, the Inspectorate does not perform an inspection at $i - 1$, then the Operator's payoff is given by

$$d - k\,f\,\alpha\,, \tag{15.86}$$

because he receives payoff d for an untimely inspection and the k inspections may all lead to a false alarm. The payoff to the Inspectorate is given by (15.85) and (15.86) with the replacements $A \to -B, d \to -c$ and $f \to g$. Using (15.85) and (15.86), the payoff to the Operator resp. the Inspectorate is, in analogy to (15.48) and (15.49), for any $\mathbf{r} \in J_{L,k}$ given by

$$\widetilde{Op}_{L,k}(i, \mathbf{r}) = \begin{cases} d - A\,r_{i-1} - f\,\alpha\,(k - r_{i-1}) & \text{for} \quad i = L, L-1, \ldots, 2, 1 \\ -k\,f\,\alpha & \text{for} \quad i = \text{le} \end{cases}$$

and

$$\widetilde{In}_{L,k}(i, \mathbf{r}) = \begin{cases} B\,r_{i-1} - c - g\,\alpha\,(k - r_{i-1}) & \text{for} \quad i = L, L-1, \ldots, 2, 1 \\ -k\,g\,\alpha & \text{for} \quad i = \text{le} \end{cases}\,.$$

Thus, using (15.50) and (15.52), the (expected) payoff to the Operator is, for all $i = L, L - 1, \ldots, 1$ and any $\tilde{\mathbf{q}} = (\tilde{q}_{\mathbf{r}})^T_{\mathbf{r} \in J_{L,k}} \in \tilde{Q}_{L,k}$, given by

$$d - (A - f\,\alpha) \sum_{\mathbf{r} \in J_{L,k}} r_{i-1}\,\tilde{q}_{\mathbf{r}} - k\,f\,\alpha = d - (A - f\,\alpha)\,q_{i-1} - k\,f\,\alpha =: \widetilde{Op}_{L,k}(i, \mathbf{q})\,,$$

and finally for any $\mathbf{p} \in P_L$

$$\begin{aligned} \widetilde{Op}_{L,k}(\mathbf{p}, \mathbf{q}) &= \sum_{i=1}^{L} p_i\,\widetilde{Op}_{L,k}(i, \mathbf{q}) - p_{\text{le}}\,k\,f\,\alpha \\ &= \sum_{i=1}^{L} p_i\,\Big[d - (A - f\,\alpha)\,q_{i-1}\Big] - k\,f\,\alpha\,. \end{aligned} \tag{15.87}$$

The Inspectorate's (expected) payoff is, for any $\mathbf{p} \in P_L$ and any $\mathbf{q} \in Q_{L,k}$, given by

$$\widetilde{In}_{L,k}(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{L} p_i\,\Big[-c - (B - g\,\alpha)\,q_{i-1}\Big] - k\,g\,\alpha\,. \tag{15.88}$$

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Theorem 15.2. *Given the No-No inspection game under assumption (iv") with $L > k$ periods and k inspections, errors of the first and second kind, and an unbiased test procedure. The Operator's set of mixed strategies is given by (15.45), the Inspectorate's strategy set by (15.51), and the payoffs to both players by (15.87) and (15.88).*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $\widetilde{Op}^*_{L,k} := \widetilde{Op}_{L,k}(\mathbf{p}^*, \mathbf{q}^*)$ and $\widetilde{In}^*_{L,k} := \widetilde{In}_{L,k}(\mathbf{p}^*, \mathbf{q}^*)$:

(i) For

$$d > \frac{k}{L}\,(A - f\,\alpha) \tag{15.89}$$

the Operator behaves illegally and an equilibrium strategy is given by

$$p_i^* = \frac{1}{L}\,, \quad i = L, \ldots, 1 \qquad \textit{and} \qquad p^*_{\textit{le}} = 0\,. \tag{15.90}$$

An equilibrium strategy of the Inspectorate is given by

$$q_j^* = \frac{k}{L}\,, \qquad j = L-1, \ldots, 0\,. \tag{15.91}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$\begin{aligned} \widetilde{Op}^*_{L,k} &= d - \frac{k}{L}\Big(A + (L-1)\,f\,\alpha\Big) \qquad \textit{and} \\ \widetilde{In}^*_{L,k} &= -c - \frac{k}{L}\Big(B + (L-1)\,g\,\alpha\Big)\,. \end{aligned} \tag{15.92}$$

(ii) For

$$d < \frac{k}{L}\,(A - f\,\alpha) \tag{15.93}$$

the Operator behaves legally, i.e., $p^*_L = \ldots = p^*_1 = 0$ and $p^*_{le} = 1$. The Inspectorate's set of equilibrium strategies is given by

$$0 \geq d - (A - f\,\alpha)\,q_{i-1}\,, \quad i = L, \ldots, 1, \quad \textit{and} \qquad \sum_{j=0}^{L-1} q_j^* = k\,. \tag{15.94}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{L,k} = -k\,f\,\alpha \qquad \textit{and} \qquad In^*_{L,k} = -k\,g\,\alpha\,.$$

Proof: Ad (i): The strategies given by (15.90) and (15.91) obviously belong to P_L and $Q_{L,k}$, respectively. Using (15.91) and (15.92), (15.87) yields for any $\mathbf{p} \in P_L$

$$\widetilde{Op}_{L,k}(\mathbf{p}, \mathbf{q}^*) = d - (A - f\,\alpha)\,\frac{k}{L} - k\,f\,\alpha = \widetilde{Op}^*_{L,k}\,,$$

and (15.90) together with (15.88) leads for any $\mathbf{q} \in Q_{L,k}$ to

$$\widetilde{In}_{L,k}(\mathbf{p}^*, \mathbf{q}) = \frac{1}{L}\Big[-c\,L - (B - g\,\alpha)\,k\Big] - k\,g\,\alpha = \widetilde{In}^*_{L,k}\,,$$

i.e., the Nash equilibrium conditions are fulfilled as equality.

Ad (ii): (15.94) follows directly from the Operator's Nash equilibrium condition. □

Let us comment the results of Theorem 15.2: First, it is impressive how the game theoretical solution changes with the change of assumption (iv') to (iv"). We will observe a similar sensitivity to a modelling assumption between the Dresher-Höpfinger and the *original* Thomas-Nisgav inspection game: In the first one *at most* one illegal activity can be performed, while in the latter one (exactly) one illegal activity must be performed. This little change leads to a considerable change in the game theoretical solutions; see (16.11) – (16.13) for $b = d = 1$, $c = 1$ and $a = -1$ in contrast to (17.31) – (17.33) for $b = d = 1$ and $\beta = 0$.

Second, the equilibrium strategy (15.91) of the Inspectorate in case of illegal behaviour of the Operator is a robust equilibrium strategy, because it also fulfils (15.94), i.e., the Inspectorate can just play (15.91) and does not need to check whether (15.89) or (15.93) is valid; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

Third, if we choose a uniform distribution over the set $J_{L,k}$, see (15.46), i.e.,

$$\tilde{q}_{\mathbf{r}} = \binom{L}{k}^{-1} \qquad \text{for any} \qquad \mathbf{r} \in J_{L,k}\,,$$

we get

$$q_j = \binom{L}{k}^{-1} \binom{L-1}{k-1} = \frac{k}{L} \qquad \text{for all} \qquad j = L-1, \ldots, 0\,,$$

i.e., q_j^* given by (15.91).

Third, Rinderle (1996) analysed the Se-Se inspection game under assumption (iv"), and he obtained in case of $d - A > -f\,\alpha$ equilibrium strategies which can be transformed into (15.90) and (15.91), and equilibrium payoffs that coincide with (15.92). Note that $d - A > -f\,\alpha$ implies that (15.89) is fulfilled. This equivalence between the game theoretical solutions of the No-No inspection game discussed in this section and Rinderle's Se-Se inspection game is remarkable because under assumption (iv') such an equivalence only exists in case of $L = 2$ periods/steps and $k = 1$ inspection/control; see case (i) of Lemmata 15.1 and 16.3 and p. 351.

As mentioned at the beginning of this section, the main reason for assumption (iv'), as opposed to assumption (iv"), was to demonstrate the sensitivity of solutions to small changes in the assumptions. In fact, we think that assumption (iv') – no false alarms and related costs after any timely or untimely detected illegal activity – in most cases meets reality best. However, formal agreements and jurisdiction sometimes may have odd consequences if, for example, as it happened, the State has to grant some compensation to convicted subjects for treatment against formal rules. In that wider sense assumption (iv") may appropriately describe strange real situations.

15.5 Choice of the false alarm probability

Again, like in Sections 9.5 and 12.4, we ask for the optimal value of the false alarm probability α and limit our considerations to one inspection in $L = 2$ periods. Throughout this section we assume that (9.69) is fulfilled again. The proceeding goes along the same lines as in Sections 9.5 and 12.4, as only the analytical form of the equilibrium payoffs to both players in case of illegal behaviour of the Operator differs.

According to (15.12) and (15.15) the equilibrium payoff to the Operator is given by

$$Op^*_{2,1}(\alpha) := \begin{cases} Op^*_{2,1} & \text{for Operator's illegal behaviour} \\ -f\,\alpha & \text{for Operator's legal behaviour} \end{cases}$$

$$= \begin{cases} d - \dfrac{A^2}{2\,A - f\,\alpha} & \text{for} \quad d > \dfrac{(A - f\,\alpha)^2}{2\,A - f\,\alpha} \\ -f\,\alpha & \text{for} \quad d < \dfrac{(A - f\,\alpha)^2}{2\,A - f\,\alpha} \end{cases} . \tag{15.95}$$

Defining $F(\alpha)$ for any $\alpha \in [0,1]$ by

$$F(\alpha) := d - \frac{(b+d)^2\,(1-\beta(\alpha))^2}{2\,(b+d)\,(1-\beta(\alpha)) - f\,\alpha}\,, \tag{15.96}$$

we see that $F(\alpha) = Op^*_{2,1}$ if and only if $d > (A - f\,\alpha)^2/(2\,A - f\,\alpha)$. Using (9.69), (14.2) and (15.96) we have

$$F(0) = d \qquad \text{and} \qquad F(1) = d - \frac{(b+d)^2}{2\,(b+d) - f} < F(0)\,.$$

Like in Sections 9.5 and 12.4, $F(\alpha)$ is a monotone decreasing function on $[0,1]$: Define

$$\tilde{F}(\alpha,\beta) := d - \frac{(b+d)^2\,(1-\beta)^2}{2\,(b+d)\,(1-\beta) - f\,\alpha}\,.$$

Then we have $F(\alpha) = \tilde{F}(\alpha, \beta(\alpha))$ and get for any $\alpha \in (0,1)$ assuming that $\beta(\alpha)$ is a differentiable function on $(0,1)$

$$\frac{d}{d\alpha}\,F(\alpha) = \left(\frac{\partial}{\partial\alpha}\,\tilde{F}(\alpha,\beta), \frac{\partial}{\partial\beta}\,\tilde{F}(\alpha,\beta)\right)\Big|_{\alpha=\alpha,\beta=\beta(\alpha)} \begin{pmatrix} 1 \\ \beta'(\alpha) \end{pmatrix}$$

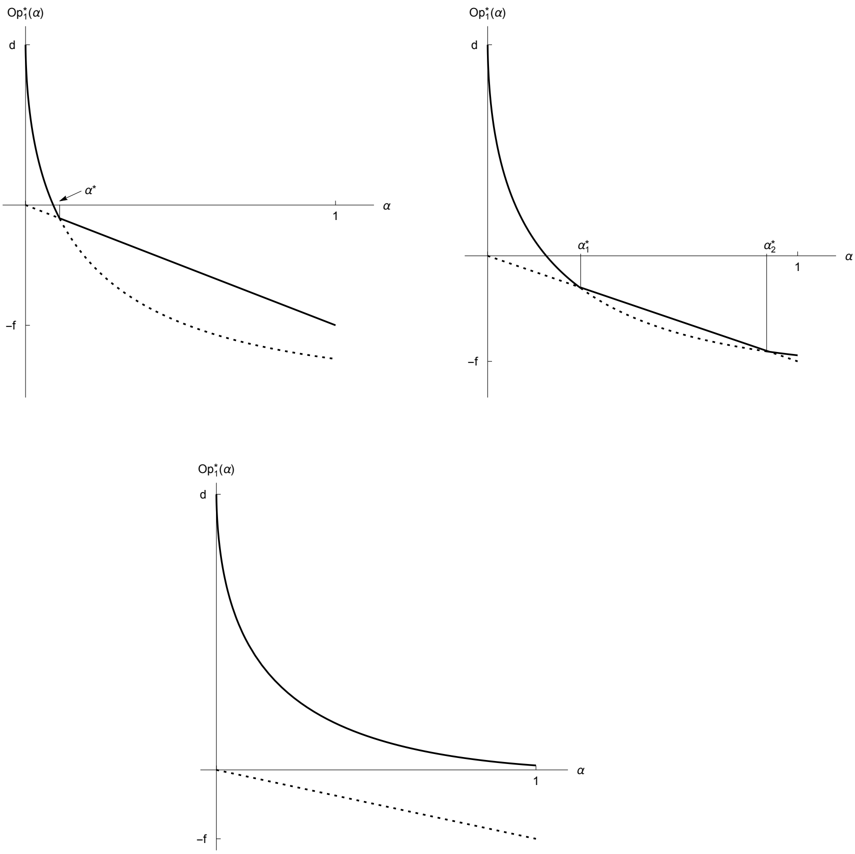
$$= \frac{(b+d)^2\,(1-\beta(\alpha))}{(2\,A - f\,\alpha)^2}\,\left(-f\,(1-\beta(\alpha)) + 2\,(A - f\,\alpha)\,\beta'(\alpha)\right)\Big|_{A=(b+d)\,(1-\beta(\alpha))}\,,$$

which is less than zero, because of $\beta'(\alpha) < 0$ and (15.4).

Figure 15.6 represents $F(\alpha)$ and $-f\,\alpha$ as well as the resulting $Op^*_{2,1}(\alpha)$ using (9.74) with $(\mu_1 - \mu_0)/\sigma = 1.5$. Again, depending on the regions of definition, see (15.95), $F(\alpha)$ and $-f\,\alpha$ are solid or dashed, and $Op^*_{2,1}(\alpha)$ is solid for any $\alpha \in [0,1]$. We choose here $b = 10$ and $f = 3$; the three graphs correspond again to $d = 4$ (left top), $d = 6$ (right top) and $d = 12$ (bottom), which fulfil (14.2).

As in Sections 9.5 and 12.4, we distinguish the cases (i) and (ii) from (9.75) with the special cases (9.76) and (9.77), because $F(\alpha)$ is a monotone decreasing function on $[0,1]$. In case (ii) and no intersection point, the Operator will behave illegally for all values of α (bottom graph), because we have $F(\alpha) > -f\,\alpha$ for any $\alpha \in [0,1]$.

Figure 15.6 The equilibrium payoff (15.95) to the Operator for $b = 10, f = 3$ and $d = 4$ (top left), $d = 6$ (top right) and $d = 12$ (bottom).



To determine the optimal value of α, we use the Inspectorate Leadership Principle again: According to (15.12) and (15.15), the equilibrium payoff to the Inspectorate is

$$In_{2,1}^* := \begin{cases} In_{2,1}^* & \text{for Operator's illegal behaviour} \\ -g\,\alpha & \text{for Operator's legal behaviour} \end{cases}$$

$$= \begin{cases} G(\alpha) & \text{for} \quad d > \dfrac{(A - f\,\alpha)^2}{2\,A - f\,\alpha} \\ -g\,\alpha & \text{for} \quad d < \dfrac{(A - f\,\alpha)^2}{2\,A - f\,\alpha} \end{cases}, \tag{15.97}$$

where $G(\alpha)$ is, using (15.2) and (15.12), for any $\alpha \in [0, 1]$ defined by

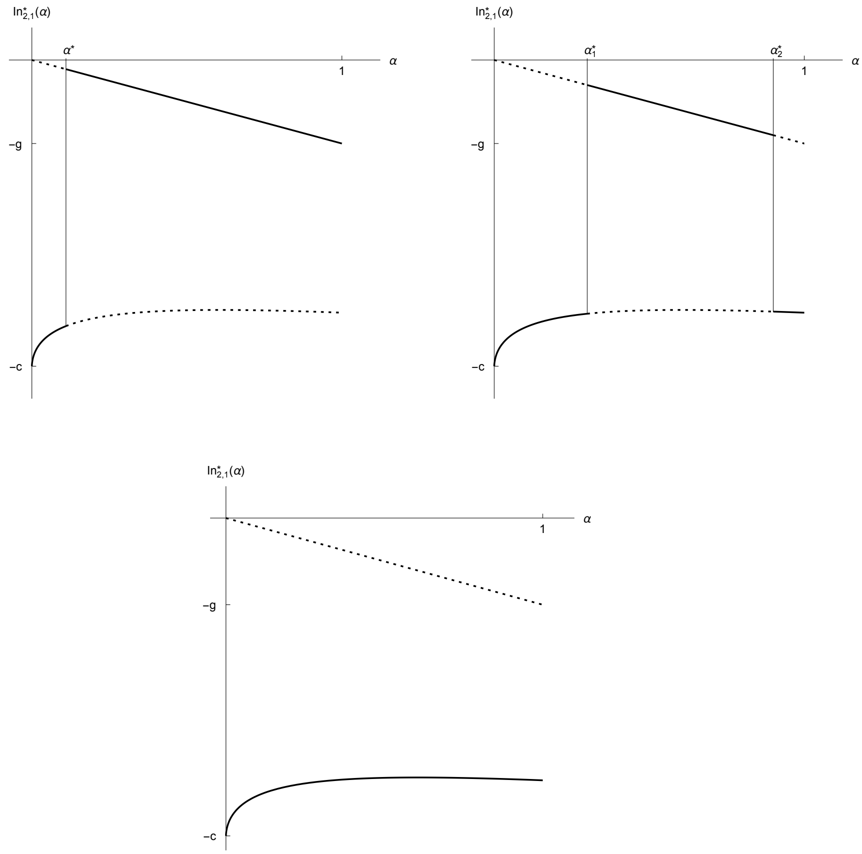
$$G(\alpha) := -c + \frac{(c - a)^2\,(1 - \beta(\alpha))^2}{2\,(c - a)\,(1 - \beta(\alpha)) + g\,\alpha}\,. \tag{15.98}$$

Again, $G(\alpha)$ is coincides with $In^*_{2,1}$, see (15.97), if and only if $d > (A - f\,\alpha)^2/(2\,A - f\,\alpha)$. Using (9.69), (14.2) and (15.98) we get

$$G(0) = -c \qquad \text{and} \qquad G(1) = -c + \frac{(c-a)^2}{2\,(c-a)+g} < -g\,.$$

Figure 15.7 illustrates $In^*_{2,1}(\alpha)$ (solid curve) for the sets of parameters used in Figure 15.6 and $c = 11, a = 10$ and $g = 3$, which fulfil (14.2).

Figure 15.7 The equilibrium payoff (15.97) to the Inspectorate for the sets of parameters used in Figure 15.6 and $c = 11, a = 10, g = 3$.



For $d = 4$ and $d = 6$ (top row), we see that for $\alpha = \alpha^*$ resp. $\alpha = \alpha_1^*$ and legal behaviour of the Operator the Inspectorate's payoff is maximized. This is, as outlined in Section 9.4, the optimal choice of both players in which the Operator is deterred from behaving illegally.

Again, in case (ii) and no intersection point in (9.75), i.e., the bottom graph in Figure 15.7, the application of the Inspector Leadership Principle does not result in the deterrence of the Operator.

Chapter 16

Se-Se inspection game: Dresher-Höpfinger model and extensions

The earliest known inspection game over time has been described and analysed by Dresher (1962). In the terminology of Table 2.1 it is a critical time Se-Se inspection game. Whereas Dresher formulated it as a zero-sum game with idealized payoffs, Höpfinger (1971) considered a non-zero-sum game with general payoffs. The model and its solution have become very influential both from the methodological point of view and because of its wide range of applications. Therefore we will call this game in honour of both authors the Dresher-Höpfinger model.

Whereas in all previous chapters we used the term inspections we will now, for historical reasons and in view of the next chapter, replace this term by *control*. Also, as mentioned in Chapter 14, we will use the term *step* instead of critical time. Thus, quite generally and having in mind all assumptions in Chapter 14, we consider L steps in *at most one* of which the Operator behaves illegally, i.e., performs an illegal activity, and in which the Inspectorate performs k controls.

In this chapter, assumptions (vi) and (x) of Chapter 14 are specified as follows:

(vi') The Operator decides at the beginning, i.e., at step L, whether to behave illegally at that step. If he behaves legally at steps $L, \ldots, \ell + 1$ ($1 \leq \ell \leq L - 1$), then the Operator decides whether to behave illegally at step ℓ; and so on. The Operator does not need to behave illegally throughout the game; see assumption (iii).

The Inspectorate decides at the beginning whether to control at step L. If it has still controls at its disposal, then the Inspectorate decides at step $L - 1$ whether to control at that step; and so on.

(x') The game ends either at the step at which the Operator behaves illegally, or at that step at which the number of controls left is zero, or at that step at which the number of controls left is equal to the number of steps left, or at step 1.

Assumptions (iv) and (viii) will be specified in the following sections, while the remaining assumptions of Chapter 14 except (ix) hold throughout this chapter. Regarding assumption (x') we note that if the Operator behaves illegally at step i, $i = L, ..., 1$, then the game ends at step i regardless whether the illegal behaviour is detected at that step or not. In the latter case, the Operator has successfully performed his illegal activity and thus, the game ends as well.

In Section 16.1 we start as usually with a special case, and present thereafter the general Dresher-Höpfinger model with the complete proof of the game theoretical solution. In Section 16.2 we take into account errors of the second kind, and in the Section 16.3 errors of the first and second kind. In the latter section a solution has been found only for $k = 1$ control during L steps, but it is interesting, nevertheless, because of its relation to the No-No inspection game of Chapter 15 and its game theoretical solution. Finally, we discuss in Section 16.4 for the Dresher-Höpfinger inspection game with $L = 2$ steps and $k = 1$ control the choice of the best value of the false alarm probability α.

Let us specify what we said already in the introduction to this Part III: For the models considered in this and the next chapter, time is a priori no issue; the steps introduced here may be days, nights or weeks, and the objectives of the players are just the gains and losses in case of detected or undetected illegal behaviour of the inspected side. Nevertheless, let us repeat, we call this Part III *Critical Time* because these models describe also critical time conflicts.

16.1 Any number of steps and controls

The inspection game analysed in this section is based on the specifications:

(iv') During a control the Inspectorate does not commit any errors of the first and second kind, i.e., if the illegal activity is carried through at the same step at which a control is performed, then it is detected with certainty at that step.

(viii') The payoffs to the two players (Operator, Inspectorate) are given by

$$\begin{array}{rl} (d, -c) & \text{for an untimely control and illegal behaviour} \\ (-b, -a) & \text{for a timely control and illegal behaviour} \\ (0, 0) & \text{for legal behaviour}\,, \end{array} \tag{16.1}$$

where the parameters satisfy the conditions

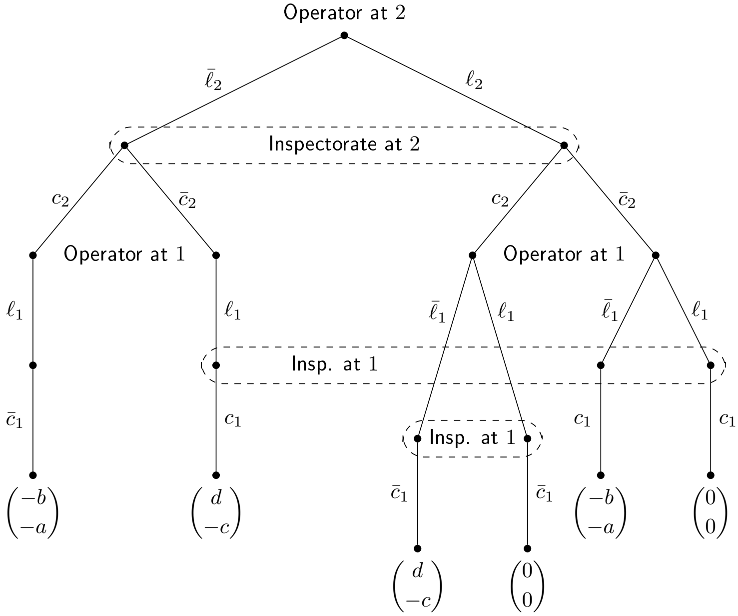
$$0 < min(b, d) \qquad \text{and} \qquad 0 < a < c\,. \tag{16.2}$$

If the Operator behaves illegally, then the case "untimely control" means that no control is performed at the step at which the Operator behaves illegally.

Consider first the Dresher-Höpfinger model with $L = 2$ steps and $k = 1$ control which is abbreviated by $\Gamma(2, 1)$, and the extensive form of which is presented in Figure 16.1. Note that according to the comment on p. 50, all extensive form games in this chapter start with the Operator's decision at L. Since it is the first of many subsequent ones, we describe it in major detail.

At step 2, i.e., at the top of the tree, the Operator decides to behave illegally immediately ($\bar{\ell}_2$) or not (ℓ_2). In the latter case he decides at step 1 to behave illegally immediately ($\bar{\ell}_1$) or not (ℓ_1). Also at step 2 the Inspectorate decides, not knowing the Operator's decision, to control (c_2) or not ($\bar{c}_2$). At step 1 it cannot control ($\bar{c}_1$) any more if it does so at step 2 (c_2), and it

Figure 16.1 Extensive form of the Dresher-Höpfinger inspection game $\Gamma(2,1)$.

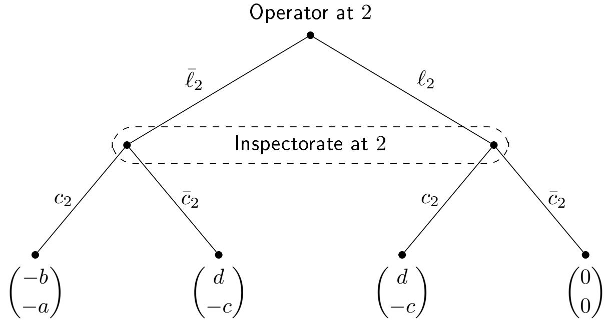


has to control (c_1) if it does not at step 2 ($\bar{c}_2$). If the Inspectorate does not control at step 2 ($\bar{c}_2$), then it does not know at step 1 if the Operator behaved illegally at step 2 ($\bar{\ell}_2$) or not (ℓ_2), and if he behaves illegally at step 1 ($\bar{\ell}_1$) or not at all (ℓ_1) (middle information set). If the Operator chooses ℓ_2 and the Inspectorate c_2, then the Inspectorate does not know at step 1 whether the Operator behaves illegally at step 1 (lowest information set). The payoffs to the two players are given at the end nodes of the tree and are, using (16.1), self-explaining. Note that – as mentioned in the introduction to this chapter – the Operator does not necessarily behave illegally.

Let us comment the two information sets of the Inspectorate at step 1: They represent the state of information of the Inspectorate, but they have no operational meaning since at all nodes contained in these two sets there are no alternatives. Thus, if one is only interested in the decisions of the Inspectorate, one can omit these information sets. If one, furthermore, omits the strictly dominated strategy ℓ_1 resp. $\bar{\ell}_1$ of the Operator in the subgames starting after the moves $\ell_2 c_2$ resp. $\ell_2 \bar{c}_2$, then one arrives at the reduced extensive form game and its normal form represented in Figure 16.2. Note that in all subsequent extensive form games we will no longer consider information sets without alternatives at their nodes, in other words, we will consider only the reduced extensive forms corresponding to Figure 16.2 for the game $\Gamma(2,1)$.

Let the probability of behaving illegally at step 2 ($\bar{\ell}_2$) be $\bar{p}_{2,1}$. For the Inspectorate, let $q_{2,1}$ be the probability to control at step 2 (c_2). If the Operator behaves legally at step 2 (ℓ_2), then – depending on the Inspectorate's behaviour at step 2 – the subgame $\Gamma(1,1)$ or $\Gamma(1,0)$ at step 1 is reached. In both games the Inspectorate has no strategic alternatives because in

Figure 16.2 Reduced extensive form and the corresponding normal form of the inspection game in Figure 16.1.



	c_2	$\bar{c}_2$
$\bar{\ell}_2$	$-b$; $-a$	d ; $-c$
ℓ_2	d ; $-c$	0 ; 0

$\Gamma(1,1)$ it has to control, i.e., $q_{1,1} = q^*_{1,1} = 1$, and in $\Gamma(1,0)$ it cannot perform a control, i.e., $q_{1,0} = q^*_{1,0} = 0$. Thus, the probabilities $q_{1,1}$ and $q_{1,0}$ are fixed, and are therefore excluded from the Inspectorate's set of behavioural strategies $Q_{2,1}$ defined below. Formally, and in contrast to the Inspectorate, the Operator has strategic alternatives in the games $\Gamma(1,1)$ and $\Gamma(1,0)$: He can choose any probability $\bar{p}_{1,1} \in [0,1]$ resp. $\bar{p}_{1,0} \in [0,1]$, where in equilibrium we have $\bar{p}^*_{1,1} = 0$ and $\bar{p}^*_{1,0} = 1$ because of the strict dominance. Therefore, $\bar{p}_{1,1}$ and $\bar{p}_{1,0}$ are excluded from Operator's set of behavioural strategies $P_{2,1}$. Summing up, we have

$$P_{2,1} = \{\bar{p}_{2,1} : \bar{p}_{2,1} \in [0,1]\} \quad \text{and} \quad Q_{2,1} = \{q_{2,1} : q_{2,1} \in [0,1]\}\,. \tag{16.3}$$

Let us note that contrary to the notation of the strategies for the Se-Se inspection game in Part I, we use here p and q instead of g and h, because we transform the extensive form games into normal form games in which the notation p and q is utilized in Part I. Also note that in order to be consistent with the notation in Chapter 17, we already use here the notation $\bar{p}_{2,1}$.

The Operator's (expected) payoff is, for any $\bar{p}_{2,1} \in P_{2,1}$ and any $q_{2,1} \in Q_{2,1}$, using the bimatrix in Figure 16.2 and (19.3), given by

$$Op_{2,1}(\bar{p}_{2,1}, q_{2,1}) = (\bar{p}_{2,1}, 1-\bar{p}_{2,1}) \begin{pmatrix} -b & d \\ d & 0 \end{pmatrix} \begin{pmatrix} q_{2,1} \\ 1-q_{2,1} \end{pmatrix} \tag{16.4}$$

and that of the Inspectorate, using (19.4), by

$$In_{2,1}(\bar{p}_{2,1}, q_{2,1}) = (\bar{p}_{2,1}, 1-\bar{p}_{2,1}) \begin{pmatrix} -a & -c \\ -c & 0 \end{pmatrix} \begin{pmatrix} q_{2,1} \\ 1-q_{2,1} \end{pmatrix}\,. \tag{16.5}$$

The game theoretical solution of this inspection game, see Canty et al. (2001), is presented in

Lemma 16.1. *Given the Se-Se inspection game with* $L = 2$ *steps and* $k = 1$ *control, i.e.,* $\Gamma(2, 1)$. *The sets of behavioural strategies are given by (16.3), and the payoffs to both players by (16.4) and (16.5).*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{2,1} := Op_{2,1}(\bar{p}^*_{2,1}, q^*_{2,1})$ *and* $In^*_{2,1} := In_{2,1}(\bar{p}^*_{2,1}, q^*_{2,1})$*:*

The Operator behaves illegally at step 2 *with probability*

$$\bar{p}^*_{2,1} = \frac{c}{2\,c - a}\,, \tag{16.6}$$

and the Inspectorate controls at step 2 *with probability*

$$q^*_{2,1} = \frac{d}{2\,d + b}\,. \tag{16.7}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{2,1} = \frac{d^2}{2\,d + b} \qquad \text{and} \qquad In^*_{2,1} = -\,\frac{c^2}{2\,c - a}\,. \tag{16.8}$$

Proof. Consider the normal form given in Figure 16.2. According to (16.2), the preference directions are cyclic, so that there exists a unique equilibrium in mixed strategies. Then the method of rendering the adversary indifferent as regards to the choice of his strategy leads to the equilibrium strategies given by (16.6) and (16.7) and the corresponding equilibrium payoffs (16.8); see Theorem 19.1. □

This equilibrium is quite different from the solution of the No-No inspection game; see (15.10) to (15.12) with $\alpha = \beta = 0$. First, as one would expect, the ability to make use of the information available at step 1 gives the Operator an advantage with respect to the No-No inspection game. Indeed, subtracting the Operator's equilibrium payoff in the No-No inspection game from that for the Se-Se inspection game, we obtain, using (15.12) and (16.8),

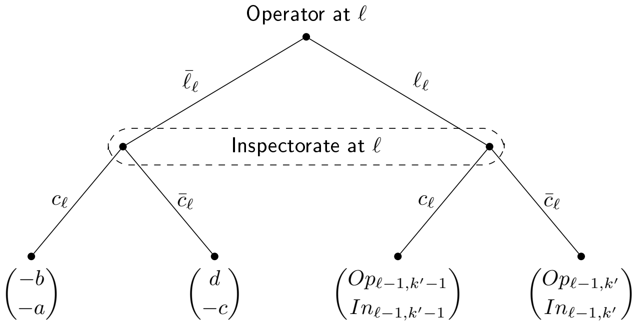
$$\frac{d^2}{2\,d + b} - \frac{d - b}{2} = \frac{b}{2}\,\frac{b + d}{2\,d + b} > 0.$$

Second, because $\bar{p}^*_{2,1} < 1$, the Operator will behave legally with positive probability, irrespective of his payoff parameters, i.e., he cannot be deterred from behaving illegally.

Let us now turn to the general case of any number L of steps and k controls which is denoted by $\Gamma(L, k)$, and which has been analysed first by Dresher (1962). He considered a zero-sum game with payoffs to the Inspectorate as given by $+1$ for detected illegal behaviour, -1 for undetected illegal behaviour, i.e., $d = b = 1$ in (16.1), and 0 for legal behaviour of the Operator. As mentioned on p. 315, Dresher's model and its recursive treatment represents a landmark in the area of inspection games over time and has been modified and extended in the decades after its invention as we will show later on.

The payoffs in Dresher's model have been generalized by Höpfinger (1971). He assumed that the Operator's gain for a successful illegal activity need not equal his loss if he is caught, and also solved the resulting recurrence equation explicitly. Furthermore, zero-sum payoffs are not fully adequate since a caught illegal activity, compared to behaving legally throughout, is usually undesirable for *both* players since for the Inspectorate this demonstrates a failure of his surveillance system. The recursive form of the inspection game as analysed by Höpfinger (1971) and later by Rinderle (1996) is given in Figure 16.3 both in extensive and in normal form, where we have already presented the subgame which is reached in case the Operator behaves legally at steps $L, \ldots, \ell+1$ (for $\ell < L$), and ℓ steps as well as k' controls are left. The variables ℓ and k' are subject to $2 \leq \ell \leq L$ and $1 \leq k' \leq \min(\ell-1, k)$. The cases $\ell = 1$ as well as $k' = 0$ and $k' = \ell$ are excluded from the game theoretical analysis; see the explanations before (16.9).

Figure 16.3 Recursive extensive form and corresponding recursive normal form of the subgame $\Gamma(\ell, k')$ of the Dresher-Höpfinger inspection game $\Gamma(L, k)$, if ℓ steps and k' controls are left, and the Operator behaves legally at steps $L, \ldots, \ell+1$ ($2 \leq \ell \leq L$, $1 \leq k' \leq \min(\ell-1, k)$).



	c_ℓ	$\bar{c}_\ell$
$\bar{\ell}_\ell$	$-a$ $-b$	$-c$ d
ℓ_ℓ	$In_{\ell-1,k'-1}$ $Op_{\ell-1,k'-1}$	$In_{\ell-1,k'}$ $Op_{\ell-1,k'}$

The payoffs in both games in Figure 16.3 can be explained as follows: Suppose the Operator behaves illegal at step ℓ ($\bar{\ell}_\ell$). If the Inspectorate performs a control at step ℓ (c_ℓ), then the illegal activity is detected leading to the payoffs $(-b, -a)$, and if it does not control at step ℓ ($\bar{c}_\ell$), then the illegal behaviour has been successful and the payoffs are $(d, -c)$. If, on contrary, the Operator behaves legal at step ℓ (ℓ_ℓ), then depending on the Inspectorate's decision (c_ℓ or $\bar{c}_\ell$) two subgames are reached the (expected) payoff in which are abbreviated as follows: Let $Op_{\ell-1,k'}$ and $In_{\ell-1,k'}$ resp. $Op_{\ell-1,k'-1}$ and $In_{\ell-1,k'-1}$ denote the (expected) payoffs to both players in the subgame with $\ell-1$ steps and k' resp. $k'-1$ controls.

A remark on the notation: In Parts I and II the illegal and legal behaviour are denoted by $\bar{\ell}$ and ℓ, respectively. In Part III we keep this notation for consistency reasons, however, because L is the number of steps, it is appealing to abbreviate any step between $L, \ldots, 1$ by the letter ℓ as n and N in the step by step game of Section 5.3. This implies that the legal behaviour at step ℓ is denoted by ℓ_ℓ. We do not expect the reader to be confused by this notation.

Two pairs of (ℓ, k')-values deserve a special attention. Note that we always have $k' \leq \ell$. First, if $k' = \ell$ then the Inspectorate controls at any of the remaining ℓ steps and thus, the Operator will behave legally in all these steps because $0 > -b$, i.e., we have for the equilibrium payoffs

$Op^*_{\ell,\ell} = 0$ and $In^*_{\ell,\ell} = 0$ for all $1 \leq \ell \leq L$. Second, if $k' = 0$ then there is no control left and thus, the Operator will behave illegally in one of the remaining steps because $d > 0$. Here we get for the equilibrium payoffs $Op^*_{\ell,0} = d$ and $In^*_{\ell,0} = -c$ for all $1 \leq \ell \leq L$. Thus, the equilibrium payoffs need to fulfil for any $1 \leq \ell \leq L$ the boundary conditions

$$Op^*_{\ell,\ell} = 0\,, \qquad In^*_{\ell,\ell} = 0 \qquad \text{and} \qquad Op^*_{\ell,0} = d\,, \qquad In^*_{\ell,0} = -c\,. \tag{16.9}$$

Let $\bar{p}_{L,k}$ denote the probability of behaving illegally at step L ($\bar{\ell}_L$) and let $q_{L,k}$ be the probability to control at step L (c_L). Suppose the Operator behaves legally at steps $L, \ldots, \ell{+}1$, $1 \leq \ell < L$, then the game has reached step ℓ, i.e., ℓ steps are left, and suppose that the Inspectorate has still k' controls at its disposal, $1 \leq k' \leq \min(\ell - 1, k)$. Then $\bar{p}_{\ell,k'}$ denotes the probability to behave illegally at step ℓ ($\bar{\ell}_\ell$) and $q_{\ell,k'}$ denotes the probability to control at step ℓ (c_ℓ). Although for a complete description of the game the sets of behavioural strategies of both players needs to be defined, we omit this here, as it is cumbersome, because each subgame contains two further subgames, and is not further used.

The game theoretical solution of this inspection game, see Rinderle (1996), is presented in

Theorem 16.1. *Given the Se-Se inspection game with $L > k$ steps and k controls, i.e., $\Gamma(L,k)$, the recursive extensive and normal forms of which are represented in Figure 16.3. The payoffs to both players are defined recursively using the recursive normal form representation in Figure 16.3, and the equilibrium payoffs to both players fulfil the boundary conditions (16.9).*

Suppose ℓ steps, $2 \leq \ell \leq L$, and k' controls, $1 \leq k' \leq \min(\ell - 1, k)$, are left, and the Operator behaves legally at steps $L, \ldots, \ell{+}1$, i.e., the subgame $\Gamma(\ell, k')$ is reached. Define the functions

$$f(\ell,k') := \sum_{i=0}^{k'} \binom{\ell}{i} \left(\frac{b}{d}\right)^{k'-i} \qquad \text{and} \qquad g(\ell,k') := \sum_{i=0}^{k'} \binom{\ell}{i} \left(-\frac{a}{c}\right)^{k'-i}. \tag{16.10}$$

Then a Nash equilibrium in the subgame $\Gamma(\ell,k')$ is given by the following equilibrium strategies and payoffs $Op^*_{\ell,k'}$ and $In^*_{\ell,k'}$:

The Operator behaves illegally at step ℓ with probability

$$\bar{p}^*_{\ell,k'} = 1 - \frac{1 - \dfrac{a}{c}}{-\dfrac{\binom{\ell-2}{k'}}{g(\ell-1,k')} + \dfrac{\binom{\ell-2}{k'-1}}{g(\ell-1,k'-1)} + 1 - \dfrac{a}{c}}\,, \tag{16.11}$$

where $\binom{\ell-2}{\ell-1} := 0$, and the Inspectorate controls at step ℓ with probability

$$q^*_{\ell,k'} = \frac{f(\ell-1,k'-1)}{f(\ell,k')}\,. \tag{16.12}$$

The equilibrium payoffs to the Operator and to the Inspectorate in the subgame $\Gamma(\ell,k')$ are

$$Op^*_{\ell,k'} = d\,\frac{\binom{\ell-1}{k'}}{f(\ell,k')} \qquad \text{and} \qquad In^*_{\ell,k'} = -c\,\frac{\binom{\ell-1}{k'}}{g(\ell,k')}\,, \tag{16.13}$$

which – for $\ell = L$ and $k' = k$ – are the equilibrium payoffs of the entire game $\Gamma(L,k)$.

Proof. We proceed in four steps.

1. In order to show that (16.11) and (16.12) constitute probabilities, we first prove that $g(\ell, k')$ satisfies the two recursive relations:

$$g(\ell, k') = g(\ell-1, k') + g(\ell-1, k'-1) \qquad \text{and} \tag{16.14}$$

$$g(\ell, k') = -\frac{a}{c}\, g(\ell, k'-1) + \binom{\ell}{k'}. \tag{16.15}$$

In fact, with the well-known binomial addition formula

$$\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}, \qquad 1 \le i \le n, \tag{16.16}$$

and an appropriate change of summation we get by (16.10)

$$\begin{aligned}
g(\ell, k') &= \Big(-\frac{a}{c}\Big)^{k'} + \sum_{i=1}^{k'} \binom{\ell}{i} \Big(-\frac{a}{c}\Big)^{k'-i} \\
&= \binom{\ell-1}{0} \Big(-\frac{a}{c}\Big)^{k'} + \sum_{i=1}^{k'} \left(\binom{\ell-1}{i} + \binom{\ell-1}{i-1}\right) \Big(-\frac{a}{c}\Big)^{k'-i} \\
&= \sum_{i=0}^{k'} \binom{\ell-1}{i} \Big(-\frac{a}{c}\Big)^{k'-i} + \sum_{i=1}^{k'} \binom{\ell-1}{i-1} \Big(-\frac{a}{c}\Big)^{k'-i} \\
&= \sum_{i=0}^{k'} \binom{\ell-1}{i} \Big(-\frac{a}{c}\Big)^{k'-i} + \sum_{i=0}^{k'-1} \binom{\ell-1}{i} \Big(-\frac{a}{c}\Big)^{k'-(i+1)} \\
&= g(\ell-1, k') + g(\ell-1, k'-1),
\end{aligned}$$

i.e., (16.14), and

$$\begin{aligned}
g(\ell, k') &= \sum_{i=0}^{k'} \binom{\ell}{i} \Big(-\frac{a}{c}\Big)^{k'-i} = \sum_{i=0}^{k'-1} \binom{\ell}{i} \Big(-\frac{a}{c}\Big)^{k'-i} + \binom{\ell}{k'} \\
&= \Big(-\frac{a}{c}\Big) \sum_{i=0}^{k'-1} \binom{\ell}{i} \Big(-\frac{a}{c}\Big)^{k'-i-1} + \binom{\ell}{k'} \\
&= \Big(-\frac{a}{c}\Big)\, g(\ell, k'-1) + \binom{\ell}{k'},
\end{aligned}$$

i.e., (16.15). $f(\ell, k')$ also fulfils (16.14) and (16.15) if one replaces $-a/c$ by b/d. Using (16.10) we get for all $1 \le \ell \le L$

$$g(\ell, \ell) = \Big(1 - \frac{a}{c}\Big)^{\ell} \qquad \text{and} \qquad g(\ell, 0) = 1,$$

and thus, (16.14) yields $g(\ell, k') > 0$, i.e., the ratios regarding $g(\ell-1, k')$ and $g(\ell-1, k'-1)$ in (16.11) are well-defined. To prove that $\bar{p}^*_{\ell,k'} \in (0,1)$, it is sufficient to show that

$$\frac{\binom{\ell-2}{k'-1}}{g(\ell-1,k'-1)} > \frac{\binom{\ell-2}{k'}}{g(\ell-1,k')},$$

which is, using (16.14), equivalent to

$$g(\ell-1,k'-1)\,\frac{\ell-1}{k'} < g(\ell,k')\,. \tag{16.17}$$

We prove this inequality by induction. For $k'=1$ we have by (16.10)

$$g(\ell-1,0)\;(\ell-1) = \ell-1 < \ell - \frac{a}{c} = g(\ell,1)$$

which holds because of $a < c$; see (16.2). Assume now $k' > 1$ and assume that (16.17) holds for all $1, \ldots, k'-1$. Using (16.15) with $\ell-1$ and adding it to (16.14) we get

$$g(\ell,k') = \left(1-\frac{a}{c}\right)\, g(\ell-1,k'-1) + \binom{\ell-1}{k'}$$

and shifting both variables by -1

$$g(\ell-1,k'-1) = \left(1-\frac{a}{c}\right)\, g(\ell-2,k'-2) + \binom{\ell-2}{k'-1}.$$

Multiplying the second equation with $(\ell-1)/k'$ and subtracting it from the first we get, keeping in mind that $\ell-1 \geq k' > 1$ implies $(\ell-1)/k' \leq (\ell-2)/(k'-1)$,

$$\begin{aligned} g(\ell,k') - g(\ell-1,k'-1)\,\frac{\ell-1}{k'} &= \left(1-\frac{a}{c}\right)\left(g(\ell-1,k'-1) - g(\ell-2,k'-2)\,\frac{\ell-1}{k'}\right) \\ &\geq \left(1-\frac{a}{c}\right)\left(g(\ell-1,k'-1) - g(\ell-2,k'-2)\,\frac{\ell-2}{k'-1}\right). \end{aligned}$$

Due to the induction assumption, the right hand side is larger than zero, i.e., (16.17) is shown, and we have $\bar{p}^*_{\ell,k'} \in (0,1)$.

Because $f(\ell,k')$ is larger than 0 by definition, and because (16.14) for $f(\ell,k')$ instead of $g(\ell,k')$ implies $f(\ell,k') > f(\ell-1,k'-1)$, we see, using (16.12), that $q^*_{\ell,k'} \in (0,1)$.

2. For any ℓ with $2 \leq \ell \leq L$ and any k' with $1 \leq k' \leq \min(\ell-1,k)$ we consider the normal form given in Figure 16.3. In part 4 of the proof it is shown that the payoffs are cyclic, see (16.31) and (16.32), which implies that there exists a unique Nash equilibrium in mixed strategies which is determined with the help of the indifference principle, see Theorem 19.1,

$$Op^*_{\ell,k'} = q^*_{\ell,k'}\,(-b) + (1-q^*_{\ell,k'})\,d = q^*_{\ell,k'}\,Op^*_{\ell-1,k'-1} + (1-q^*_{\ell,k'})\,Op^*_{\ell-1,k'} \tag{16.18}$$

and

$$In^*_{\ell,k'} = -\bar{p}^*_{\ell,k'}\,a + (1-\bar{p}^*_{\ell,k'})\,In^*_{\ell-1,k'-1} = -\bar{p}^*_{\ell,k'}\,c + (1-\bar{p}^*_{\ell,k'})\,In^*_{\ell-1,k'}\,. \tag{16.19}$$

This leads to

$$q^*_{\ell,k'} = \frac{d - Op^*_{\ell-1,k'}}{Op^*_{\ell-1,k'-1} - Op^*_{\ell-1,k'} + b + d} \tag{16.20}$$

and

$$\bar{p}^*_{\ell,k'} = \frac{In^*_{\ell-1,k'} - In^*_{\ell-1,k'-1}}{In^*_{\ell-1,k'} - In^*_{\ell-1,k'-1} + c - a}\,. \tag{16.21}$$

Inserting (16.20) into (16.18) and (16.21) into (16.19), we get the following recursive relations for $Op^*_{\ell,k'}$ and $In^*_{\ell,k'}$:

$$Op^*_{\ell,k'} = \frac{d\,Op^*_{\ell-1,k'-1} + b\,Op^*_{\ell-1,k'}}{Op^*_{\ell-1,k'-1} - Op^*_{\ell-1,k'} + b + d} \tag{16.22}$$

and

$$In^*_{\ell,k'} = \frac{c\,In^*_{\ell-1,k'-1} - a\,In^*_{\ell-1,k'}}{In^*_{\ell-1,k'} - In^*_{\ell-1,k'-1} + c - a}\,. \tag{16.23}$$

Two observations are important: First, if we replace b by a and d by $-c$ in the right hand side of (16.22), then we get the right hand side of (16.23), which means that the two recursive relations have the same structure. Second, if we write (16.23) in the form

$$\frac{In^*_{\ell,k'}}{c} = \frac{\dfrac{In^*_{\ell-1,k'-1}}{c} - \dfrac{a}{c}\,\dfrac{In^*_{\ell-1,k'}}{c}}{\dfrac{In^*_{\ell-1,k'}}{c} - \dfrac{In^*_{\ell-1,k'-1}}{c} + 1 - \dfrac{a}{c}}\,, \tag{16.24}$$

then we see that $In^*_{\ell,k'}/c$ depends only on the single parameter a/c which simplifies the determination of the solution of the recursive relations considerably.

3. With the transformations

$$\tilde{I}^*_{\ell,k'} := \frac{In^*_{\ell,k'}}{c} \qquad \text{and} \qquad \tilde{a} := -\frac{a}{c}\,,$$

we have to show that

$$\tilde{I}^*_{\ell,k'} = -\frac{\binom{\ell-1}{k'}}{\tilde{g}(\ell,k')} \qquad \text{with} \qquad \tilde{g}(\ell,k') = \sum_{i=0}^{k'} \binom{\ell}{i}\,\tilde{a}^{k'-i} \tag{16.25}$$

fulfils by (16.24) the recursive relation

$$\tilde{I}^*_{\ell,k'} = \frac{\tilde{I}^*_{\ell-1,k'-1} + \tilde{a}\,\tilde{I}^*_{\ell-1,k'}}{\tilde{I}^*_{\ell-1,k'} - \tilde{I}^*_{\ell-1,k'-1} + 1 + \tilde{a}} \tag{16.26}$$

together with the boundary conditions, see (16.9),

$$\tilde{I}^*_{\ell,\ell} = 0 \qquad \text{and} \qquad \tilde{I}^*_{\ell,0} = -1\,. \tag{16.27}$$

We see immediately that (16.27) is fulfilled. In order to show that the recursive relation (16.26) is fulfilled we use (16.14) and (16.15) which imply that $\tilde{g}(\ell, k')$ satisfies

$$\tilde{g}(\ell, k') = \tilde{g}(\ell-1, k') + \tilde{g}(\ell-1, k'-1) \tag{16.28}$$

and

$$\tilde{g}(\ell, k') = \tilde{a}\,\tilde{g}(\ell, k'-1) + \binom{\ell}{k'}. \tag{16.29}$$

Substituting $\tilde{I}^*_{\ell,k'}$ from (16.25) into (16.26) we get

$$-\frac{\binom{\ell-1}{k'}}{\tilde{g}(\ell,k')} = \frac{-\dfrac{\binom{\ell-2}{k'-1}}{\tilde{g}(\ell-1,k'-1)} - \tilde{a}\,\dfrac{\binom{\ell-2}{k'}}{\tilde{g}(\ell-1,k')}}{-\dfrac{\binom{\ell-2}{k'}}{\tilde{g}(\ell-1,k')} + \dfrac{\binom{\ell-2}{k'-1}}{\tilde{g}(\ell-1,k'-1)} + 1 + \tilde{a}}$$

or equivalently,

$$-\frac{\binom{\ell-1}{k'}}{\tilde{g}(\ell,k')} = \tag{16.30}$$

$$= \frac{\binom{\ell-2}{k'-1}\tilde{g}(\ell-1,k') + \tilde{a}\binom{\ell-2}{k'}\tilde{g}(\ell-1,k'-1)}{\binom{\ell-2}{k'}\tilde{g}(\ell-1,k'-1) - \binom{\ell-2}{k'-1}\tilde{g}(\ell-1,k') - (1+\tilde{a})\,\tilde{g}(\ell-1,k'-1)\,\tilde{g}(\ell-1,k')}.$$

The nominator on the right hand side of (16.30) is by (16.16) and (16.29)

$$\begin{aligned}
&\binom{\ell-2}{k'-1}\tilde{g}(\ell-1,k') + \tilde{a}\binom{\ell-2}{k'}\tilde{g}(\ell-1,k'-1)\\
&= \binom{\ell-2}{k'-1}\left(\tilde{a}\,\tilde{g}(\ell-1,k'-1) + \binom{\ell-1}{k'}\right) + \tilde{a}\binom{\ell-2}{k'}\tilde{g}(\ell-1,k'-1)\\
&= \tilde{a}\,\tilde{g}(\ell-1,k'-1)\left(\binom{\ell-2}{k'-1} + \binom{\ell-2}{k'}\right) + \binom{\ell-2}{k'-1}\binom{\ell-1}{k'}\\
&= \tilde{a}\,\tilde{g}(\ell-1,k'-1)\binom{\ell-1}{k'} + \binom{\ell-2}{k'-1}\binom{\ell-1}{k'}\\
&= \left(\tilde{a}\,\tilde{g}(\ell-1,k'-1) + \binom{\ell-2}{k'-1}\right)\binom{\ell-1}{k'}.
\end{aligned}$$

Therefore, (16.30) is equivalent to

$$-\left(\tilde{a}\,\tilde{g}(\ell-1,k'-1) + \binom{\ell-2}{k'-1}\right)\tilde{g}(\ell,k')$$

$$= \binom{\ell-2}{k'}\tilde{g}(\ell-1,k'-1) - \binom{\ell-2}{k'-1}\tilde{g}(\ell-1,k'-1) - (1+\tilde{a})\,\tilde{g}(\ell-1,k'-1)\,\tilde{g}(\ell-1,k')$$

or, using (16.28), equivalent to

$$-\left(\tilde{a}\,\tilde{g}(\ell-1,k'-1) + \binom{\ell-2}{k'-1}\right)\left(\tilde{g}(\ell-1,k') + \tilde{g}(\ell-1,k'-1)\right)$$

$$= \binom{\ell-2}{k'} \tilde{g}(\ell-1,k'-1) - \binom{\ell-2}{k'-1} \tilde{g}(\ell-1,k'-1) - (1+\tilde{a})\, \tilde{g}(\ell-1,k'-1)\, \tilde{g}(\ell-1,k')$$

which is equivalent to

$$- \left(\tilde{a}\, \tilde{g}(\ell-1,k'-1) + \binom{\ell-2}{k'-1}\right) \tilde{g}(\ell-1,k'-1)$$
$$= \binom{\ell-2}{k'} \tilde{g}(\ell-1,k'-1) - \tilde{g}(\ell-1,k'-1)\, \tilde{g}(\ell-1,k')$$

or, finally, after diving by $\tilde{g}(\ell-1,k'-1)(>0)$ and using (16.16), equivalent to (16.29), i.e., $\tilde{I}^*_{\ell,k'}$ from (16.25) fulfils the recursive relation (16.26).

With the remark given above, the validity of $Op^*_{\ell,k'}$ can be shown in the same way.

In order to obtain the equilibrium strategies (16.11) and (16.12), we substitute (16.13) into the left hand equalities (16.18) and (16.19) and use, in case of (16.12), the relations (16.14) and (16.15) for $f(\ell,k')$ instead of $g(\ell,k')$. At this point we see the reason for the different complexities of the optimal strategies (16.11) and (16.12): Whereas with (16.18) we can relate $q^*_{\ell,k'}$ just to $Op^*_{\ell,k'}$, according to (16.19) the probability $\bar{p}^*_{\ell,k'}$ is related to both $In^*_{\ell,k'}$ and $In^*_{\ell-1,k'-1}$.

4. Let us come back to the preference directions which are given by

$$Op^*_{\ell-1,k'-1} > -b \qquad \text{and} \qquad Op^*_{\ell-1,k'} < d \tag{16.31}$$

and furthermore,

$$In^*_{\ell-1,k'-1} < In^*_{\ell-1,k'}\,. \tag{16.32}$$

Note that $-c < -a$ holds by definition; see (16.2). Because $f(\ell,k') > 0$ for all $2 \le \ell \le L$ and all k' with $1 \le k' \le \min(\ell-1,k)$, we have $Op^*_{\ell-1,k'-1} > 0 > -b$. Further we get from (16.10)

$$f(\ell,k') = \sum_{i=0}^{k'} \binom{\ell}{i} \left(\frac{b}{d}\right)^{k'-i} > \binom{\ell}{k'} > \binom{\ell-1}{k'},$$

which implies by (16.13) that $Op^*_{\ell-1,k'} < d$. Relation (16.32) is by (16.13) equivalent to

$$g(\ell-1,k'-1)\,\frac{\ell-1-k'}{k'} < g(\ell-1,k') \tag{16.33}$$

or, using (16.14), equivalent to

$$g(\ell-1,k'-1)\,\frac{\ell-1}{k'} < g(\ell,k')\,.$$

The validity of this inequality, however, has already been proven after (16.17), which completes the proof. □

On p. 319 we have shown for $L=2$ steps and $k=1$ control that the equilibrium payoff of the Se-Se inspection game is larger than that of the No-No inspection game. This statement is also true for any $L > 1$ steps and $k = 1$ control, because, using case (i) in Corollary 15.1 for $\beta = 0$ and (16.13), we get

$$d\,\frac{\binom{L-1}{1}}{f(L,1)} - \left(d - (b+d)\,\frac{1}{L}\right) = \frac{b\,(b+d)}{L\,(L\,d+b)} > 0\,.$$

Furthermore, the statement is true for any $L > k > 1$: Because (16.15) for b/d instead of $-a/c$ implies

$$f(L,k) = f(L-1,k) + f(L-1,k-1)\,,$$

and because (16.33) yields for f instead of g

$$f(L,k-1)\,\frac{L-k}{k} < f(L-1,k)\,,$$

we get

$$\begin{aligned} d\,\frac{\binom{L-1}{k}}{f(L,k)} - d + (b+d)\,\frac{k}{L} &= d\,\frac{k}{L}\,\frac{b}{d}\,\left(-\frac{L-k}{k}\,\frac{f(L,k-1)}{f(L,k)} + 1\right) \\ &= \frac{k}{L}\,\frac{b}{f(L,k)}\,\left(\left(1-\frac{L}{k}\right)\,f(L,k-1) + f(L,k)\right) > 0\,, \end{aligned}$$

i.e., the equilibrium payoff of the Se-Se inspection game is larger than that of the No-No inspection game. Note that the Operator will behave legally in equilibrium with positive probability since $\bar{p}^*_{\ell,k'} < 1$, however, the Inspectorate has no deterring strategy.

In Theorem 16.1 we have determined a Nash equilibrium using the recursive structure of the game. What happens if we solve the extensive form game instead? Do we get additional equilibrium strategies and if yes, are they payoff equivalent to those already found? Here, these questions have not yet been addressed. In Chapter 17, however, we show that in fact the equilibrium strategies of the generalized Thomas-Nisgav inspection game, which is also solved using its recursive structure, are no longer unique. There are more equilibrium strategies of the second player (called Customs in Section 17.1), if one considers the extensive form as a whole, but they are all payoff equivalent.

As mentioned on p. 315, the Dresher-Höpfinger model has become very influential. A slightly different and very interesting variant will be discussed in Chapter 17: In the *original* Thomas-Nisgav inspection game it is assumed that the Smuggler must perform an illegal activity, in contrast to the Dresher-Höpfinger inspection game where the Operator can perform at most one illegal activity. This change of the modelling assumption leads to a considerable change in the game theoretical solutions; see (16.11) – (16.13) for $b = d = 1$, $c = 1$ and $a = -1$ in contrast to (17.31) – (17.33) for $b = d = 1$ and $\beta = 0$ and also p. 368. A numerical comparison can be found in Krieger and Avenhaus (2018b).

16.2 Any number of steps and controls; errors of the second kind

In case of attribute sampling procedures errors of the second kind may occur, in other words an illegal activity will, even in case the control is timely, only be detected with probability $1 - \beta$. Thus, the inspection game analysed in this section is based on the specifications:

(iv') During a control the Inspectorate may commit an error of the second kind, i.e., if the Operator behaves illegally at the same step at which the Inspectorate performs a control, then the illegal activity is not detected with probability β. This non-detection probability is the same for all controls.

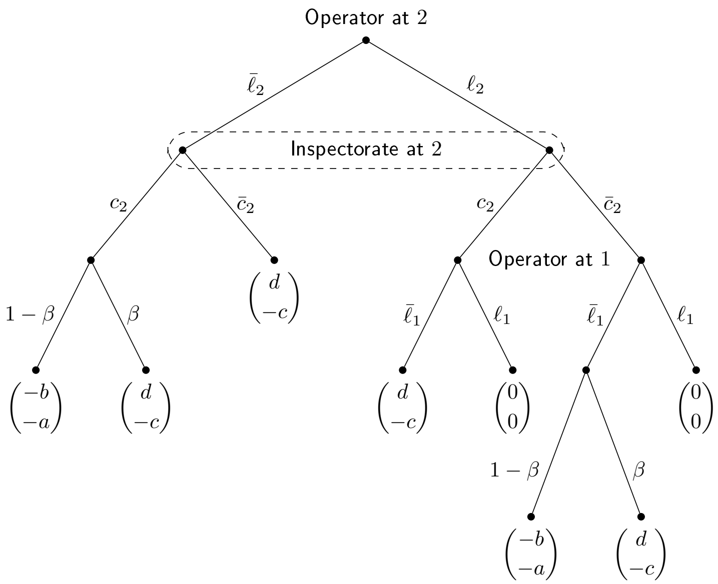
(viii') The payoffs to the two players (Operator, Inspectorate) are given by

$$\begin{array}{rl} (d,-c) & \text{for an untimely control and illegal behaviour, or} \\ & \text{a timely control and no detection of the illegal behaviour} \\ (-b,-a) & \text{for a timely control and detection of the illegal behaviour} \\ (0,0) & \text{for legal behaviour,} \end{array} \quad (16.34)$$

where the parameters satisfy (16.2).

In Figure 16.4 the extensive form of the Dresher-Höpfinger inspection game $\Gamma(2,1)$ with errors of the second kind is represented for the case of $L = 2$ steps and $k = 1$ control. The chance moves are not explicitly named, but can be identified via the probabilities $1-\beta$ and β.

Figure 16.4 Extensive form of the Dresher-Höpfinger inspection game $\Gamma(2,1)$ with errors of the second kind.



Let us compare this new extensive form with that of Figure 16.1: As announced on p. 317 we have omitted now the information sets of the Inspectorate at step 1 and consequently, we have omitted its choices $\bar{c}_1$ and c_1. Instead, a chance move is introduced after a control the outcomes of which lead to different payoffs than before.

Also we see that the Operator decides at step 1 to eventually act illegally even in case that there will be a control since it may not be detected. If we introduce the payoffs[1]

$$-\tilde{b} := -\,b\,(1-\beta) + d\,\beta \qquad \text{and} \qquad -\,\tilde{a} := -a\,(1-\beta) - c\,\beta\,, \tag{16.35}$$

then we see that on the right hand side of the game in Figure 16.4 the Operator will decide for

$$\begin{matrix}\bar{\ell}_1\\ \ell_1\end{matrix} \quad \text{for} \quad -\,\tilde{b} \gtrless 0 \qquad \text{or} \qquad \beta \gtrless \frac{b}{b+d}\,. \tag{16.36}$$

Thus, if we replace in Figure 16.4 the chance moves by the payoffs (16.35) and if we omit the strictly dominated strategy ℓ_1 in the subgame starting after the moves $\ell_2 c_2$ and if we, depending on (16.36), omit the strictly dominated strategy $\bar{\ell}_1$ resp. ℓ_1 in the subgame starting after the moves $\ell_2\bar{c}_2$, then we arrive at a reduced extensive form of this game – which is not presented here – and its normal forms which are given in Table 16.1 and which correspond to that given in Figure 16.3.

Table 16.1 Normal forms of the inspection game given in Figure 16.4. Left: $\beta < b/(b+d)$, Right: $\beta > b/(b+d)$.

	c_2	$\bar{c}_2$
$\bar{\ell}_2$	$-\tilde{a}$; $-\tilde{b}$	$-c$; d
ℓ_2	$-c$; d	0; 0

	c_2	$\bar{c}_2$
$\bar{\ell}_2$	$-\tilde{a}$; $-\tilde{b}$	$-c$; d
ℓ_2	$-c$; d	$-\tilde{a}$; $-\tilde{b}$

The only difference of the left hand normal form in Table 16.1 to that of Figure 16.2 is that we have replaced the payoffs a and b by $\tilde{a}$ and $\tilde{b}$. Using (16.2) and (16.35), we see that $-\tilde{b} < d$ and $-c < -\tilde{a}$. Therefore, both normal forms have a cyclic payoff structure and thus, a unique equilibrium in mixed strategies. Note that after the moves $\ell_2\bar{c}_2$, the Operator decides at step 1 according to (16.36).

The game theoretical solution of this inspection game is presented in

Lemma 16.2. *Given the Se-Se inspection game with $L = 2$ steps, $k = 1$ control, and with errors of the second kind, i.e., $\Gamma(2,1)$. The sets of behavioural strategies are given by (16.3) and the payoffs to both players by (16.4) and (16.5) appropriately modified according to the normal form representations in Table 16.1.*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{2,1} := Op_{2,1}(\bar{p}^*_{2,1}, q^*_{2,1})$ and $In^*_{2,1} := In_{2,1}(\bar{p}^*_{2,1}, q^*_{2,1})$:

(i) For

$$1 > \frac{1}{1-\beta}\,\frac{1}{1+b/d}$$

[1] In Chapter 15 we have introduced the abbreviations A and B in (15.2) which have a similar meaning as $\tilde{a}$ and $\tilde{b}$; in fact we have $\tilde{a} = c - B$ and $\tilde{b} = A - d$. Comparing the payoff matrix in Figure 16.3 to them in Table 16.1, it is mandatory to identify $-b$ resp. $-a$ with $-\tilde{b}$ and $-\tilde{a}$. Thus, the definitions (16.35) are natural for the inspection game discussed in this section. In Chapter 15, however, the notation A and B for easier reference to the original paper is maintained.

the Operator behaves illegally at step 2 *with probability*

$$\bar{p}^*_{2,1} = \frac{c}{2\,c - \tilde{a}}\,,$$

and the Inspectorate controls at step 2 *with probability*

$$q^*_{2,1} = \frac{d}{2\,d + \tilde{b}}\,.$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{2,1} = \frac{d^2}{2\,d + \tilde{b}} \qquad \text{and} \qquad In^*_{2,1} = -\,\frac{c^2}{2\,c - \tilde{a}}\,.$$

(ii) For

$$1 < \frac{1}{1-\beta}\,\frac{1}{1+b/d}$$

the Operator behaves illegally at step 2 *with probability*

$$\bar{p}^*_{2,1} = \frac{1}{2}\,,$$

and the Inspectorate controls at step 2 *with probability*

$$q^*_{2,1} = \frac{1}{2}\,.$$

The equilibrium payoffs to the Operator and to the Inspectorate are

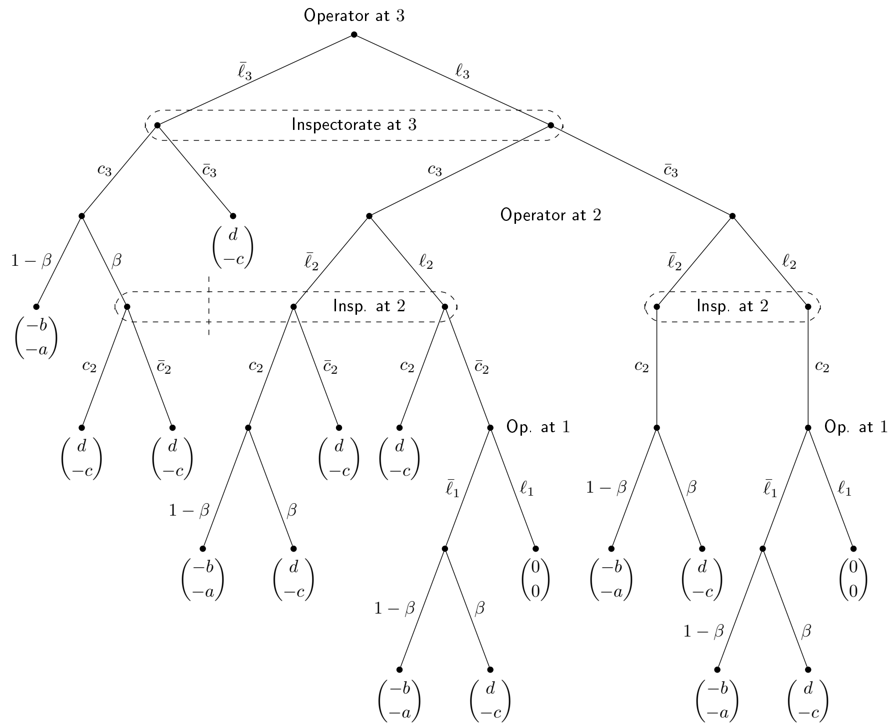
$$Op^*_{2,1} = \frac{1}{2}\,(-\tilde{b} + d) \qquad \text{and} \qquad In^*_{2,1} = \frac{1}{2}\,(-\tilde{a} - c)\,.$$

Proof. Because $-\tilde{b} < d$ and $-c < -\tilde{a}$, both normal forms in Table 16.1 have a cyclic payoff structure and thus, a unique equilibrium in mixed strategies which can be found with the help of the indifference principle; see Theorem 19.1. Alternatively the Nash equilibrium conditions (19.6) can be easily seen. □

Now let us turn to the general Dresher-Höpfinger inspection game with errors of the second kind. It would be tempting – at least for $\beta < b/(b+d)$ – to immediately replace a and b by $\tilde{a}$ and $\tilde{b}$ in the reduced extensive and normal forms as given in Figure 16.3 and modify Theorem 16.1 appropriately. This is however not so easily done since the information structure of the Dresher-Höpfinger inspection game with errors of the second kind and more than one control is more complicated than that with just $k = 1$ control. In order to illustrate this, we consider the game $\Gamma(3,2)$ with $L = 3$ steps and $k = 2$ controls. Its extensive form is given in Figure 16.5.

Figure 16.5 can be explained as follows: First of all, and as announced on p. 317, the information sets of the Inspectorate with just one alternative and the subsequent moves are omitted. Second, and this is new, we see that if the Inspectorate controls at step 3 and in case the illegal activity of the Operator is not detected, the Inspectorate does not know at step 2 if an illegal activity did take place and was not detected, or if it did not yet take place. Thus, the second (left) information set of the Inspectorate includes the node following the moves $\bar{\ell}_3 c_3 \beta$ which then

Figure 16.5 Extensive form of the Dresher-Höpfinger inspection game $\Gamma(3,2)$ with errors of the second kind.

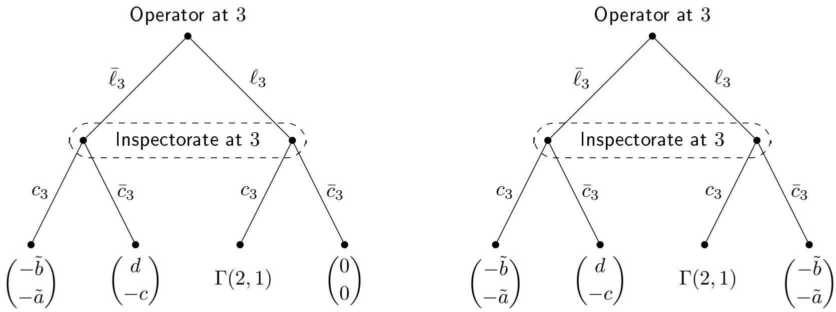


leads to the same payoffs $(d,-c)$ independently of how the Inspectorate decides. This fact permits us to cut the information set in the way indicated in Figure 16.5. Third, following the moves $\ell_3\bar{c}_3\bar{\ell}_2$ we get the payoffs $(-\tilde{b},-\tilde{a})$, whereas following the moves $\ell_3\bar{c}_3\ell_2$ we get either $(-\tilde{b},-\tilde{a})$ or $(0,0)$: Using (16.36), $\bar{\ell}_1$ is strictly dominated by ℓ_1 if and only if $-\tilde{b}<0$ and vice versa. Thus, for $-\tilde{b}>0$ we get for both $\bar{\ell}_2$ and ℓ_2 the payoffs $(-\tilde{b},-\tilde{a})$, whereas for $-\tilde{b}<0$ the decision $\bar{\ell}_2$ is strictly dominated by ℓ_2, and thus, the payoffs are $(0,0)$.

In sum, having cut the second (left) information set of the Inspectorate we can replace the moves following ℓ_3c_3 by the subgame $\Gamma(2,1)$, and the moves following $\ell_3\bar{c}_3$ by the payoffs as described above. This way we arrive at the reduced extensive forms of the Dresher-Höpfinger inspection game $\Gamma(3,2)$ with errors of the second kind which are represented in Figure 16.6.

We see that the recursive extensive form of the game $\Gamma(3,2)$ is for the case $\beta<b/(b+d)$ just a special case of the one given in Figure 16.3 if we replace a and b by $\tilde{a}$ and $\tilde{b}$. Thus the solution of this game is just given by Theorem 16.1 for $L=3$ steps and $k=2$ controls and $\tilde{a}$ and $\tilde{b}$ instead of a and b. Generalizing this procedure to the analysis of the general Dresher-Höpfinger inspection game $\Gamma(L,k)$ with errors of the second kind, we arrive immediately at its game theoretical solution.

Figure 16.6 Recursive extensive form of the inspection game in Figure 16.5. Left: $\beta < b/(b+d)$, Right: $\beta > b/(b+d)$.



The case $\beta > b/(b+d)$ is different. Having cut the information set as described before, we can formulate the general recursive game $\Gamma(L,k)$ in the same way as in Figure 16.3, but now, other than before, with the new boundary conditions for the equilibrium payoffs

$$Op^*_{\ell,k'} = \begin{cases} -\tilde{b} & \text{for} \quad k' = \ell \\ d & \text{for} \quad k' = 0 \end{cases} \qquad \text{and} \qquad In^*_{\ell,k'} = \begin{cases} -\tilde{a} & \text{for} \quad k' = \ell \\ -c & \text{for} \quad k' = 0 \end{cases} \tag{16.37}$$

for any $1 \le \ell \le L$. Note that for both cases the recursive normal form is represented in Table 16.2.

Table 16.2 Recursive normal form of the subgame $\Gamma(\ell,k')$ of the Dresher-Höpfinger inspection game $\Gamma(L,k)$ with errors of the second kind, if ℓ steps and k' controls are left, and the Operator behaves legally at steps $L,\ldots,\ell+1$ ($2 \le \ell \le L$, $1 \le k' \le \min(\ell-1,k)$).

	c_ℓ	$\bar{c}_\ell$
$\bar{\ell}_\ell$	$-\tilde{b}$; $-\tilde{a}$	d ; $-c$
ℓ_ℓ	$Op_{\ell-1,k'-1}$; $In_{\ell-1,k'-1}$	$Op_{\ell-1,k'}$; $In_{\ell-1,k'}$

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Theorem 16.2. *Given the Se-Se inspection game with $L > k$ steps, k controls, and with errors of the second kind, i.e., $\Gamma(L,k)$, the recursive normal form of which is represented in Table 16.2. The payoffs to both players are defined recursively using the recursive normal form representation in Table 16.2, and the equilibrium payoffs to both players fulfil the boundary conditions (16.9) in case (i) and (16.37) in case (ii).*

(i) For

$$1 > \frac{1}{1-\beta}\,\frac{1}{1+b/d} \tag{16.38}$$

a Nash equilibrium and the corresponding payoffs are given by Theorem 16.1 in which a and b are replaced by $\tilde{a}$ and $\tilde{b}$ as defined by (16.35).

(ii) For

$$1 < \frac{1}{1-\beta}\,\frac{1}{1+b/d} \tag{16.39}$$

and in case ℓ steps, $2 \leq \ell \leq L$, and k' controls, $1 \leq k' \leq \min(\ell-1, k)$, are left, and the Operator behaves legally at steps $L, \ldots, \ell+1$, i.e., the subgame $\Gamma(\ell, k')$ is reached, a Nash equilibrium in the subgame $\Gamma(\ell, k')$ is given by the following equilibrium strategies and payoffs $Op^*_{\ell,k'}$ and $In^*_{\ell,k'}$:

The Operator behaves illegally at step ℓ with probability

$$\bar{p}^*_{\ell,k'} = \frac{1}{\ell}\,, \tag{16.40}$$

and the Inspectorate controls at step ℓ with probability

$$q^*_{\ell,k'} = \frac{k'}{\ell}\,. \tag{16.41}$$

The equilibrium payoffs to the Operator and to the Inspectorate in the subgame $\Gamma(\ell, k')$ are

$$Op^*_{\ell,k'} = d - (1-\beta)\,(b+d)\,\frac{k'}{\ell} \qquad \text{and} \qquad In^*_{\ell,k'} = -c + (1-\beta)\,(c-a)\,\frac{k'}{\ell}\,, \tag{16.42}$$

which – for $\ell = L$ and $k' = k$ – are the equilibrium payoffs of the entire game $\Gamma(L, k)$.

Proof. As already indicated we need only to prove (ii). We proceed as in the proof of Theorem 16.1.

1. It is obvious that (16.40) and (16.41) constitute probabilities.

2. We will see in step 4 that the preference directions of the bimatrix in Table 16.2 are cyclic. This means that there exists a unique equilibrium in mixed strategies. It is determined by the following recursive relations which we have written for convenience in a way slightly different from that in the proof of Theorem 16.1, see (16.22) and (16.23):

$$Op^*_{\ell,k'} = d - (\tilde{b}+d)\,\frac{d - Op^*_{\ell-1,k'}}{Op^*_{\ell-1,k'-1} - Op^*_{\ell-1,k'} + \tilde{b} + d} \tag{16.43}$$

and

$$In^*_{\ell,k'} = -c + (c-\tilde{a})\,\frac{c + In^*_{\ell-1,k'}}{In^*_{\ell-1,k'} - In^*_{\ell-1,k'-1} + c - \tilde{a}}\,. \tag{16.44}$$

Also we get, using (16.20) and (16.21),

$$q^*_{\ell,k'} = \frac{d - Op^*_{\ell-1,k'}}{Op^*_{\ell-1,k'-1} - Op^*_{\ell-1,k'} + \tilde{b} + d} \quad \text{and} \quad \bar{p}^*_{\ell,k'} = \frac{In^*_{\ell-1,k'} - In^*_{\ell-1,k'-1}}{In^*_{\ell-1,k'} - In^*_{\ell-1,k'-1} + c - \tilde{a}}\,,$$

which simplify by (16.42) to (16.40) and (16.41).

3. We see that (16.42) satisfies both the boundary condition (16.37) and the recursive relation (16.43): In fact, the right hand side of (16.43) is, using the left hand side of (16.42) and (16.35),

$$d - (\tilde{b} + d)\,\frac{(1-\beta)\,\dfrac{k'}{\ell-1}}{(1-\beta) - (1-\beta)\left(\dfrac{k'-1}{\ell-1} - \dfrac{k'}{\ell-1}\right)} = d - (\tilde{b}+d)\,\frac{k'}{\ell}$$

$$= d - (1-\beta)\,(b+d)\,\frac{k'}{\ell} = Op^*_{\ell,k'}\,,$$

which is the left hand side of (16.43). Similarly, the right hand side of (16.42) fulfils (16.44) and the boundary condition (16.37):

$$-c + (c - \tilde{a})\,\frac{\dfrac{k'}{\ell-1}}{\dfrac{k'}{\ell-1} - \dfrac{k'-1}{\ell-1} + 1} = -c + (c-\tilde{a})\,\frac{k'}{\ell} = In^*_{\ell,k'}\,.$$

4. We show that the preference directions of the bimatrix in Table 16.2 are cyclic: From (16.2) and (16.35) we get $-c < -\tilde{a}$. Because $\tilde{b} + d = (1-\beta)\,(b+d) > 0$ we have by (16.42) that $Op^*_{\ell-1,k} < d$ and, because $k' < \ell$,

$$Op^*_{\ell-1,k-1} = d - (\tilde{b}+d)\,\frac{k'-1}{\ell-1} > d - (b+d)\,(1-\beta) = -\tilde{b}\,.$$

From (16.42) we also get

$$In^*_{\ell-1,k-1} < In^*_{\ell-1,k}\,,$$

which completes the proof. □

Let us comment the results of Theorem 16.2: First, and most importantly, it is surprising that a slight change of the parameter β from (16.38) to (16.39) changes the type of the solution so fundamentally, but this can be understood as follows: This change is equivalent to a change of $-b\,(1-\beta) + d\,\beta$ from larger to smaller than zero, i.e., it is equivalent to a change of the expected payoff to the Operator in case of a timely inspection from larger to smaller than zero. Whereas in the first case the solution is still of the Dresher-Höpfinger type solution, in the second case it is totally different and much simpler. In particular, the equilibrium strategies of both players do not depend on the payoff parameters a, b, c and d. In fact, the structure of the equilibrium strategy of the Operator is the same as that for the Se-No and Se-Se inspection game in Part I and for the Se-No and Se-Se inspection game with $\beta = 0$ resp. $\alpha = \beta = 0$ in Part II. Thus, this structure shows a universal character. We will return to this second type of solution in part (i) of Theorem 17.1 and in Theorem 17.2.

Although the equilibrium payoffs are so different under (i) and (ii), they coincide – as expected – for β to $b/(b+d)$ from the left and the right hand side. We show this statement only for the Operator's equilibrium payoff: For (i) we get by (16.35), that $\tilde{b} = 0$ for $\beta = b/(b+d)$. Thus, we obtain from (16.10) with $\tilde{b}$ instead of b

$$f(L,k) = \sum_{i=0}^{k} \binom{L}{i} \left(\frac{\tilde{b}}{d}\right)^{k-i} = \binom{L}{k}$$

and, using (16.13),

$$Op^*_{L,k} = d\,\frac{\binom{L-1}{k}}{f(L,k)} = d\,\frac{\binom{L-1}{k}}{\binom{L}{k}} = d\left(1 - \frac{k}{L}\right).$$

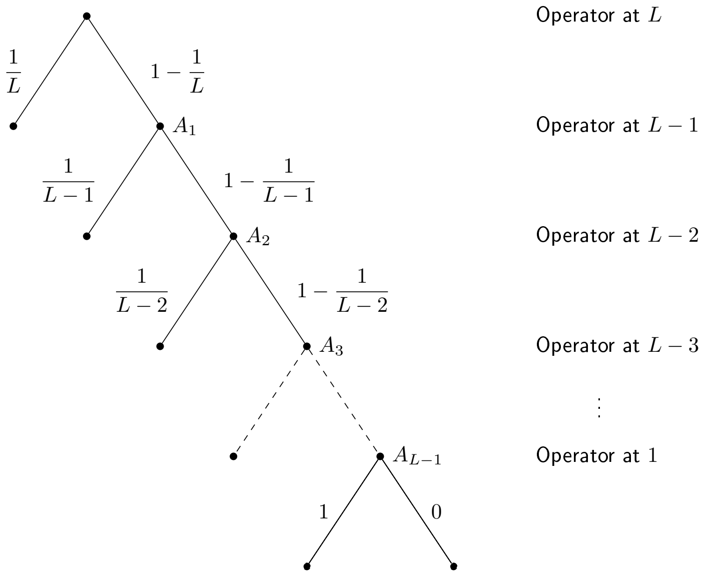
Thus, (16.42) yields for $\beta = b/(b+d)$

$$Op^*_{L,k} = d - (1-\beta)\,(b+d)\,\frac{k}{L} = d\left(1 - \frac{k}{L}\right),$$

i.e., both expressions coincide.

Second, because (16.38) is equivalent to $\beta < b/(b+d)$, the Operator behaves – according to case (i) – legally with positive probability for small β, whereas in case (ii) he behaves illegally with certainty because of the decision $\bar{\ell}_1$ in (16.36). Figure 16.7 illustrates this property explicitly.

Figure 16.7 Illustration of $\mathbb{P}$(behaving illegally) $= 1$ for case (ii) of Theorem 16.2.



The property $\mathbb{P}$(behaving illegally) $= 1$ can be shown as follows: Let A_i be the event that the Operator does not behave illegally before step $L-i$, $i = L-1, \ldots, 1$. Using Figure 16.7, we get for $i = L-1, \dots, 1$

$$\mathbb{P}(A_i) = \prod_{j=1}^{i} \left(1 - \frac{1}{L-(j-1)}\right) = \frac{L-i}{L}$$

and therewith

$$\mathbb{P}(\text{behaving illegally}) = \frac{1}{L} + \sum_{i=1}^{L-1} \frac{1}{L-i}\, \mathbb{P}(A_i) = \frac{1}{L} + \frac{L-1}{L} = 1\,.$$

Therefore, because there exists a probability distribution for the step at which the Operator behaves illegally, we can determine its expected step:

$$\begin{aligned}\mathbb{E}_{\mathbf{p}^*}(S) &= 1\,\frac{1}{L} + 2\left(\frac{L-1}{L}\,\frac{1}{L-1}\right) + 3\left(\frac{L-1}{L}\,\frac{L-2}{L-1}\,\frac{1}{L-2}\right) + \dots + L\left(\frac{L-1}{L}\,\dots\,1\right)\\ &= \frac{1}{L}\sum_{i=1}^{L} i = \frac{L+1}{2}\,,\end{aligned}$$

which, after all, is not surprising.

Third, in case (ii) the equilibrium strategies of both players do neither depend on the payoff parameters nor on the detection probability $1-\beta$ which again is surprising and which makes this solution attractive for practitioners. Note that the Operator's probability $\bar{p}^*_{\ell,k'}$ as given by (16.40) only depends on the number of steps left, and thus, they form a harmonic progression; see also Table 4.1 on p. 72 for an overview of inspection games with this property.

Finally, the equilibrium payoffs to both players in case (ii) of Theorem 16.2 are the same as that in case (i) of Corollary 15.1; see p. 300. Also the equilibrium strategies can be transformed into each other: The probability that the Operator behaves illegally at step ℓ, $\ell = L, \dots, 1$, is, using (16.40) or Figure 16.7, given by $1/L$ which is p^*_i given by (15.76) in Corollary 15.1. For Inspectorate's probability to control at steps $L, L-1$ and $L-2$ respectively we get for $k \geq 3$:

$$q^*_{L,k} = \frac{k}{L} = q^*_{L-1}\,,$$

$$\frac{k}{L}\,\frac{k-1}{L-1} + \left(1-\frac{k}{L}\right)\,\frac{k}{L-1} = \frac{k}{L} = q^*_{L-2}\,,$$

$$\begin{aligned}&\left(1-\frac{k}{L}\right)\left[\left(1-\frac{k}{L-1}\right)\frac{k}{L-2} + \frac{k}{L-1}\,\frac{k-1}{L-2}\right]\\ &\qquad + \frac{k}{L}\left[\left(1-\frac{k-1}{L-1}\right)\frac{k-1}{L-2} + \frac{k-1}{L-1}\,\frac{k-2}{L-2}\right] = \frac{k}{L} = q^*_{L-3}\,,\end{aligned}$$

where the $q^*_\cdot$ on the right hand side are the probabilities (15.77). It can be conjectured that for all remaining steps this equivalence holds as well. Note however, that although the equilibrium payoffs to both players coincide and the equilibrium strategies can be transformed into each other, the conditions (15.75) and (16.38) are not identical. Nevertheless, this result is very surprising and remarkable since it has no equivalent in Parts I and II.

16.3 Any number of steps and one control; errors of the first and second kind

Consider a last time inspections where variable sampling procedures are used which means that errors of the first and second kind may occur. Thus, the inspection game analysed in this section is based on the specification:

(iv') During a control the Inspectorate may commit errors of the first and second kind with probabilities α and β. These error probabilities are the same for all controls. Only controls which are performed before an illegal activity may incur false alarm costs.

Assumption (iv') is justified on p. 282. Because false alarms have not yet been considered in this chapter, the payoffs to the Operator and Inspectorate as given by (16.34) need to be extended to the payoffs (14.1) with (14.2) as to include the payoffs in case of legal behaviour of the Operator and a false alarm.

Throughout this section we use again the quantities A and B that are defined by (15.2), i.e.,

$$A = (b + d)\,(1 - \beta) > 0 \qquad \text{and} \qquad B = (c - a)\,(1 - \beta) > 0\,.$$

Also we assume again that the test procedure is unbiased, i.e., $\alpha + \beta < 1$, which implies $A - f\,\alpha > 0$; see also (15.4).

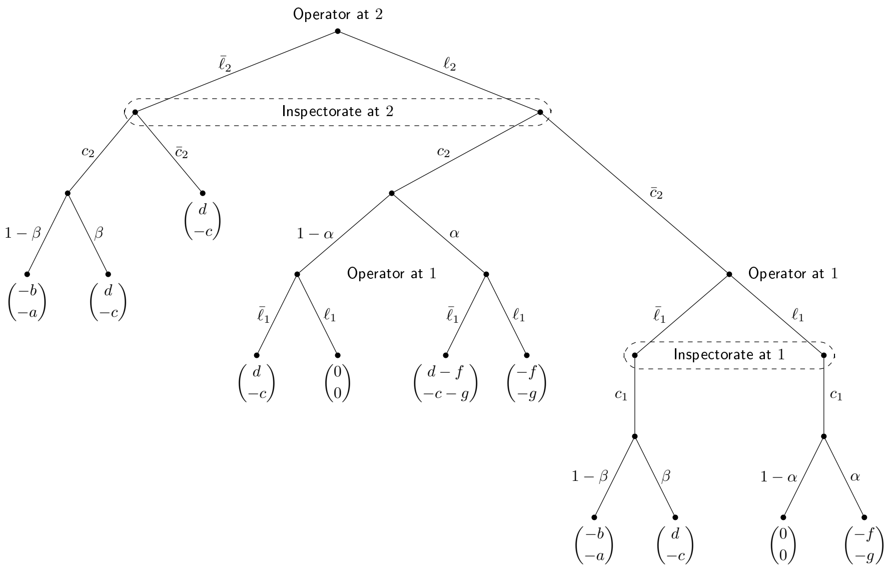
As in Section 16.2 we start with the most simple case of the Se-Se inspection game with errors of the first and second kind: $L = 2$ steps and $k = 1$ control, the extensive form of which is shown in Figure 16.8. Again, the chance moves in Figures 16.8 and 16.10 are not explicitly named, but can be identified via the probabilities $1 - \beta$ and β as well as $1 - \alpha$ and α.

Let us describe Figure 16.8 in some detail and compare it with Figure 16.1. Due to the more complicated structure we have reduced the tree wherever possible.

At step 2, i.e., at the top of the tree, the Operator decides to behave illegally immediately ($\bar{\ell}_2$) or not (ℓ_2). At step 1 he has to behave legally if he behaves illegally at step 2 (left branch), otherwise (right branch) he decides to behave illegally at step 1 ($\bar{\ell}_1$) or not (ℓ_1).

Also at step 2, the Inspectorate decides, not knowing the Operator's decision, to control (c_2) or not ($\bar{c}_2$). At step 1 it cannot control any more if it does so at step 2 (c_2), and it has to control if it does not at step 2 ($\bar{c}_2$). In that case it does not know if the Operator behaves illegally at step 1 ($\bar{\ell}_1$) or not (ℓ_1); in fact it also does not know whether the Operator behaved illegally already at step 2. Thus, formally the three nodes reached after the moves $\bar{\ell}_2\bar{c}_2$, $\ell_2\bar{c}_2\bar{\ell}_1$ and $\ell_2\bar{c}_2\ell_1$, and which are all followed by c_1 should be in one information set. This situation is not displayed in Figure 16.8, but see Figure 16.1 which indicates the overlapping information set. If the Inspectorate controls, then a chance move has to be considered: In case the Operator behaves illegally, this will be detected with probability $1 - \beta$ or not with probability β. In case the Operator behaves legally, a false alarm will be raised with probability α or not with probability $1 - \alpha$. A subtle modelling aspect, which already occurred in the No-No inspection game, see assumption (iv') and p. 282, has to be highlighted: If the Operator behaves illegally at step 2 and the Inspectorate controls at step 1, formally a false alarm may be raised at step 1. We ignore this possibility since in this case we consider the game to be finished with step 2, see assumption (x) of Chapter 14: The Operator has successfully completed the illegal activity.

Figure 16.8 Extensive form of the Dresher-Höpfinger inspection game $\Gamma(2,1)$ with errors of the first and second kind.



The payoffs to the two players are given at the end nodes of the tree and are, using (14.2), self-explaining. Like in all games of this chapter, the Operator does not necessarily behave illegally.

If both players wait till step 1, then the Operator will behave illegally at step 1 if $d-A > -f\,\alpha$, i.e., if his expected gain for behaving illegally is larger than that for legal behaviour. Thus, depending on $d-A \gtrless -f\,\alpha$, the reduced game can be represented in normal form as shown in Table 16.3.

Table 16.3 Normal forms of the inspection game given in Figure 16.8. Left: $d-A < -f\,\alpha$. Right: $d-A > -f\,\alpha$.

	c_2	$\bar{c}_2$
$\bar{\ell}_2$	$B-c$ $d-A$	$-c$ d
ℓ_2	$-c-g\,\alpha$ $d-f\,\alpha$	$-g\,\alpha$ $-f\,\alpha$

	c_2	$\bar{c}_2$
$\bar{\ell}_2$	$B-c$ $d-A$	$-c$ d
ℓ_2	$-c-g\,\alpha$ $d-f\,\alpha$	$B-c$ $d-A$

The normal form in case of $d-A > -f\,\alpha$ is identical to that of the No-No inspection game in Table 15.1; the pure strategy 1 dominates the legal strategy. Note that conceptually, a timely detection of an illegal activity can only happen in the No-No inspection game when an inspection is performed at the time point after the illegal activity, while in the Se-Se inspection game the control has to happen at the same step. Thus, if the critical time t, see p. 275, is set to zero, the strategies of both inspection games can be compared: The numbers 1 and 0 in the first row of the normal form in Table 15.1 would then be shifted to 2 and 1, respectively.

As in Section 16.1, let $\bar{p}_{2,1}$ denotes the Operator's probability to behave illegally at step 2 ($\bar{\ell}_2$). For the Inspectorate, let $q_{2,1}$ be the probability to control at step 2 (c_2). With the same argumentation as on p. 317 we reach – in case of legal behaviour at step 2 (ℓ_2) – the subgame $\Gamma(1,1)$ or $\Gamma(1,0)$ at step 1, and the respective equilibrium strategies are given by

$$\bar{p}^*_{1,1} = \begin{cases} 1 & \text{for} \quad d - A > -f\,\alpha \\ 0 & \text{for} \quad d - A < -f\,\alpha \end{cases} \qquad \text{and} \qquad \begin{aligned} \bar{p}^*_{1,0} &= 1 \\ q^*_{1,0} &= 0\,. \end{aligned} \tag{16.45}$$
$$q^*_{1,1} = 1$$

Again, as on p. 317, we exclude the probabilities $\bar{p}_{1,1}, \bar{p}_{1,0}$ and $q_{1,1}, q_{1,0}$ from the sets of behavioural strategies (16.3). The (expected) payoffs two both players are given by (16.4) and (16.5) using the bimatrices in Table 16.3.

The game theoretical solution of this inspection game, see Canty et al. (2001), is presented in

Lemma 16.3. *Given the Se-Se inspection game with $L = 2$ steps, $k = 1$ control, errors of the first and second kind, and an unbiased test procedure, i.e., $\Gamma(2,1)$, the extensive and the normal forms of which are represented in Figure 16.8 and Table 16.3. The sets of behavioural strategies are given by (16.3), and the payoffs to both players by (16.4) and (16.5) appropriately modified according to the normal form representations in Table 16.3.*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{2,1} := Op_{2,1}(\bar{p}^*_{2,1}, q^*_{2,1})$ and $In^*_{2,1} := In_{2,1}(\bar{p}^*_{2,1}, q^*_{2,1})$:

(i) For

$$d - A < -f\,\alpha$$

the Operator behaves illegally at step 2 with probability

$$\bar{p}^*_{2,1} = \frac{c}{c+B}\,, \tag{16.46}$$

and the Inspectorate controls at step 2 with probability

$$q^*_{2,1} = \frac{d + f\,\alpha}{d + A}\,. \tag{16.47}$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{2,1} = d - A\,\frac{d + f\,\alpha}{d + A} < d \qquad \text{and} \qquad In^*_{2,1} = -c + B\,\frac{c - g\,\alpha}{c + B} > -c\,. \tag{16.48}$$

(ii) For

$$d - A > -f\,\alpha$$

the Operator behaves illegally at step 2 with probability

$$\bar{p}^*_{2,1} = \frac{B + g\,\alpha}{2\,B + g\,\alpha}\,, \tag{16.49}$$

and the Inspectorate controls at step 2 with probability

$$q^*_{2,1} = \frac{A}{2\,A - f\,\alpha}\,. \qquad (16.50)$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{2,1} = d - \frac{A^2}{2\,A - f\,\alpha} < d \qquad \text{and} \qquad In^*_{2,1} = -c + \frac{B^2}{2\,B + g\,\alpha} > -c\,. \qquad (16.51)$$

Proof. In both bimatrices in Table 16.3 the preference directions are cyclic which means, that there exists a unique Nash equilibrium in mixed strategies which can be found with the help of the indifference principle; see Theorem 19.1. □

Let us comment the results of Lemma 16.3: First, because the game $\Gamma(2,1)$ in this section is a generalization of the corresponding game with only errors of the second kind, see Section 16.2, the results of Lemma 16.3 coincide in case of $\alpha = 0$ with the results of Lemma 16.2.

Second, (16.45) implies that in case $d - A > -f\,\alpha$ the Operator will behave illegally with certainty because we have $\bar{p}^*_{2,1} + (1 - \bar{p}^*_{2,1})\,\bar{p}^*_{1,1} = 1$, while in case $d - A < -f\,\alpha$ he behaves illegally with probability $\bar{p}^*_{2,1} \in (0,1)$. Thus, the Operator cannot be induced to behaving legally with certainty.

Third, it has been mentioned on p. 338 that in case of $d - A > -f\,\alpha$ the normal form of the No-No inspection game is equivalent to the normal form of the Se-Se inspection game. Thus, the equilibrium strategies and payoffs should coincide. Indeed, if $d - A > -f\,\alpha$, then condition (15.9) is fulfilled:

$$\frac{(A - f\,\alpha)^2}{2\,A - f\,\alpha} = (A - f\,\alpha)\,\frac{A - f\,\alpha}{2\,A - f\,\alpha} < (A - f\,\alpha) < d\,.$$

Comparing (15.10) with (16.49) and (15.11) with (16.50) we get

$$p^*_2 = \bar{p}^*_{2,1}\,, \quad p^*_1 = (1 - \bar{p}^*_{2,1})\,\bar{p}^*_{1,1} = 1 - \bar{p}^*_{2,1}$$

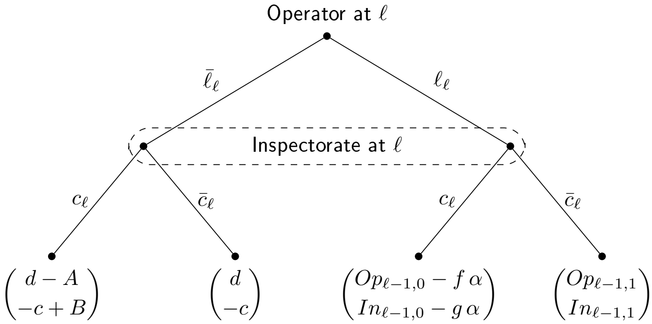
$$q^*_1 = q^*_{2,1}\,, \quad q^*_0 = 1 - q^*_{2,1}\,,$$

where the results of Lemma 15.1 are displayed on the left hand side of the equal sign. The equilibrium payoffs are obviously the same. This surprising result may be understood as follows: $d - A > -f\,\alpha$ implies that the Operator's (expected) payoff in case of a timely detection is larger than for legal behaviour. Therefore, he will behave illegally regardless of the Inspectorate's behaviour. Thus, given that he behaved legally at step 2, knowing the Inspectorate's decision at that step (c_2 or $\bar{c}_2$) does not add any valuable information to the Operator. So, he could just pick a step for the illegal activity right before the game starts.

In case $d - A < -f\,\alpha$ and condition (15.9) is fulfilled, the Operator has an advantage in the Se-Se inspection game, because $d - A < -f\,\alpha$ is equivalent to the fact that $Op^*_{2,1}$ given by (15.12) is smaller than that given by (16.48).

Let us now consider the Dresher-Höpfinger inspection game with errors of the first and second kind for any number L of steps but $k = 1$ control, i.e., $\Gamma(L,1)$. The recursive form of this inspection game is presented in Figure 16.9 both in extensive and in normal form, where we have already presented the subgame which is reached if the Operator behaves legally at steps

Figure 16.9 Recursive extensive form and corresponding recursive normal form of the subgame $\Gamma(\ell, 1)$ of the Dresher-Höpfinger inspection game $\Gamma(L, 1)$ with errors of the first and second kind, if ℓ steps and 1 control are left, and the Operator behaves legally at steps $L, \ldots, \ell + 1$ $(2 \leq \ell \leq L)$.



	c_ℓ	$\bar{c}_\ell$
$\bar{\ell}_\ell$	$B - c$ $d - A$	$-c$ d
ℓ_ℓ	$In_{\ell-1,0} - g\,\alpha$ $Op_{\ell-1,0} - f\,\alpha$	$In_{\ell-1,1}$ $Op_{\ell-1,1}$

$L, \ldots, \ell + 1$ (for $\ell < L$), and ℓ steps as well as one control are left ($2 \leq \ell \leq L$). The payoffs $Op_{\ell-1,\cdot}$ and $In_{\ell-1,\cdot}$ are explained below.

Let $\bar{p}_{L,1}$ denote the probability to behave illegally at step L ($\bar{\ell}_L$), and let $q_{L,1}$ be the probability to control at step L (c_L). Suppose the game has reached step ℓ with $2 \leq \ell < L$ and the Inspectorate has still the only control at its disposal. Then $\bar{p}_{\ell,1}$ denotes the probability to behave illegally at step ℓ if the Operator does not do so before ($\bar{\ell}_\ell$) and $q_{\ell,1}$ denotes the probability to control at step ℓ (c_ℓ). In case of $\ell = 1$, the equilibrium strategies are again given by (16.45). Again, the sets of behavioural strategies are omitted; see the comment on p. 321.

The (expected) payoffs to both players is defined recursively using the normal form representation in Figure 16.9, where $Op_{\ell-1,\cdot}$ and $In_{\ell-1,\cdot}$ denote the (expected) payoffs to both players in the subgame with $\ell - 1$ steps and 1 or 0 controls left. The equilibrium payoffs need to fulfil the boundary conditions

$$\begin{aligned} &Op^*_{2,1} \quad \text{and} \quad In^*_{2,1} \quad \text{according to} \quad (16.48) \quad \text{or} \quad (16.51) \\ &Op^*_{\ell,0} = d \qquad \text{and} \qquad In^*_{\ell,0} = -c \qquad \text{for any} \qquad 1 \leq \ell \leq L\,. \end{aligned} \tag{16.52}$$

The game theoretical solution of this inspection game, see Canty et al. (2001), is presented in

Theorem 16.3. *Given the Se-Se inspection game with $L \geq 3$ steps, $k = 1$ control, errors of the first and second kind, and an unbiased test procedure, i.e., $\Gamma(L, 1)$, the recursive extensive and the recursive normal forms of which are represented in Figure 16.9. The payoffs to both* *players are defined recursively using the recursive normal form representation in Figure 16.9, and the equilibrium payoffs to both players fulfil the boundary conditions (16.52).*

Suppose ℓ steps, $3 \leq \ell \leq L$, are left and the Inspectorate has still the only control at its disposal, i.e., the subgame $\Gamma(\ell, 1)$ is reached.

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{\ell,1}$ and $In^*_{\ell,1}$:

The Operator behaves illegally at step ℓ with probability

$$\bar{p}^*_{\ell,1} = \frac{In^*_{\ell-1,1} + c + g\,\alpha}{In^*_{\ell-1,1} + c + g\,\alpha + B}\,, \tag{16.53}$$

and the Inspectorate controls at step ℓ with probability

$$q^*_{\ell,1} = \frac{d - Op^*_{\ell-1,1}}{d - f\,\alpha - Op^*_{\ell-1,1} + A}\,. \tag{16.54}$$

The equilibrium payoffs to the Operator and to the Inspectorate in the subgame $\Gamma(\ell, 1)$ are, using (16.48) or (16.51) depending on the respective case,

$$Op^*_{\ell,1} = d - \frac{f\,\alpha}{1 - \left(1 - \dfrac{f\,\alpha}{d - Op^*_{2,1}}\right)\left(1 - \dfrac{f\,\alpha}{A}\right)^{\ell-2}} \tag{16.55}$$

and

$$In^*_{\ell,1} = -c - \frac{g\,\alpha}{1 - \left(1 + \dfrac{g\,\alpha}{c + In^*_{2,1}}\right)\left(1 + \dfrac{g\,\alpha}{B}\right)^{\ell-2}}\,, \tag{16.56}$$

which – for $\ell = L$ – are the equilibrium payoffs of the entire game $\Gamma(L, 1)$.

Proof. It is obvious that $\bar{p}^*_{\ell,1} \in [0, 1]$ and, because of (15.4), also $q^*_{\ell,1} \in [0, 1]$.

For any ℓ with $2 \leq \ell \leq L$ we consider the recursive normal form representation of the subgame $\Gamma(\ell, 1)$ in Figure 16.9. The preference directions of the payoffs are cyclic again, which will be shown at the end of the proof and is assumed to be given at present. Because of this cyclic structure there exists a unique Nash equilibrium in mixed strategies which is determined, using (16.52), with the help of the indifference principle, see Theorem 19.1:

$$Op^*_{\ell,1} = q^*_{\ell,1}\,(d - A) + (1 - q^*_{\ell,1})\,d = q^*_{\ell,1}\,(d - f\,\alpha) + (1 - q^*_{\ell,1})\,Op^*_{\ell-1,1} \tag{16.57}$$

and

$$In^*_{\ell,1} = \bar{p}^*_{\ell,1}\,(B - c) + (1 - \bar{p}^*_{\ell,1})\,(-c - g\,\alpha) = -\bar{p}^*_{\ell,1}\,c + (1 - \bar{p}^*_{\ell,1})\,In^*_{\ell-1,1}\,, \tag{16.58}$$

which leads to (16.53) and (16.54). Inserting (16.53) into (16.58) and (16.54) into (16.57), we obtain the following recursive relations for $Op^*_{\ell,1}$ and $In^*_{\ell,1}$:

$$Op^*_{\ell,1} = d - A\,q^*_{\ell,1} = d - A\,\frac{d - Op^*_{\ell-1,1}}{d - f\,\alpha - Op^*_{\ell-1,1} + A} \tag{16.59}$$

and

$$In^*_{\ell,1} = \frac{-c\,(c+g\,\alpha)+(B-c)\,In^*_{\ell-1,1}}{In^*_{\ell-1,1}+c+g\,\alpha+B} = -c+B\,\frac{c+In^*_{\ell-1,1}}{In^*_{\ell-1,1}+c+g\,\alpha+B}\,. \qquad (16.60)$$

With some elementary manipulations it can be shown that (16.55) and (16.56) fulfil (16.59) and (16.60), respectively.

To prove that the preference direction are cyclic we start with the case $\ell = 3$: (16.48) and (16.51) both imply $Op^*_{2,1} < d$ and $In^*_{2,1} > -c$. Furthermore, we have $B - c > -c$, and, using (15.4), $d - A < d - f\,\alpha$. Thus, the preference directions are cyclic for $\ell = 3$. The proof is now conducted by induction using the fact that $Op^*_{\ell,1}$ as given by (16.59) is smaller than d (induction hypothesis is $Op^*_{\ell-1,1} < d$), and $In^*_{\ell,1}$ as given by (16.60) larger than $-c$ (induction hypothesis is $In^*_{\ell-1,1} > -c$), which completes the proof. □

Let us comment the results of Theorem 16.3: First, the distinction of the cases $d - A \gtrless -f\,\alpha$ which can be expected from Theorem 16.2 and Lemma 16.3 enter Theorem 16.3 via the boundary conditions (16.48) or (16.51) that are used in (16.55) and (16.56). Note for ease notation we have presented the equilibrium strategies as a function of the equilibrium payoffs. For further discussion we now derive the explicit expressions as well as their limiting behaviour for $\alpha \to 0$. Let x and y be defined as in (15.28), i.e.,

$$x = 1 - \frac{f\,\alpha}{A} \quad (\in (0,1)) \qquad \text{and} \qquad y = 1 + \frac{g\,\alpha}{B} \quad (> 1)\,.$$

We obtain in case of $d - A < -f\,\alpha$ for (16.55) and (16.56), using (16.48), the expressions

$$\begin{aligned} Op^*_{\ell,1} &= d - A\,\frac{1-x}{1-\dfrac{d}{d+A\,(1-x)}\,x^{\ell-1}} \quad \text{and} \\ In^*_{\ell,1} &= -c + B\,\frac{y-1}{\dfrac{c}{c-(y-1)\,B}\,y^{\ell-1}-1}\,, \end{aligned} \qquad (16.61)$$

which simplify for $\alpha \to 0$, or equivalently $x \to 1$ and $y \to 1$, using L'Hospital's rule to

$$Op^*_{L,1} = d - A\,\frac{d}{A+d\,(L-1)} \qquad \text{and} \qquad In^*_{L,1} = -c + B\,\frac{c}{B+c\,(L-1)}\,. \qquad (16.62)$$

Furthermore, (16.62) implies, using (16.53) and (16.54), for $\alpha \to 0$

$$\bar{p}^*_{\ell,1} = \frac{c}{B+c\,(\ell-1)} \qquad \text{and} \qquad q^*_{\ell,1} = \frac{d}{A+d\,(\ell-1)}\,. \qquad (16.63)$$

In case of $d - A > -f\,\alpha$, we obtain for (16.55) and (16.56), using (16.51), the expressions

$$Op^*_{\ell,1} = d - A\,\frac{1-x}{1-x^{\ell}} \qquad \text{and} \qquad In^*_{\ell,1} = -c + B\,\frac{y-1}{y^{\ell}-1}\,, \qquad (16.64)$$

which lead, using (16.53) and (16.54), to the explicit forms of the equilibrium strategies

$$\bar{p}^*_{\ell,1} = \frac{y-1}{y^{\ell}-1}\,y^{\ell-1} \qquad \text{and} \qquad q^*_{\ell,1} = \frac{1-x}{1-x^{\ell}}\,. \qquad (16.65)$$

(16.64) and (16.65) simplify for $\alpha \to 0$, which is again equivalent to $x \to 1$ and $y \to 1$, using L'Hospital's rule to

$$\bar{p}^*_{\ell,1} = q^*_{\ell,1} = \frac{1}{\ell} \quad \text{and} \quad Op^*_{L,1} = d - \frac{A}{L} \quad \text{and} \quad In^*_{L,1} = -c + \frac{B}{L}\,. \tag{16.66}$$

Second, for $\alpha = \beta = 0$ the solution of Theorem 16.3 coincides with those of Theorem 16.1 for $k = 1$ control: (16.10) implies with $k' = 1$

$$f(L,1) = \sum_{i=0}^{1} \binom{\ell}{i} \left(\frac{b}{d}\right)^{1-i} = \frac{b}{d} + \ell \quad \text{and} \quad g(\ell,1) = \sum_{i=0}^{1} \binom{\ell}{i} \left(-\frac{a}{c}\right)^{1-i} = -\frac{a}{c} + \ell$$

and thus, we get for (16.11), (16.12) and (16.13), using $g(\ell-1,0) = 1$ and $f(\ell-1,0) = 1$,

$$\bar{p}^*_{\ell,1} = \frac{c}{-a + c\,\ell} \quad \text{and} \quad q^*_{\ell,1} = \frac{d}{b + d\,\ell}\,, \tag{16.67}$$

with the equilibrium payoffs

$$Op^*_{L,1} = d\,\frac{\binom{L-1}{1}}{f(L,1)} = d^2\,\frac{L-1}{b + d\,L} \quad \text{and} \quad In^*_{L,1} = -c\,\frac{\binom{L-1}{1}}{g(L,1)} = -c^2\,\frac{L-1}{-a + c\,L}\,. \tag{16.68}$$

Because $\alpha = \beta = 0$ implies $d - A = -b < 0$, we see, using the definition of A and B, that (16.67) and (16.68) coincide with (16.63) and (16.62). Thus, the game theoretical solution of the Se-Se inspection game discussed in this section is for $\alpha = \beta = 0$ the same as that for Theorem 16.1 and $k = 1$ control, as expected.

Third, for $\alpha = 0$ but $\beta > 0$ we see that the condition $d - A < 0$ is equivalent to $\beta < b/(b+d)$, i.e., case (i) in Theorem 16.2 is valid. Then, using (16.67) and (16.68) with $\tilde{a}$ and $\tilde{b}$ defined by (16.35) instead of a and b, respectively, we get

$$\bar{p}^*_{\ell,1} = \frac{c}{-(a\,(1-\beta) + c\,\beta) + c\,\ell} \quad \text{and} \quad q^*_{\ell,1} = \frac{d}{(b\,(1-\beta) - d\,\beta) + d\,\ell}\,, \tag{16.69}$$

with the equilibrium payoffs

$$Op^*_{L,1} = d^2\,\frac{L-1}{(b\,(1-\beta) - d\,\beta) + d\,L} \quad \text{and} \quad In^*_{L,1} = -c^2\,\frac{L-1}{-(a\,(1-\beta) + c\,\beta) + c\,L}\,. \tag{16.70}$$

Using the definition of A and B, (16.69) and (16.70) are seen to be equivalent to (16.63) and (16.62).

Because the condition $d - A < 0$ is equivalent to $\beta < b/(b+d)$, case (ii) in Theorem 16.2 is valid. Again, it can be seen by simple comparison that (16.66) coincide with (16.40), (16.41) and (16.42). In sum, we have shown that the game theoretical solution of the Se-Se inspection game discussed in this section is for $\alpha = 0$ but $\beta > 0$ the same as that for Theorem 16.2 and $k = 1$ control, again as expected.

Fourth, as in the special case $L = 2$ steps, the Operator cannot be induced to behaving legally with certainty. While in the case of $d - A > -f\,\alpha$ he behaves, using of (16.45), illegally with certainty, in case of $d - A < -f\,\alpha$ he behaves legally with positive probability.

Finally, we draw the attention to the relation of the Se-Se to the No-No inspection game already mentioned on p. 340 for the case $L = 2$ steps: For $d - A > -f\,\alpha$ we have

$$1 + \frac{f\,\alpha}{d} = 1 + \frac{f\,\alpha}{A}\,\frac{1}{\frac{d}{A}} < 1 + \frac{f\,\alpha}{A}\,\frac{1}{1 - \frac{f\,\alpha}{A}} = \frac{1}{1 - \frac{f\,\alpha}{A}}\,,$$

and thus, for all $L > 2$,

$$\left(1+\frac{f\,\alpha}{d}\right)\left(1-\frac{f\,\alpha}{A}\right)^{L} < \left(1-\frac{f\,\alpha}{A}\right)^{L-1} < 1\,.$$

Using (15.28), we get

$$x^L < \left(1+\frac{f\,\alpha}{d}\right)^{-1} = \left(1+\frac{A}{d}\,(1-x)\right)^{-1}\,,$$

i.e., (15.55) is fulfilled for $k = 1$ control. Obviously, the equilibrium payoffs (15.58) coincide with (16.64). Furthermore, the equilibrium strategies can be transformed into each other: The probability that the Operator behaves illegally at step L is the same; see (16.65) for $\ell = L$ and p_L^* as given by (15.56). Furthermore, the probability that he behaves illegal at step $\ell, 1 \leq \ell \leq L-1$, is, using (16.65), given by

$$(1-\bar{p}^*_{L,1})\,(1-\bar{p}^*_{L-1,1})\ldots(1-\bar{p}^*_{\ell+1,1})\,\bar{p}^*_{\ell,1} = \bar{p}^*_{\ell,1}\prod_{i=\ell+1}^{L}(1-\bar{p}^*_{\ell,1})$$

$$= \frac{y-1}{y^\ell-1}\,y^{\ell-1}\prod_{i=\ell+1}^{L}\left(1-\frac{y-1}{y^i-1}\,y^{i-1}\right) = \frac{y-1}{y^\ell-1}\,y^{\ell-1}\prod_{i=\ell+1}^{L}\frac{y^{i-1}-1}{y^i-1} = \frac{y-1}{y^L-1}\,y^{\ell-1}\,,$$

i.e., p_i^* as given by (15.56) for $i = \ell$. For the Inspectorate similar considerations can be made: For $\ell = L$, (16.65) coincides with (15.57) for $j = L-1$. Also, the Inspectorate's probability of controlling at step $\ell, 1 \leq \ell \leq L-1$, is, using (16.65), in analogy to the last equation

$$(1-q^*_{L,1})\,(1-q^*_{L-1,1})\ldots(1-q^*_{\ell+1,1})\,q^*_{\ell,1} = \frac{1-x}{1-x^L}\,x^{L-\ell}\,,$$

i.e., q_j^* as given by (15.57) for $j = \ell + 1$.

In sum, for $d - A > -f\alpha$ the game theoretical solution of the Se-Se inspection game coincide resp. can be transformed into the game theoretical solution of the No-No inspection game for $k = 1$ control. We have observed this remarkable property already for $k > 1$ inspections and any number L of steps in case of $\alpha = 0$; see p. 336. We will see that for $k > 1$ and $\alpha > 0$ this does not hold any more; see p. 351.

As for the case $L = 2$ steps, the Operator has an advantage in the Se-Se inspection game in case of $d - A < -f\,\alpha$ and (15.55) is fulfilled, because $d - A < -f\,\alpha$ is equivalent to the inequality

$$d - A\,\frac{1-x}{1-\dfrac{d}{d+A\,(1-x)}\,x^{L-1}} > d - A\,\frac{1-x}{1-x^L}\,,$$

i.e., the equilibrium payoff $Op^*_{L,1}$ to the Operator in the No-No inspection game given by (15.58) is smaller than that in the Se-Se inspection game given by (16.61).

Due to the complexity of conditions to be taken into account, further generalizations of the above result to $k > 1$ inspections at $L > 1$ steps in the sense of Theorems 16.1 or 16.2 seem infeasible. Therefore, we present two observations and consider then the special case $\Gamma(3,2)$

of the Dresher-Höpfinger inspection game with errors of the first and second kind in order to illustrate the complexity of the general game $\Gamma(L, k)$:

First, if $k = L$, then the Operator will behave legally at any step provided $d - A < -L\, f\, \alpha$. Otherwise he will behave illegally right at step L in order to avoid false alarm costs, because they are – by assumption – excluded after an illegal activity; see assumption (iv').

Second, if $k < L$, then in the course of the game all possible situations are reached with positive probability in which an illegal activity has not yet occurred and the same number of steps and controls remain. Therefore, the cases

$$d - A \gtrless -k'\, f\, \alpha \qquad \text{for} \qquad k' = k, k-1, \dots, 1$$

have all to be considered.

We now analyse the Dresher-Höpfinger inspection game $\Gamma(3, 2)$ with errors of the first and second kind the reduced extensive form of which is presented in Figure 16.10.

Let us describe Figure 16.10. At step 3, i.e., at the top of the tree, the Operator decides to behave illegally immediately ($\bar{\ell}_3$) or not (ℓ_3). At step 2, he has to behave legally, if he behaves illegally at step 3 (left branch), otherwise (right branch) he decides to behave illegally ($\bar{\ell}_2$) or not (ℓ_2) at step 2. The same holds then for step 1 in case he behaves legally at step 2.

Also at step 3, the Inspectorate decides, not knowing the Operator's decision, to control (c_3) or not ($\bar{c}_3$). If it decides to control at step 3, then it decides at step 2 to control (c_2) or not ($\bar{c}_2$). In the first case it cannot control any more at step 1, in the second case it has to control at step 1. If the Inspectorate decides not to control at step 3, it has to control at steps 2 (c_2) and 1 (c_1). The information sets of the Inspectorate at steps 3 and 2 are shown in Figure 16.10, its information set at 1 is not shown for simplicity since here it has to control which leads to the payoffs shown in the Figure. Note that the subtle modelling aspect described in assumption (iv') and on p. 337 regarding false alarms holds here as well: If, for example, the Operator behaves illegally at step 3 and the Inspectorate does not control at step 3, then formally false alarms may be raised at steps 2 and 1. We ignore this possibility since in this case we consider the game to be finished with step 3: The Operator has successfully completed the illegal activity. Chance moves are no longer shown in this reduced form of the game instead, the error first and second kind probabilities are included in the payoffs to both players at the end nodes. As in Figure 16.8, not all information sets are depicted in Figure 16.10. For example: the three nodes reached after the moves $\bar{\ell}_3\bar{c}_3$, $\ell_3\bar{c}_3\bar{\ell}_2$ and $\ell_3\bar{c}_3\ell_2$, and which are all followed by the strategy c_2c_1 should be in one information set.

In order to solve this inspection game, the three cases

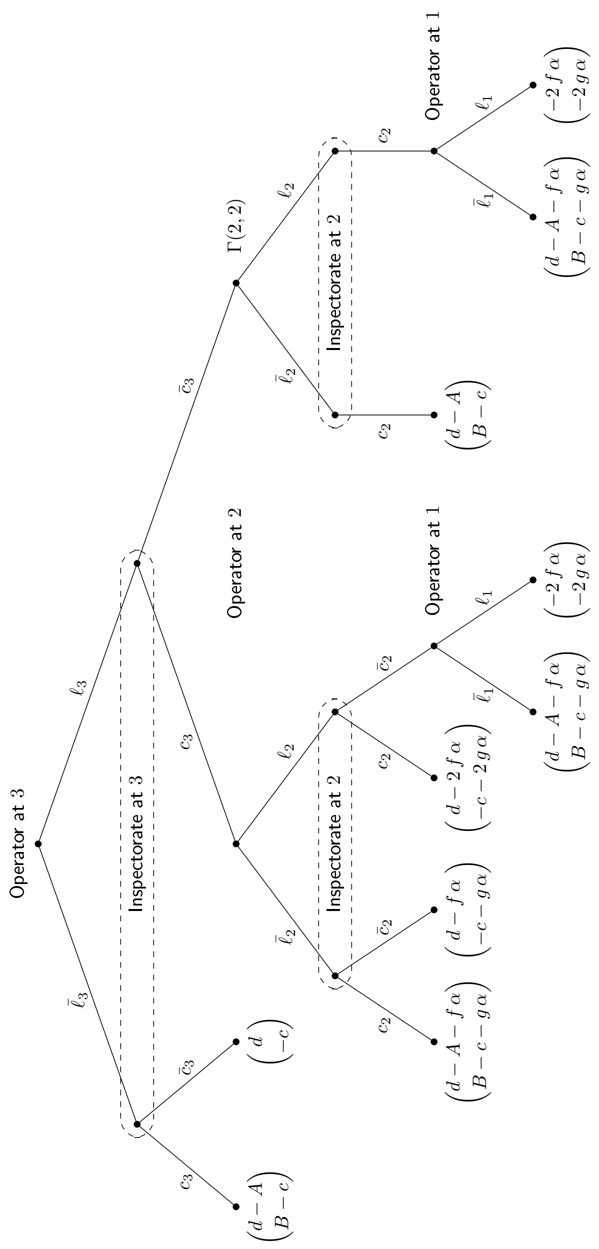
(i): $d - A < -2\, f\, \alpha$ (ii): $-\,2\, f\, \alpha < d - A < -f\, \alpha$ and (iii): $-\, f\, \alpha < d - A$

have to be distinguished. The solution is different for each one of the above, and each case involves subcases.

The game theoretical solution of this inspection game is presented in Lemma 16.4, where only case (iii) is treated in Canty et al. (2001) explicitly.

Lemma 16.4. *Given the Se-Se inspection game with $L = 3$ steps, $k = 2$ controls, errors of the first and second kind, and an unbiased test procedure, i.e., $\Gamma(3, 2)$, the extensive form of which is represented in Figure 16.10.*

Figure 16.10 Reduced extensive form of the Dresher-Höpfinger inspection game $\Gamma(3,2)$ with errors of the first and second kind.



Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{3,2}$ *and* $In^*_{3,2}$*:*

(i) For

$$d - A < -2\, f\, \alpha$$

an equilibrium strategy of the Operator is given by

$$\bar{p}^*_{3,2} = c\, \frac{c - g\,\alpha}{(B + c)^2 - c\,(B + g\,\alpha)}\,, \qquad \bar{p}^*_{2,k'} = \begin{cases} 0 & \text{for} \quad k' = 2 \\ \text{(16.46)} & \text{for} \quad k' = 1 \end{cases}\,,$$

and an equilibrium strategy of the Inspectorate by

$$q^*_{3,2} = \frac{(A + d)\,(d + 2\, f\,\alpha)}{(A + d)^2 - d\,(A - f\,\alpha)}\,, \qquad q^*_{2,k'} = \begin{cases} 1 & \text{for} \quad k' = 2 \\ \text{(16.47)} & \text{for} \quad k' = 1 \end{cases}\,.$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{3,2} = d - A\, q^*_{3,2} \qquad \text{and} \qquad In^*_{3,2} = -c\,\bar{p}^*_{3,2} - 2\, g\,\alpha\,(1 - \bar{p}^*_{3,2})\,.$$

(ii) For

$$-2\, f\,\alpha < d - A < -f\,\alpha \tag{16.71}$$

and

$$\frac{d + f\,\alpha}{d + A} + \frac{f\,\alpha}{A} < 1 \tag{16.72}$$

an equilibrium strategy of the Operator is given by

$$\bar{p}^*_{3,2} = 1 - B\,\frac{B + c}{(2\,B + c)\,(B + g\,\alpha)}\,, \qquad \bar{p}^*_{2,k'} = \begin{cases} 1 & \text{for} \quad k' = 2 \\ \text{(16.46)} & \text{for} \quad k' = 1 \end{cases}\,,$$

and an equilibrium strategy of the Inspectorate by

$$q^*_{3,2} = A\,\frac{A + d}{(2\,A + d)\,(A - f\,\alpha)}\,, \qquad q^*_{2,k'} = \begin{cases} 1 & \text{for} \quad k' = 2 \\ \text{(16.47)} & \text{for} \quad k' = 1 \end{cases}\,.$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{3,2} = d - A\, q^*_{3,2} \qquad \text{and} \qquad In^*_{3,2} = -c + B\,(1 - \bar{p}^*_{3,2})\,.$$

If (16.72) is not fulfilled, the Operator behaves illegally at step 3 *with probability one, i.e.,* $\bar{p}^*_{3,2} = 1$*, and the Inspectorate controls at step* 3 *with probability one, i.e.,* $q^*_{3,2} = 1$*. The equilibrium payoffs to the Operator and to the Inspectorate are* $Op^*_{3,2} = d - A$ *and* $In^*_{3,2} = B - c$*.*

(iii) For

$$-f\,\alpha < d - A \tag{16.73}$$

and

$$\left(1 - \frac{f\,\alpha}{A}\right)\left(2 - \frac{f\,\alpha}{A}\right) > 1 \tag{16.74}$$

an equilibrium strategy of the Operator is given by

$$\bar{p}^*_{3,2} = 1 - B\,\frac{2\,B + g\,\alpha}{(2\,B + g\,\alpha)^2 - B^2}\,, \qquad \bar{p}^*_{2,k'} = \begin{cases} 1 & \textit{for} \quad k' = 2 \\ \textit{(16.49)} & \textit{for} \quad k' = 1 \end{cases}\,,$$

and an equilibrium strategy of the Inspectorate by

$$q^*_{3,2} = A\,\frac{2\,A - f\,\alpha}{(2\,A - f\,\alpha)^2 - A^2}\,, \qquad q^*_{2,k'} = \begin{cases} 1 & \textit{for} \quad k' = 2 \\ \textit{(16.50)} & \textit{for} \quad k' = 1 \end{cases}\,.$$

The equilibrium payoffs to the Operator and to the Inspectorate are

$$Op^*_{3,2} = d - A\,q^*_{3,2} \qquad \textit{and} \qquad In^*_{3,2} = -c + B\,(1 - \bar{p}^*_{3,2})\,.$$

If (16.74) is not fulfilled, the Operator behaves illegally at step 3 *with probability one, i.e.,* $\bar{p}^*_{3,2} = 1$*, and the Inspectorate controls at step* 3 *with probability one, i.e.,* $q^*_{3,2} = 1$*. The equilibrium payoffs to the Operator and to the Inspectorate are* $Op^*_{3,2} = d - A$ *and* $In^*_{3,2} = B - c$*.*

Proof. Ad (i): Using Figure 16.10, the Operator chooses in this case after the moves $\ell_3 c_3 \ell_2 \bar{c}_2$ the strategy ℓ_1 and after the moves $\ell_3 \bar{c}_3$ the strategy ℓ_2. Thus, we arrive at the recursive normal form given in Table 16.4, where the equilibrium payoffs $Op^*_{2,1}$ and $In^*_{2,1}$ are given by (16.48).

Table 16.4 Recursive normal form of the Dresher-Höpfinger inspection game $\Gamma(3,2)$ with errors of the first and second kind for case (i).

	c_3	$\bar{c}_3$
$\bar{\ell}_3$	$B - c$ (Inspectorate); $d - A$ (Operator)	$-c$ (Inspectorate); d (Operator)
ℓ_3	$In^*_{2,1} - g\,\alpha$ (Inspectorate); $Op^*_{2,1} - f\,\alpha$ (Operator)	$-2\,g\,\alpha$ (Inspectorate); $-2\,f\,\alpha$ (Operator)

The non-trivial requirements for cyclic preferences for the bimatrix of Table 16.4 are then given by

$$d - A < Op^*_{2,1} - f\,\alpha \qquad \text{and} \qquad In^*_{2,1} - g\,\alpha < -2\,g\,\alpha\,. \tag{16.75}$$

The first requirement is fulfilled, because $d - A < -2\,f\,\alpha$ and $f\,\alpha < A$ imply $d\,f\,\alpha - A^2 < -2\,f\,\alpha\,A$, which is equivalent to

$$\frac{d + f\,\alpha}{d + A} + \frac{f\,\alpha}{A} < 1\,,$$

which is equivalent, using (16.48), to the left inequality in (16.75). Using (16.48) again, the second requirement is equivalent to $c > 0$, which is true because of (14.2). Thus, the game has cyclic preferences, and the equilibrium strategies and payoffs can be found again with the help of the indifference principle; see Theorem 19.1.

Ad (ii): In this case, the Operator chooses after the moves $\ell_3 c_3 \ell_2 \bar{c}_2$ the strategy ℓ_1 like in case (i), but after the moves $\ell_3 \bar{c}_3$ the strategy $\bar{\ell}_2$; see Figure 16.10. The recursive normal form of this game is given in Table 16.5, where again the equilibrium payoffs (16.48) are used.

Table 16.5 Recursive normal form of the Dresher-Höpfinger inspection game $\Gamma(3,2)$ with errors of the first and second kind for cases (ii) and (iii).

	c_3	$\bar{c}_3$
$\bar{\ell}_3$	$B - c$ $d - A$	$-c$ d
ℓ_3	$In^*_{2,1} - g\,\alpha$ $Op^*_{2,1} - f\,\alpha$	$B - c$ $d - A$

To obtain cyclic preference directions, the inequalities

$$d - A < Op^*_{2,1} - f\,\alpha \qquad \text{and} \qquad In^*_{2,1} - g\,\alpha < B - c \tag{16.76}$$

have to be valid. The first requirement is, using (16.48), equivalent to (16.72). Table 16.3 implies that the best Inspectorate's payoff in the game $\Gamma(2,1)$ is $B - c$. Thus, the second condition holds as well. Therefore, the preference directions are cyclic and again the indifference principle leads to the equilibrium strategies and payoffs.

If condition (16.72) is not fulfilled, then the preferences are not cyclic and the game has a unique equilibrium in pure strategies in which the Operator behaves illegally right at step 3 and the Inspectorate inspects at step 3, i.e., $\bar{p}^*_{3,2} = q^*_{3,2} = 1$. Their respective payoffs are $d - A$ and $B - c$.

For illustration, if $\alpha = 0.5$, $\beta = 0.25$, $f = 5$, and $d = b = 8$, then (16.71) and (16.72) are fulfilled. If $\alpha = 0.95$, $\beta = 0.04$, $f = 20$, and $d = b = 22$, however, only (16.71) is valid.

Ad (iii): Using Figure 16.10, the Operator chooses after the moves $\ell_3 c_3 \ell_2 \bar{c}_2$ the strategy $\bar{\ell}_1$. and after the moves $\ell_3 \bar{c}_3$ he chooses $\bar{\ell}_2$ like in case (ii). The recursive normal form is also given in Table 16.5, where – instead of (16.48) – the equilibrium payoffs $Op^*_{2,1}$ and $In^*_{2,1}$ are given by (16.51).

The non-trivial requirements for cyclic preferences are given by (16.76). The first condition is, using (16.51), explicitly given by (16.74). The second requirement holds again, because $B - c$ is the best Inspectorate's payoff in the game $\Gamma(2,1)$. The equilibrium strategies and payoffs can be found again with the help of the indifference principle. The argumentation in case (16.74) is not fulfilled goes along the same lines as in case (ii).

For illustration, if $\alpha = 0.7$, $\beta = 0.25$, $f = 5$, and $d = b = 6$, then (16.73) and (16.74) are fulfilled. If $\alpha = 0.95$, $\beta = 0.04$, $f = 20$, $d = 41$, and $b = 21$, however, only (16.73) is valid.

This completes the proof. □

As remarked on p. 282, taking into account the possibility of errors first and second kind in time-critical inspection problems leads to a large number of model variants. For example: If instead of assumption (iv') on p. 337, assumption (iv") on p. 307 is used, then the game theoretical solution of the No-No and the Se-Se inspection game can be transformed into each other; see the comment on p. 311.

Let us conclude with what has been announced on p. 345: Other than in the case of $k = 1$ control, the game theoretical solution of the Dresher-Höpfinger inspection game $\Gamma(3,2)$ with errors of the first and second kind given in Lemma 16.4 is very different from that of the corresponding No-No inspection game given in Lemma 15.2.

16.4 Choice of the false alarm probability

Like in Sections 9.5, 12.4 and 15.5 we ask for the optimal value of the false alarm probability α, and again we limit our considerations to $k = 1$ control at $L = 2$ steps, and assume that (9.69) is fulfilled again. The situation here, however, is different from the previous ones because the Operator will behave illegally with positive probability for all values of α; see Lemma 16.3. Thus, we are just looking for that value of α which maximizes the Inspectorate's equilibrium payoff.

Because $\beta(\alpha)$ is assumed to be a monotone decreasing function of α, see (9.69), the function $d - (b + d)\,(1 - \beta(\alpha))$ is also a decreasing function with the value $d > 0$ for $\alpha = 0$ and, using (14.2), $-b < -f$ for $\alpha = 1$. Thus, there exists a unique $\alpha^* \in (0,1)$ being the solution of

$$d - (b + d)\,(1 - \beta(\alpha^*)) = -f\,\alpha^* \,.$$

According to (16.48) and (16.51), the Inspectorate's equilibrium payoff is given by

$$In^*_{2,1}(\alpha) = \begin{cases} -c + \dfrac{B^2}{2\,B + g\,\alpha} & \text{for} \quad \alpha < \alpha^* \\ -c + B\,\dfrac{c - g\,\alpha}{c + B} & \text{for} \quad \alpha > \alpha^* \end{cases} \,, \tag{16.77}$$

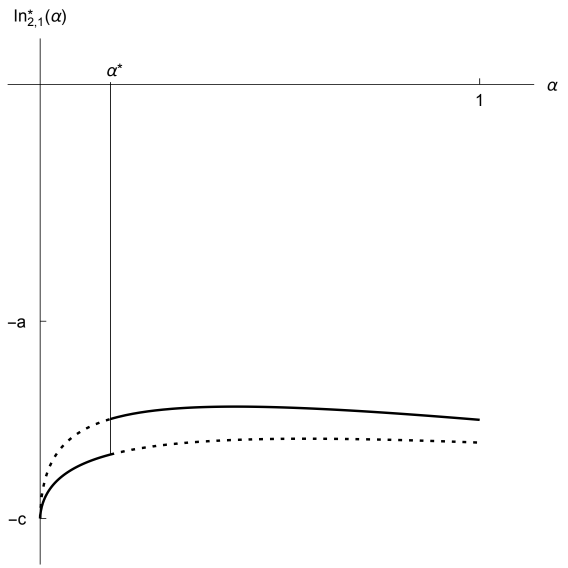
and thus, using $B = (c - a)\,(1 - \beta)$, $c > a$ and $g > 0$,

$$In^*_{2,1}(0) = -c \qquad \text{and} \qquad In^*_{2,1}(1) = -c + \frac{(c-a)\,(c-g)}{2\,c - a} < -a \,.$$

Because of (9.69), the function $(c - a)\,(1 - \beta(\alpha)) + g\,\alpha$ is monotone increasing from 0 for $\alpha = 0$ to $c - a + g < c$ for $\alpha = 1$, we have $(c - a)\,(1 - \beta(\alpha)) + g\,\alpha < c$ for any $\alpha \in [0,1]$ which is equivalent to

$$-c + \frac{B^2}{2\,B + g\,\alpha} < -c + B\,\frac{c - g\,\alpha}{c + B} \,.$$

Figure 16.11 The equilibrium payoff (16.77) to the Inspectorate for the parameters $d = 12, b = 6, f = 3$ and $c = 11, a = 6, g = 3$.



In Figure 16.11, the solid curve represents $In^*_{2,1}(\alpha)$ according to (16.77) for the parameters $d = 12, b = 6, f = 3$ (for finding α^*) and $c = 11, a = 6, g = 3$, which fulfil (14.2).

It depends on $\beta(\alpha)$ where the maximum α^{**} of $In^*_{2,1}(\alpha)$ is attained: at $\alpha^{**} = \alpha^*$, at $\alpha^{**} \in (\alpha^*, 1)$ or even at $\alpha^{**} = 1$. In the latter case the Inspectorate has to call for an alarm under any circumstances at the control, which means that the detection system needs to be revised.

Chapter 17

Strait control and models with multiple illegal activities

There is a large number of models which deal with Smugglers who want to cross a strait with some contraband and with Customs who want to catch them. In principle, these models can also be interpreted in terms of critical times and inspections, but here this wording sounds somewhat superficial. In addition, and this is new here, there are models dealing with multiple illegal activities. Therefore and, as mentioned already on p. III, because of the large literature, we consider in this chapter this class of models separately.

In doing so, we do not just present the models and their game theoretical solutions. Rather we discuss the assumptions underlying these models, since they have not been formulated explicitly in many cases, and we show that some limiting assumptions need not be made, in particular in view of the information states of the players in the course of the game; see von Stengel's inspection game in Section 17.2.

Note that in this chapter and other than so far, we name the games by their authors. Also it should be mentioned that according to our best knowledge von Stengel (1991) was the first who discussed carefully the issue of information in recursive inspection games.

In Section 17.1 we introduce the generalized Thomas-Nisgav inspection model and derive its game theoretical solution which is published in this monograph for the first time. Thereafter, we describe and analyse the inspection game by Baston and Bostock (1991), in which a situation is considered where Customs has two patrol boats that can be used during the same night. Section 17.2 deals with inspection models with multiple illegal activities.

Note that the assumptions of Chapter 14 are specified separately in Sections 17.1 and 17.2. While most of these assumptions are rather unproblematic, assumption (v) needs a justification because in conflict situations between Customs and Smuggler no agreed rules, formal agreements or international treaties exist the Smuggler has to adhere: How does the smuggler know the number k of controls? Assumption (v) can be justified keeping in mind that Customs has to obey rules given by its State, and that Smugglers can observe the long term activities of Customs; see also pp. 18 and 276.

Note that in Chapter 24 a Se-No critical time inspection game with an *expected* number of inspections in one facility is analysed, see also p. 18, which has surprising relations to the generalized Thomas-Nisgav inspection game treated in Section 17.1.

17.1 Any number of nights and controls; errors of the second kind: Generalized Thomas-Nisgav model, Models by Baston and Bostock and by Garnaev

Let us start with the work of Thomas and Nisgav (1976) which deals with problems the most simple one is close to that analysed by Dresher (1962). But let us mention already now that there is a fundamental difference in the assumptions between these two model which lead to completely different solutions. Let us quote Thomas and Nisgav:

> We consider a long, narrow strait where smuggling activity is taking place. Let side A represent a patrol unit whose objective is to capture or reduce the value of contraband held by side B, the infiltrator or smuggler seeking escape by crossing the strait to exit from side A's territory. The contraband held by side B is perishable with a lifetime of M time units; consequently, he must make his escape within M time units in order to benefit from his infiltration. An example of the type of contraband is intelligence information. Side A is under a single command equipped with speedboats containing search radar and communication units. Side B is an individual unit with small motorboats.
>
> Although side A has search radar, due to the narrowness of the strait, side B's radar echo will be shadowed by land, thus making radar detection near the shore virtually impossible. Thus, A can detect B only if B is sufficiently far from shore. For obvious reasons, side B only attempts escape at night, and he departs from a point near a village or parallel to a village located on the other side of the strait. Although the patrol boats are much faster than B's, the fact that the strait is long and narrow gives side B a chance to cross successfully without being detected.

Let us return to our convention, i.e., let the first player represent the Infiltrator or *Smuggler*, seeking to escape by crossing the strait to exit from the second player's territory. This activity is in the following called *illegal activity* or *smuggling*. The second player is the patrol unit, in the following called *Customs*, whose objective is to capture or reduce the value of contraband held by the first player. The contraband itself is perishable with a lifetime of L time units; consequently, he *must* make his escape within L time units in order to benefit from his infiltration.

Thomas and Nisgav (1976) analyse first a zero-sum game where each side, Smuggler and Customs, has a single boat each and where the Smuggler is detected with probability $1-\beta$ when the patrol boat controls the strait. They assume the payoff to Customs to be 1 if it catches the Smuggler and -1 if it does not.

In this section the inspection model of Thomas and Nisgav is generalized in two ways: First, it is assumed that the Smuggler decides at the beginning of the L nights either to behave illegally during these L nights with certainty or not at all. Second, a non-zero-sum game is considered with the payoffs given below; see (17.1) and (17.2).

Furthermore, it is assumed that there are no errors of the first kind, that the Customs's resources are limited to $k \leq L$ night patrols, which are then distributed on the L nights, that the Smuggler's success requires a single crossing of the strait during one of the L nights, and that both players know the values of k and L.

In this section, assumptions (iii), (iv), (vi), (viii) and (x) of Chapter 14 are specified as follows:

(iii') The Smuggler may behave illegally at most once at the steps $L, \dots, 1$.

(iv') During a control Customs may commit an error of the second kind, i.e., if the Smuggler behaves illegally at the same step at which Customs performs a control, then the illegal activity is not detected with probability β. This non-detection probability is the same for all controls.

(vi') The Smuggler decides at the beginning, i.e., at step L, whether to behave legally throughout the game or not. In the latter case he decides whether to behave illegally at step L. If the Smuggler behaves legally at steps $L, \dots, \ell+1$ $(1 \leq \ell \leq L-1)$, then he decides whether to behave illegally at step ℓ; and so on. The Smuggler behaves illegally latest at step 1, if he decided at the beginning to behave illegally.

Customs decides at the beginning whether to control at step L. If it has still controls at its disposal, then Customs decides at step $L-1$ whether to control at that step; and so on.

(viii') The payoffs to the two players (Smuggler, Customs) are given by

$$\begin{aligned} (d, -c) &\quad \text{for undetected smuggling} \\ (-b, -a) &\quad \text{for detected smuggling} \\ (0, 0) &\quad \text{for no smuggling}\,, \end{aligned} \tag{17.1}$$

where the parameters satisfy the conditions

$$0 < min(b, d) \qquad \text{and} \qquad 0 < a < c\,. \tag{17.2}$$

(x') The game ends either at step L in case the Smuggler behaves legally throughout the game, or at the step at which the Smuggler behaves illegally, or at that step at which the number of controls left is zero, or at that step at which the number of controls left is equal to the number of steps left, or at step 1.

The remaining assumptions of Chapter 14 except (ix) hold throughout this section.

Four comments on the above assumptions and the wording in this section: First, in the following the term "step" is used synonymously to the term "night" referring to the application the original Thomas-Nisgav inspection game was developed for.

Second, assumption (vi') indicates that the generalized Thomas-Nisgav inspection game is a Se-Se inspection game. Between the three Se-Se inspection models analysed in Chapters 12, 16, and this section the following important difference is emphasized: While in the Avenhaus-Canty inspection game and in the Dresher-Höpfinger inspection game, the Operator decides to behave legally or illegally in the course of the game, in the generalized Thomas-Nisgav inspection game discussed here the Smuggler makes this decision only at the very beginning of the game.

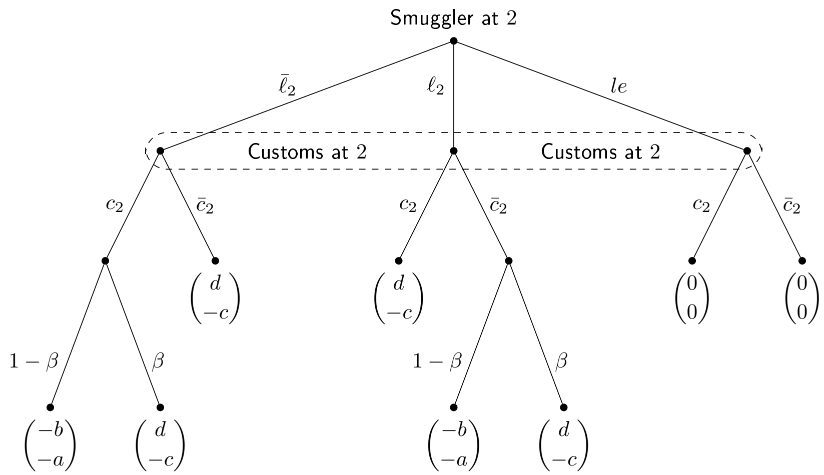
Third, note that "undetected smuggling" means "untimely control or a timely control and no detection of the smuggling", and that "detected smuggling" means "timely control and detection of the smuggling".

Fourth, regarding assumption (x') we note that if the Smuggler behaves illegally at step i, $i = L, ..., 1$, then the game ends at step i regardless whether the illegal behaviour is detected at that step or not. In the latter case, the Operator has successfully performed his illegal activity and thus, the game ends as well.

In order to obtain the payoffs for the above described original Thomas-Nisgav inspection game, we have to choose $a = -1$, $c = 1$, and thus, $b = d = 1$ because it is a zero-sum game. Note that at first sight this is not literally a special case of the generalized Thomas-Nisgav inspection game since $a = -1$ contradicts (17.2). Because, however, the "no smuggling" case in (17.1) is excluded in the original Thomas-Nisgav inspection game, the normalization $(0, 0)$ for "no smuggling" disappears, and only the condition $a < c$ remains in (17.2), which is fulfilled because $a = -1 < 1 = c$.

In order to understand the information structure of the generalized Thomas-Nisgav inspection game – which we abbreviate by $\Gamma(L, k)$ – we consider first the special cases $\Gamma(2, 1)$ and $\Gamma(4, 2)$. Figure 17.1 represents the extensive form of the game $\Gamma(2, 1)$. Note that according to the comment on p. 50, all extensive form games in this section start with the Smuggler, and that chance moves are not explicitly named, but can be identified via the probabilities $1 - \beta$ and β.

Figure 17.1 Extensive form of the generalized Thomas-Nisgav inspection game $\Gamma(2, 1)$.



Before the first night the Smuggler decides either to smuggle during the first night ($\bar{\ell}_2$), postpone it to the next night (ℓ_2) or decide not to smuggle at all (le stands for legal behaviour). Not knowing the Smugglers decision at step 2, Customs can either decide to patrol in the first night (c_2 stands for control) or not to patrol in the first night ($\bar{c}_2$ stands for *not* control) which means that it needs to patrol in the second night. With the abbreviations

$$-\tilde{b} = -b\,(1-\beta) + d\,\beta \qquad \text{and} \qquad -\tilde{a} = -a\,(1-\beta) - c\,\beta\,, \tag{17.3}$$

which have already been introduced in (16.35) and are repeated here for easier reference, we get immediately the reduced normal form which is given in Table 17.1. Note that in the original work by Thomas and Nisgav (1976) not the extensive form but the (recursive) normal form of the inspection game is considered from the very beginning. We will come back to this important issue on p. 368.

In order to solve the extensive form games in this chapter, we transform them into normal form games. Thus, we use – as in Chapter 16 – from the very beginning the notation p and q instead

Table 17.1 Reduced normal form of the generalized Thomas-Nisgav inspection game $\Gamma(2,1)$.

	c_2	$\bar{c}_2$
$\bar{\ell}_2$	$-\tilde{b}$; $-\tilde{a}$	d ; $-c$
ℓ_2	d ; $-c$	$-\tilde{b}$; $-\tilde{a}$
le	0 ; 0	0 ; 0

of g and h; see also the comment on p. 318.

Because the Nash equilibria are presented in behavioural strategies, the sets of pure strategies for the players is neither introduce here for the game $\Gamma(2,1)$ nor later for the general game $\Gamma(L,k)$. Instead, let in analogy to Section 16.1, $\bar{p}_{2,1}$ denote the probability to smuggle in the first night ($\bar{\ell}_2$ at 2), $p_{2,1}$ denote the probability not to smuggle in the first night, and p_{le} be the probability not to smuggle at all. In case of illegal behaviour – which happens with probability $\bar{p}_{2,1}+p_{2,1}$ – and in case the smuggle is not performed in the first night, i.e., ℓ_2 with probability $p_{2,1}$, it has to be done in the second night independent of the number of controls k' available in the second night, i.e., $\bar{p}_{1,k'}=1$ for $k'=0,1$. This implies that $\bar{p}_{1,k'}=1$ is not a strategic variable and is thus not considered in the Smuggler's set of behavioural strategies

$$P_{2,1} := \left\{\mathbf{p} := (\bar{p}_{2,1}, p_{2,1}, p_{\mathsf{le}})^T \in [0,1]^3 : \bar{p}_{2,1} + p_{2,1} + p_{\mathsf{le}} = 1\right\} . \tag{17.4}$$

For later purpose we also introduce the Smuggler's set of behavioural strategies in case legal behaviour is not taken into account:

$$P'_{2,1} := \{\mathbf{p} := \bar{p}_{2,1} : \bar{p}_{2,1} \in [0,1]\} . \tag{17.5}$$

For Customs, let $q_{2,1}$ be the probability to control in the first night (c_2 at 2). In case of the decision $\bar{c}_2$, i.e., no control in the first night (with probability $1-q_{2,1}$), Customs has to control in the second night. In analogy to the Smuggler's $\bar{p}_{1,k'}=1$, the probability $q_{1,1}$ is 1, i.e., not a strategic variable, and is thus not considered in the Custom's set of behavioural strategies

$$Q_{2,1} := \{\mathbf{q} := q_{2,1} : q_{2,1} \in [0,1]\} . \tag{17.6}$$

The Smuggler's (expected) payoff is, for any $\mathbf{p} \in P_{2,1}$ and any $\mathbf{q} \in Q_{2,1}$, using Table 17.1, given by

$$Op_{2,1}(\mathbf{p},\mathbf{q}) := \bar{p}_{2,1}\left(-\tilde{b}\,q_{2,1} + d\,(1-q_{2,1})\right) + p_{2,1}\left(d\,q_{2,1} - \tilde{b}\,(1-q_{2,1})\right) \tag{17.7}$$

and that of Customs by

$$In_{2,1}(\mathbf{p},\mathbf{q}) := \bar{p}_{2,1}\left(-\tilde{a}\,q_{2,1} - c\,(1-q_{2,1})\right) + p_{2,1}\left(-c\,q_{2,1} - \tilde{a}\,(1-q_{2,1})\right), \tag{17.8}$$

see (19.3) and (19.4). Note that although we call in this section the players Smuggler and Customs, we remain with the old notation for the payoffs.

Comparing the normal form game in Table 17.1 for the "illegal" game, i.e., the game in which the Smuggler smuggles with certainty, with the one of the Dresher-Höpfinger inspection game with errors of the second kind for the case $\beta > b/(b+d)$, see the right hand normal form of Table 16.1, we see that both normal forms coincide.

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Lemma 17.1. *Given the generalized Thomas-Nisgav inspection game with $L = 2$ nights, $k = 1$ control, and with errors of the second kind, i.e., $\Gamma(2,1)$, the extensive and reduced normal forms of which are represented in Figure 17.1 and Table 17.1. The sets of behavioural strategies are given by (17.4) and (17.6), and the payoffs to both players by (17.7) and (17.8).*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{2,1} := Op_{2,1}(\mathbf{p}^*, \mathbf{q}^*)$ and $In^*_{2,1} := In_{2,1}(\mathbf{p}^*, \mathbf{q}^*)$:

(i) For

$$\frac{1}{2} < \frac{1}{1-\beta} \frac{1}{1+b/d} \tag{17.9}$$

the Smuggler behaves illegally in the entire game $\Gamma(2,1)$, i.e., $p^*_{le} = 0$. An equilibrium strategy of the Smuggler is given by

$$\bar{p}^*_{2,1} = p^*_{2,1} = \frac{1}{2},$$

and an equilibrium strategy of Customs by

$$q^*_{2,1} = \frac{1}{2}. \tag{17.10}$$

The equilibrium payoffs to the Smuggler and to Customs are

$$Op^*_{2,1} = d - (1-\beta)\,(b+d)\,\frac{1}{2} \qquad \text{and} \qquad In^*_{2,1} = -c + (1-\beta)\,(c-a)\,\frac{1}{2}.$$

(ii) For

$$\frac{1}{2} > \frac{1}{1-\beta} \frac{1}{1+b/d} \tag{17.11}$$

the Smuggler behaves legally in the entire game $\Gamma(2,1)$, i.e., $p^*_{le} = 1$ and $\bar{p}^*_{2,1} = p^*_{2,1} = 0$. Equilibrium strategies of Customs are given by

$$\frac{1}{1-\beta} \frac{1}{1+b/d} \le q^*_{2,1} \le 1 - \frac{1}{1-\beta} \frac{1}{1+b/d}, \tag{17.12}$$

where $q^*_{2,1}$ given by (17.10) fulfils (17.12).

The equilibrium payoffs to the Smuggler and to Customs are

$$Op^*_{2,1} = In^*_{2,1} = 0.$$

Proof. Ad (i): As mentioned before this Lemma, the normal form of the "illegal" game coincides with the one of the Dresher-Höpfinger inspection game with errors of the second kind for the case $\beta > b/(b+d)$. Thus, its solution is given by Lemma 16.2 (ii). The condition $Op_{2,1}(le, \mathbf{q}^*) = 0 < Op^*_{2,1}$ is fulfilled by virtue of (17.9).

Ad (ii): The equilibrium strategy of Customs in case of no smuggling is determined by

$$\bar{\ell}_2 : \quad 0 \geq -\tilde{b}\, q^*_{2,1} + d\,(1 - q^*_{2,1}) \qquad \text{and} \qquad \ell_2 : \quad 0 \geq d\, q^*_{2,1} - \tilde{b}\,(1 - q^*_{2,1})\,,$$

which leads, using (17.3), to (17.12). Condition (17.11) ensures that (17.12) is not empty. □

Note that (17.10) is a robust equilibrium strategy; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

Before treating the generalized Thomas-Nisgav inspection game $\Gamma(L,k)$, we consider now the game $\Gamma(4,2)$. At first sight it may be surprising that other than in the last chapter, this game is chosen instead of the simpler game $\Gamma(3,2)$. It will turn out, however, that we will encounter a new property for the solution of the game $\Gamma(4,2)$ which cannot yet be observed in the game $\Gamma(3,2)$; see p. 369.

Figure 17.2 represents the extensive form of the generalized Thomas-Nisgav inspection game $\Gamma(4,2)$.

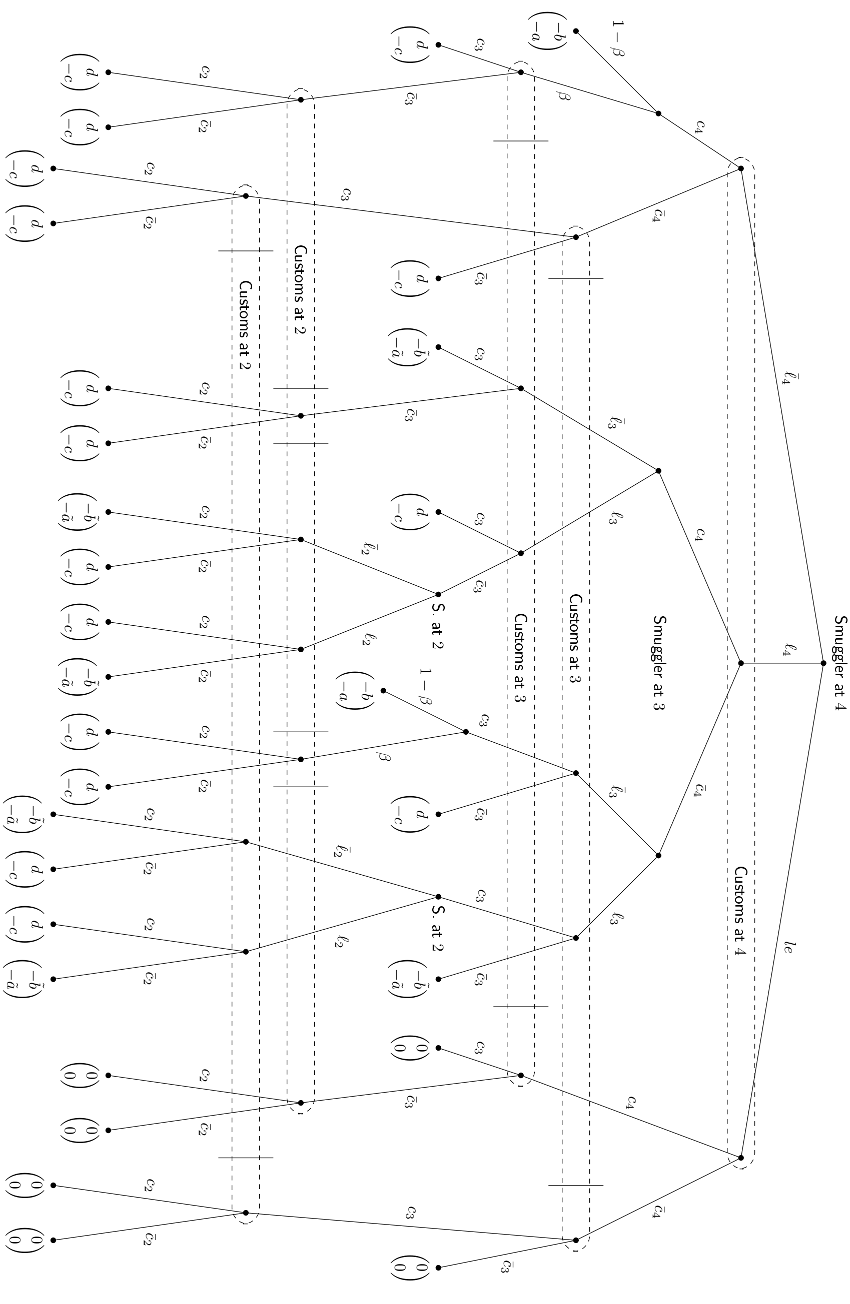
Because it is the most complex extensive form game analysed in this monograph, we comment it in some detail. The moves of the Smuggler are quickly explained: Before the first night (step 4) he decides to behave illegally immediately ($\bar{\ell}_4$), to postpone the illegal activity (ℓ_4), or to behave legally throughout the four nights (le). In the second case, he decides before the second night (step 3) to smuggle immediately ($\bar{\ell}_3$) or not (ℓ_3), and so forth. If after two nights Customs has already spent its two controls (c_4 and c_3) and the Smuggler has not yet smuggled (ℓ_4 and ℓ_3), then he does this in the third or fourth, i.e., last, night.

The situation of Customs is more complicated: Of course, if the Smuggler behaves illegally during the first night, and Customs controls and detects it (with probability $1-\beta$), then the game ends with payoffs $(-b,-a)$ to the two players. Also, it is clear that Customs, when deciding before the first night (step 4) to control in this night (c_4) or not ($\bar{c}_4$), does not know if the Smuggler will behave illegally immediately ($\bar{\ell}_4$), or if he postpones the illegal activity (ℓ_4), or if he will behave legally throughout the four nights (le). This is indicated by the information set named "Customs at 4". What is Customs information state before the second night, if, for example, it does not control during the first night ($\bar{c}_4$)? Customs does not know, if 1) Smuggler behaved illegally during the first night ($\bar{\ell}_4$), if 2) Smuggler behaved legally during the first night (ℓ_4) and will smuggle or not in the second night ($\bar{\ell}_3$ or ℓ_3), or if 3) the Smuggler will behave legally throughout the four nights. This is indicated by the upper of the two information sets named "Customs at 3". All subsequent information sets can be explained analogously.

In order not to let this figure become too complicated, some of the chance moves are not explicitly shown instead, they are replaced by the (expected) payoffs $(-\tilde{b},-\tilde{a})$. And also, as in earlier extensive form games, the moves c_1 and $\bar{c}_1$ of Customs are not explicitly shown since before the last night Customs has no longer a choice: Either it controlled already twice, then it cannot any more, or not, then it has to do so.

Looking at Figure 17.2, we see that we can simplify the extensive form game considerably: First, if the Smuggler behaves illegally during the first night then he will either be detected or not. In any case the payoffs are $(-\tilde{b},-\tilde{a})$ thus, we can cut the information sets as shown in

Figure 17.2 Extensive form of the generalized Thomas-Nisgav inspection game $\Gamma(4, 2)$.



the Figure. The same happens if the Smuggler decides to behave legally throughout the four nights: Then the payoffs are $(0, 0)$ and the information sets can be cut appropriately.

Let, in analogy to the notation in the generalized Thomas-Nisgav inspection game $\Gamma(2, 1)$, $\bar{p}_{4,2}$ be the probability to smuggle in the first night, $p_{4,2}$ be the probability not to smuggle in the first night, and p_{le} be the probability not to smuggle at all. In case of illegal behaviour (with probability $\bar{p}_{4,2} + p_{4,2}$) and in case the smuggle is not performed in the first night, a game with three nights and 1 (c_4) or 2 ($\bar{c}_4$) remaining controls is reached.

Let in the game with $\ell = 3$ nights, k', $k' = 1, 2$, denote the number of controls left. Then $\bar{p}_{3,k'}$ denotes the probability to smuggle immediately ($\bar{\ell}_3$). Because the decision to behave legally throughout the entire game can only be made at the very beginning of the game (at 4), the probability $p_{3,k'}$ of postponing the illegal activity to one of the next two nights (ℓ_3) is given by $1 - \bar{p}_{3,k'}$. If the game with $\ell = 2$ nights is reached, and $k' = 1$ control is still left, then $\bar{p}_{2,1}$ denotes the probability to smuggle immediately ($\bar{\ell}_2$). Note that in case of $\ell = 2$ nights and $k' = 2$ controls left, in both nights controls have to be performed. Thus, Smuggler's set of behavioural strategies is, using (17.5), given by

$$P_{4,2} := \left\{ (\bar{p}_{4,2}, p_{4,2}, p_{\mathsf{le}})^T \in [0,1]^3 : \bar{p}_{4,2} + p_{4,2} + p_{\mathsf{le}} = 1 \right\} \times P'_{3,1} \times P'_{3,2} \times P'_{2,1} \,. \qquad (17.13)$$

Formally two different probabilities $p_{3,2}(c_4)$ and $p_{3,2}(\bar{c}_4)$ (say) would need to be introduced: If Smuggler decides for ℓ_4 then $p_{3,2}(c_4)$ after Customs choice c_4 is potentially different from $p_{3,2}(\bar{c}_4)$ after its choice $\bar{c}_4$. Because Theorem 17.1 indicates that in equilibrium both probabilities coincide, we model them as equal from the very beginning.

For Customs, let $q_{4,2}$ resp. $1 - q_{4,2}$ be the probability to control in the first night (c_4) resp. not to control in the first night ($\bar{c}_4$). Suppose the game with $\ell = 3$ nights and k', $k' = 1, 2$, controls is reached. Then $q_{3,k'}$ denotes the probability to control in the next night (c_3). The probability $q_{3,2}$ belongs to the upper of the two information sets named "Customs at 3" in Figure 17.2, and $q_{3,1}$ belongs to the lower one. In case the game with $\ell = 2$ nights and $k' = 1$ control is reached (the case $k' = 2$ controls implies $q_{2,2} = q_{1,1} = 1$), the probability to control in the third night (c_2) is denoted by $q_{2,1}$. Note that formally two different probabilities in this case need to be introduced, because at 2 Customs has two information sets: One in the subgame starting after the moves $\ell_4 c_4 \ell_3 \bar{c}_3$, and the other one in the subgame starting after the moves $\ell_4 \bar{c}_4 \ell_3 c_3$). Figure 17.2, however, demonstrates that both subgames are identical, and therefore, only one probability $q_{2,1}$ is introduced. Thus, Custom's set of behavioural strategies is given by

$$Q_{4,2} := \{q_{4,2} : q_{4,2} \in [0,1]\} \times \{q_{3,2} : q_{3,2} \in [0,1]\} \times$$
$$\times \{q_{3,1} : q_{3,1} \in [0,1]\} \times \{q_{2,1} : q_{2,1} \in [0,1]\} \,. \qquad (17.14)$$

Let the Smuggler's illegal strategies $\bar{\ell}_4$, $\ell_4\bar{\ell}_3$, $\ell_4\ell_3\bar{\ell}_2$ and $\ell_4\ell_3\ell_2\bar{\ell}_1$ be numbered by i, $i = 4, \ldots, 1$. Using Figure 17.2, the (expected) payoff to the Smuggler is, for any $\mathbf{q} = (q_{4,2}, q_{3,2}, q_{3,1}, q_{2,1}) \in Q_{4,2}$, given by

$$Op_{4,2}(4, \mathbf{q}) = q_{4,2}\,(-\tilde{b}) + (1 - q_{4,2})\,d$$

$$Op_{4,2}(3, \mathbf{q}) = q_{4,2}\left(q_{3,1}\,(-\tilde{b}) + (1 - q_{3,1})\,d\right) + (1 - q_{4,2})\left(q_{3,2}\,(-\tilde{b}) + (1 - q_{3,2})\,d\right)$$

$$Op_{4,2}(2, \mathbf{q}) = q_{4,2}\left(q_{3,1}\,d + (1 - q_{3,1})\left(q_{2,1}\,(-\tilde{b}) + (1 - q_{2,1})\,d\right)\right)$$

$$+ (1 - q_{4,2}) \left(q_{3,2} \left(q_{2,1} (-\tilde{b}) + (1 - q_{2,1}) \, d \right) + (1 - q_{3,2}) (-\tilde{b}) \right)$$

$$Op_{4,2}(1, \mathbf{q}) = q_{4,2} \left(q_{3,1} \, d + (1 - q_{3,1}) \left(q_{2,1} \, d + (1 - q_{2,1}) (-\tilde{b}) \right) \right)$$

$$+ (1 - q_{4,2}) \left(q_{3,2} \left(q_{2,1} \, d + (1 - q_{2,1}) (-\tilde{b}) \right) \right) + (1 - q_{3,2}) (-\tilde{b}) \Big) \,,$$

or, equivalently, by

$$\begin{aligned}
\frac{d - Op_{4,2}(4, \mathbf{q})}{\tilde{b} + d} &= q_{4,2} \\
\frac{d - Op_{4,2}(3, \mathbf{q})}{\tilde{b} + d} &= q_{4,2} \, q_{3,1} + (1 - q_{4,2}) \, q_{3,2} \\
\frac{d - Op_{4,2}(2, \mathbf{q})}{\tilde{b} + d} &= q_{4,2} \, (1 - q_{3,1}) \, q_{2,1} + (1 - q_{4,2}) \left(q_{3,2} \, q_{2,1} + 1 - q_{3,2} \right) \\
\frac{d - Op_{4,2}(1, \mathbf{q})}{\tilde{b} + d} &= q_{4,2} \, (1 - q_{3,1}) \, (1 - q_{2,1}) + (1 - q_{4,2}) \left(q_{3,2} \, (1 - q_{2,1}) + 1 - q_{3,2} \right) .
\end{aligned} \tag{17.15}$$

The right hand side of (17.15) consists of probabilities only. But what do they mean? In order to answer this question, we consider the six possibilities

$$43, \, 42, \, 41, \, 32, \, 31, \, 21 \,,$$

for Customs to distribute its two controls on the four nights and furthermore, the probabilities $\tilde{q}_j$, $j = 4, \ldots, 1$, to control in the $4 - j + 1$-th night. Thus, we have

$$\begin{aligned}
\tilde{q}_4 &= \mathbb{P}(43) + \mathbb{P}(42) + \mathbb{P}(41) \,, \qquad & \tilde{q}_3 &= \mathbb{P}(43) + \mathbb{P}(32) + \mathbb{P}(31) \,, \\
\tilde{q}_2 &= \mathbb{P}(42) + \mathbb{P}(32) + \mathbb{P}(21) \,, \qquad & \tilde{q}_1 &= \mathbb{P}(41) + \mathbb{P}(31) + \mathbb{P}(21) \,.
\end{aligned} \tag{17.16}$$

Determining now the probabilities on the right hand side of (17.16) with the help of Figure 17.2, we get

$$\mathbb{P}(43) = q_{4,2} \, q_{3,1} \,, \quad \mathbb{P}(42) = q_{4,2} \, (1 - q_{3,1}) \, q_{2,1} \,, \quad \mathbb{P}(41) = q_{4,2} \, (1 - q_{3,1}) \, (1 - q_{2,1}) \,,$$

$$\mathbb{P}(32) = (1 - q_{4,2}) \, q_{3,2} \, q_{2,1} \,, \quad \mathbb{P}(31) = (1 - q_{4,2}) \, q_{3,2} \, (1 - q_{2,1}) \,,$$

$$\mathbb{P}(21) = (1 - q_{4,2}) \, (1 - q_{3,2}) \,,$$

and therefore, using (17.16),

$$\begin{aligned}
\tilde{q}_4 &= q_{4,2} \\
\tilde{q}_3 &= q_{4,2} \, q_{3,1} + (1 - q_{4,2}) \, q_{3,2} \\
\tilde{q}_2 &= q_{4,2} \, (1 - q_{3,1}) \, q_{2,1} + (1 - q_{4,2}) \left(q_{3,2} \, q_{2,1} + 1 - q_{3,2} \right) \\
\tilde{q}_1 &= q_{4,2} \, (1 - q_{3,1}) \, (1 - q_{2,1}) + (1 - q_{4,2}) \left(q_{3,2} \, (1 - q_{2,1}) + 1 - q_{3,2} \right) .
\end{aligned} \tag{17.17}$$

Thus, we see, surprisingly enough, that the terms on the right hand side of (17.15) are just the probabilities $\tilde{q}_j$, $j = 4, \ldots, 1$. Define

$$\tilde{Q}_{4,2} = \left\{ \tilde{\mathbf{q}} := (\tilde{q}_4, \tilde{q}_3, \tilde{q}_2, \tilde{q}_1) \in [0,1]^4 : \sum_{j=1}^{4} \tilde{q}_j = 2 \right\} . \tag{17.18}$$

Note that we do not transpose the vectors in (17.18) and (17.20) because we do not define the payoffs to the two players using matrix operations as in (17.7) and (17.8). Then, using (17.15) and (17.17), the Smuggler's payoff can be written in terms of $\tilde{\mathbf{q}} \in \tilde{Q}_{4,2}$:

$$Op_{4,2}(i, \tilde{\mathbf{q}}) := d - \tilde{q}_i\,(\tilde{b} + d)\,, \qquad i = 4, \ldots, 1\,. \tag{17.19}$$

Let $\tilde{p}_i$ denote the Smuggler's probability for choosing the illegal strategy i (remember that his illegal strategies $\bar{\ell}_4$, $\ell_4\bar{\ell}_3$, $\ell_4\ell_3\bar{\ell}_2$ and $\ell_4\ell_3\ell_2\bar{\ell}_1$ are numbered by i, $i = 4, \ldots, 1$), and let $\tilde{p}_{\mathsf{le}}$ denote the probability to behave legally throughout the entire game. Define

$$\tilde{P}_{4,2} := \left\{ \tilde{\mathbf{p}} := (\tilde{p}_4, \tilde{p}_3, \tilde{p}_2, \tilde{p}_1, \tilde{p}_{\mathsf{le}}) \in [0,1]^5 : \sum_{i=1}^{4} \tilde{p}_i + \tilde{p}_{\mathsf{le}} = 1 \right\} . \tag{17.20}$$

Then, using (17.19), we get for Smuggler's (expected) payoff

$$Op_{4,2}(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) = \sum_{i=1}^{4} \tilde{p}_i \, \left(d - \tilde{q}_i\,(\tilde{b} + d) \right) + \tilde{p}_{\mathsf{le}}\, 0 = d\,(1 - \tilde{p}_{\mathsf{le}}) - (\tilde{b} + d) \sum_{i=1}^{4} \tilde{p}_i\, \tilde{q}_i\,. \tag{17.21}$$

Custom's (expected) payoff can be determined from (17.21) by replacing d by $-c$ and $\tilde{b}$ by $\tilde{a}$. Thus, we have

$$In_{4,2}(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) = \sum_{i=1}^{4} \tilde{p}_i \, (-c - \tilde{q}_i\,(\tilde{a} - c)) + \tilde{p}_{\mathsf{le}}\, 0 = -c\,(1 - \tilde{p}_{\mathsf{le}}) - (\tilde{a} - c) \sum_{i=1}^{4} \tilde{p}_i\, \tilde{q}_i\,. \tag{17.22}$$

The game theoretical solution of this inspection game – based on the strategy sets (17.20) and (17.18) instead of the behavioural strategy sets (17.13) and (17.14) – which is published in this monograph for the first time, is presented in

Lemma 17.2. *Given the generalized Thomas-Nisgav inspection game with $L = 4$ nights, $k = 2$ controls, and with errors of the second kind, i.e., $\Gamma(4,2)$, the extensive form of which is represented in Figure 17.2. The Smuggler's set of mixed strategies is given by (17.20), the Inspectorate's strategy set by (17.18), and the payoffs to both players by (17.21) and (17.22).*

Then a Nash equilibrium is given by the following equilibrium strategies and payoffs $Op^*_{4,2} := Op_{4,2}(\tilde{\mathbf{p}}^*, \tilde{\mathbf{q}}^*)$ and $In^*_{4,2} := In_{4,2}(\tilde{\mathbf{p}}^*, \tilde{\mathbf{q}}^*)$:

(i) For

$$\frac{1}{2} < \frac{1}{1-\beta}\,\frac{1}{1+b/d} \tag{17.23}$$

the Smuggler behaves illegally in the entire game $\Gamma(4,2)$, i.e., $\tilde{p}^*_{\mathsf{le}} = 0$. An equilibrium strategy of the Smuggler is given by

$$\tilde{p}^*_i = \frac{1}{4}\,, \qquad i = 4, \ldots, 1\,, \tag{17.24}$$

and an equilibrium strategy of Customs by

$$\tilde{q}_j^* = \frac{1}{2}\,, \qquad j = 4, \ldots, 1\,. \tag{17.25}$$

The equilibrium payoffs to the Smuggler and to Customs are

$$Op_{4,2}^* = d - (1-\beta)\,(b+d)\,\frac{1}{2} \qquad \text{and} \qquad In_{4,2}^* = -c + (1-\beta)\,(c-a)\,\frac{1}{2}\,.$$

(ii) For

$$\frac{1}{2} > \frac{1}{1-\beta}\,\frac{1}{1+b/d}$$

the Smuggler behaves legally in the entire game $\Gamma(4,2)$, i.e., $\tilde{p}_{\mathsf{le}}^* = 1$ and $\tilde{p}_i^* = 0$, $i = 4, \ldots, 1$. Equilibrium strategies of Customs are given by

$$\frac{1}{1-\beta}\,\frac{1}{1+b/d} \le \tilde{q}_j^*\,, \quad j = 4, \ldots, 1 \quad \text{with} \quad \sum_{j=1}^{4} \tilde{q}_j^* = 2\,, \tag{17.26}$$

where $\tilde{q}_j^*$, $j = 4, \ldots, 1$, given by (17.25) fulfils (17.26).

The equilibrium payoffs to the Smuggler and to Customs are $Op_{4,2}^* = In_{4,2}^* = 0$.

Proof. Ad (i): Using (17.21), (17.23) and (17.24), we get for any $\tilde{\mathbf{p}} \in \tilde{P}_{4,2}$

$$Op_{4,2}(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}^*) = \left(d - \frac{1}{2}\,(\tilde{b}+d)\right)\,(1-\tilde{p}_{\mathsf{le}}) \begin{cases} = Op_{4,2}^* & \text{for} \quad \tilde{p}_{\mathsf{le}} = 0 \\ < Op_{4,2}^* & \text{for} \quad \tilde{p}_{\mathsf{le}} > 0 \end{cases}\,,$$

i.e., the Nash inequality for the Smuggler is fulfilled. For Customs, (17.22) and (17.24) with $\tilde{p}_{\mathsf{le}}^* = 0$ imply for any $\tilde{\mathbf{q}} \in \tilde{Q}_{4,2}$

$$In_{4,2}(\tilde{\mathbf{p}}^*, \tilde{\mathbf{q}}) = -c - \frac{1}{4}\,(\tilde{a}-c)\,\sum_{i=1}^{4} \tilde{q}_i = In_{4,2}^*\,.$$

Thus, Customer's Nash inequality is fulfilled as equality.

Ad (ii): Applying the Nash criterion for the Smuggler to (17.21) yields (17.26). □

Note that (17.25) is a robust equilibrium strategy; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

So far, we have determined the equilibrium strategy of Customs in terms of the $\tilde{q}_j$ and not in the original ones. The relation between $\tilde{q}_4, \tilde{q}_3, \tilde{q}_2, \tilde{q}_1$ and $q_{42}, q_{3,2}, q_{3,1}, q_{2,1}$ is given by (17.17). Using (17.25), the solution of (17.17) are given by

$$q_{4,2}^* = \frac{1}{2}\,, \qquad q_{3,2}^* = \frac{2}{3}\,, q_{3,1}^* = \frac{1}{3} \qquad \text{and} \qquad q_{2,1}^* = \frac{1}{2} \tag{17.27}$$

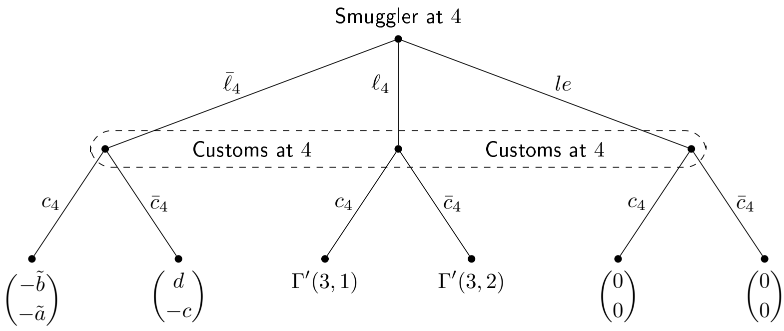
and

$$q_{4,2}^* = \frac{1}{2}\,, \quad q_{3,1}^* + q_{3,2}^* = 1\,, \quad q_{3,2}^*\,(2\,q_{2,1}^* - 1) = 0 \quad \text{and} \quad q_{3,2}^*\,(1 - 2\,q_{2,1}^*) = 0\,. \tag{17.28}$$

Thus, $(q^*_{3,2}, q^*_{3,1}) = (1/4, 3/4)$ or $(q^*_{3,2}, q^*_{3,1}) = (0, 1)$ can be chosen in (17.27). Note that only (17.27) will be covered in the recursive approach in Theorem 17.1. We will come back to this issue on p. 368.

How can we solve the generalized Thomas-Nisgav inspection game $\Gamma(L, k)$? Let us consider again the extensive form of the game $\Gamma(4, 2)$ in Figure 17.2. If we cut the information sets of Customs as described on p. 359 we can represent this game in recursive extensive form as shown in Figure 17.3, where the subgames $\Gamma'(3, 1)$ and $\Gamma'(3, 2)$ represent *only* the "illegal" parts of the game $\Gamma(3, 1)$ and $\Gamma(3, 2)$.

Figure 17.3 Reduced extensive form of the inspection game in Figure 17.2. $\Gamma'(3, 1)$ and $\Gamma'(3, 2)$ are the "illegal" parts of the game $\Gamma(3, 1)$ and $\Gamma(3, 2)$.

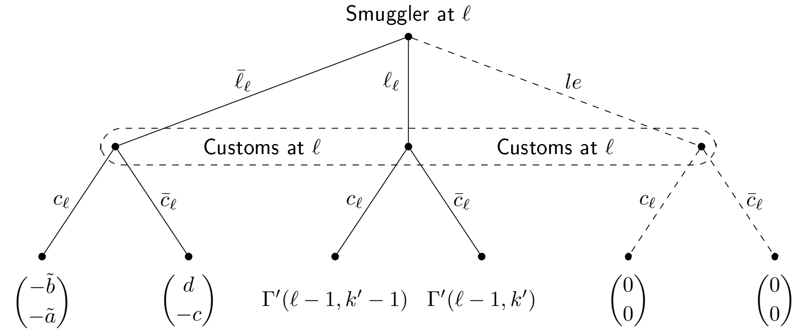


We can solve this recursive extensive form game by using the corresponding recursive normal form representation and by replacing $\Gamma'(3, 1)$ resp. $\Gamma'(3, 2)$ by their equilibrium payoffs $Op^*_{3,1}$ and $In^*_{3,1}$ resp. $Op^*_{3,2}$ and $In^*_{3,2}$. This procedure leads us to a recursive treatment of the general game $\Gamma(L, k)$ which we applied already to the Dresher-Höpfinger inspection game. Figure 17.4 represents the recursive extensive form as well as the corresponding recursive normal form of the generalized Thomas-Nisgav inspection game $\Gamma(L, k)$. Comparing this Figure with Figure 16.3 we see that the "illegal" parts are identical if we replace a and b by $\tilde{a}$ and $\tilde{b}$. Again, we do not expect that the reader is confused by the notation $\bar{\ell}_\ell$ and ℓ_ℓ; see p. 320.

Let $\bar{p}_{L,k}$ denote the probability to smuggle in the first night ($\bar{\ell}_L$ at L), $p_{L,k}$ denote the probability not to smuggle in the first night (ℓ_L at L), and p_{le} be the probability not to smuggle at all. Suppose ℓ nights with $2 \leq \ell < L$ are left for smuggling and Customs has k' controls at its disposal, $1 \leq k' \leq \min(\ell - 1, k)$. Then $\bar{p}_{\ell,k'}$ denotes the probability to smuggle in the next night, i.e., in the first of ℓ remaining nights. This implies – because behaving legally is not a choice in this subgame – that $1 - \bar{p}_{\ell,k'}$ is the probability not to smuggle in the next night. For Customs, let $q_{\ell,k'}$ be the probability to control in the next night. The cases $\ell = 1$ as well as $k' = 0$ and $k' = \ell$ are excluded from the game theoretical analysis; see p. 320 and the boundary conditions (17.29).

As argued on p. 321 we do not introduce the sets of behavioural strategies. The (expected) payoffs to both players is defined recursively using the normal form representation in Figure 17.4, where $Op_{\ell-1,\cdot}$ and $In_{\ell-1,\cdot}$ denote the (expected) payoffs to both players in the subgame with $\ell - 1$ steps and k' or $k' - 1$ controls left. For brevity, we suppress in $Op_{\ell-1,\cdot}$ and $In_{\ell-1,\cdot}$ Smuggler's and Customs strategies in the remaining ℓ nights. The same boundary conditions

Figure 17.4 Recursive extensive form and corresponding recursive normal form of the subgame $\Gamma(\ell, k')$ of the generalized Thomas-Nisgav inspection game $\Gamma(L, k)$, if ℓ nights and k' controls are left, and the Smuggler behaves legally during the nights $L, \ldots, \ell+1$ ($2 \leq \ell \leq L$, $1 \leq k' \leq \min(\ell-1, k)$). The edge labelled by le applies only in case of $\ell = L$.



		c_ℓ	$\bar{c}_\ell$
	$\bar{\ell}_\ell$	$-\tilde{a}$ $-\tilde{b}$	$-c$ d
	ℓ_ℓ	$In_{\ell-1,k'-1}$ $Op_{\ell-1,k'-1}$	$In_{\ell-1,k'}$ $Op_{\ell-1,k'}$
applies only in case of $\ell = L$	le	0 0	0 0

as that of the Dresher-Höpfinger inspection game with errors of the second kind, see (16.37), have to be met. We repeat them here for easier reference

$$Op^*_{\ell,k'} = \begin{cases} -\tilde{b} & \text{for} \quad k' = \ell \\ d & \text{for} \quad k' = 0 \end{cases} \qquad \text{and} \qquad In^*_{\ell,k'} = \begin{cases} -\tilde{a} & \text{for} \quad k' = \ell \\ -c & \text{for} \quad k' = 0 \end{cases} \tag{17.29}$$

for all $1 \leq \ell \leq L$.

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Theorem 17.1. *Given the generalized Thomas-Nisgav inspection game with $L > k$ nights, k controls, and with errors of the second kind, i.e., $\Gamma(L, k)$, the recursive extensive and normal forms of which are represented in Figure 17.4. The payoffs to both players are defined recursively using the recursive normal form representation in Figure 17.4, and the equilibrium payoffs to both players fulfil the boundary conditions (17.29).*

Suppose ℓ nights, $2 \leq \ell \leq L$, are left for smuggling, Customs has k' controls at its disposal, $1 \leq k' \leq \min(\ell-1, k)$, and the Smuggler behaves legally at steps $L, \ldots, \ell+1$, i.e., the subgame $\Gamma(\ell, k')$ is reached.

Then a Nash equilibrium in the subgame $\Gamma(\ell,k')$ is given by the following equilibrium strategies and payoffs $Op^*_{\ell,k'}$ and $In^*_{\ell,k'}$:

(i) For

$$\frac{k}{L} < \frac{1}{1-\beta}\,\frac{1}{1+b/d} \tag{17.30}$$

the Smuggler behaves illegally in the entire game $\Gamma(L,k)$, i.e., $p^*_{le}=0$. An equilibrium strategy of the Smuggler is given by

$$\bar{p}^*_{\ell,k'} = \frac{1}{\ell}\,, \tag{17.31}$$

and an equilibrium strategy of Customs by

$$q^*_{\ell,k'} = \frac{k'}{\ell}\,. \tag{17.32}$$

The equilibrium payoffs to the Smuggler and to Customs in the subgame $\Gamma(\ell,k')$ are

$$Op^*_{\ell,k'} = d-(1-\beta)\,(b+d)\,\frac{k'}{\ell} \quad \text{and} \quad In^*_{\ell,k'} = -c+(1-\beta)\,(c-a)\,\frac{k'}{\ell}\,, \tag{17.33}$$

which – for $\ell=L$ and $k'=k$ – are the equilibrium payoffs of the entire game $\Gamma(L,k)$.

(ii) For

$$\frac{k}{L} > \frac{1}{1-\beta}\,\frac{1}{1+b/d} \tag{17.34}$$

the Smuggler behaves legally in the entire game $\Gamma(L,k)$, i.e., $p^*_{le}=1$, and $\bar{p}^*_{\ell,k'}=p^*_{\ell,k'}=0$ for all $\ell=2,\ldots,L$ and all $k'=1,\ldots,\ell$. Equilibrium strategies of Customs are given by the set of inequalities

$$0 \ge Op_{L,k}(i,\mathbf{q}^*) \qquad \text{for} \qquad i=\bar{\ell}_L,\ell_L\bar{\ell}_{L-1},\ell_L\ell_{L-1}\bar{\ell}_{L-2},\ldots,\ell_L\ldots\ell_2\bar{\ell}_1\,, \tag{17.35}$$

where $q^*_{\ell,k'}$ given by (17.32) fulfils (17.35).

The equilibrium payoffs to the Smuggler and to Customs are $Op^*_{L,k}=In^*_{L,k}=0$.

Proof. Ad (i): As mentioned, comparing Figure 17.4 with Figure 16.3 we see that the "illegal" parts are identical if we replace a and b by $\tilde{a}$ and $\tilde{b}$. Thus, part (i) of Theorem 17.1 corresponds to part (ii) of Theorem 16.2.

Ad (ii): The equilibrium strategies $\mathbf{q}^*$ of Customs have to satisfy the Nash equilibrium condition for the Smuggler. Formulating this condition in terms of the pure strategies of the Smuggler, see Section 19.2, we obtain the inequalities (17.35). □

Let us comment the results of Theorem 17.1: First, comparing the equilibrium strategies and payoffs as given by (17.31) – (17.33) with (16.40) – (16.42), and with (15.76) – (15.78), we see that they coincide, which means that the equilibrium strategies and payoffs of the "illegal" generalized Thomas-Nisgav inspection game are the same as that of the Dresher-Höpfinger inspection game with errors of the second kind and $1<1/((1-\beta)(1+b/d))$ which are the same as that for the Canty-Rothenstein-Avenhaus inspection game for $\alpha=0$. In the latter case the conditions for illegal and legal behaviour are even the same as those given here; compare (15.75) with (17.30) and (15.79) with (17.34). However, they are still different games from a structural point of view! On p. 391 we will present an explanation for these surprising properties: We will show that all games mentioned above are strategically equivalent to zero-sum games with the probabilities of not detecting the illegal activity as payoffs to the Operator/Smuggler. Note that like in Section 16.2 the Smuggler's probability $\bar{p}^*_{\ell,k'}$ as given by (17.31) only depends on the number of steps left, and thus, they form a harmonic progression; see also Table 4.1 on p. 72 for an overview of inspection games with this property.

Second, the equilibrium strategy of Customs in case of the Smuggler's legal behaviour, case (ii), are given implicitly, because their explicit form appears to be infeasible for the general case. The first inequality in (17.35) for $i = \bar{\ell}_L$ is

$$0 \geq q^*_{L,k}\,(-\tilde{b}) + (1 - q^*_{L,k})\,d\,,$$

or, equivalently

$$\frac{1}{1-\beta}\,\frac{1}{1+b/d} \leq q^*_{L,k}$$

and corresponds to (17.12) and (17.26). Furthermore, the equilibrium strategy (17.32) of Customs is a robust equilibrium strategy; see also Table 7.3 on p. 142 for an overview of inspection games with a robust Inspectorate's equilibrium strategy.

Third, as mentioned on p. 356, in case of $a = -1$, $c = 1$, and $b = d = 1$ the *original* Thomas-Nisgav inspection game is considered and (17.33) implies

$$In^*_{L,k} = -1 + (1-\beta)\,2\,\frac{k}{L}\,, \tag{17.36}$$

which is the result obtained by Thomas and Nisgav (1976). Note that conditions (17.30) and (17.34) does not play any role in Thomas and Nisgav's work, because the Smuggler must behave illegally and will get – in case (17.34) is met – a negative equilibrium payoff. As mentioned on p. 327, the change of the modelling assumption "at most one illegal activity" in the Dresher-Höpfinger inspection game to the assumption "(exactly) one illegal activity" in the original Thomas-Nisgav inspection game leads to a considerable change in the game theoretical solutions; see (16.11) – (16.13) for $b = d = 1$, $c = 1$ and $a = -1$ in contrast to (17.31) – (17.33) for $b = d = 1$ and $\beta = 0$. A numerical comparison can be found in Krieger and Avenhaus (2018b).

Fourth, because Theorem 17.1 is based on the *recursive* extensive form in Figure 17.4, it remains an open problem if the game theoretical solution of the generalized Thomas-Nisgav game in *extensive form representation* has the same solution as given by Theorem 17.1. For that purpose it has to be proven that the extensive form representation can indeed be transformed in a *recursive* extensive form, i.e., if information sets can be cut in a way that leads to the recursive extensive form in Figure 17.4. For the game $\Gamma(4,2)$ this could be directly seen by inspecting Figure 17.2 which led to Figure 17.3; see also the comment on p. 371.

Finally, we realize that the recursive procedure leads in case (i) to a unique equilibrium strategy of Customs, but that the analysis of the game $\Gamma(4,2)$ shows that the solution is not unique; see (17.28). Of course, the special equilibrium strategy of Customs in Theorem 17.1 is for the game $\Gamma(4,2)$ contained in the set of equilibrium strategies given in Lemma 17.2. We encountered the same situation in Chapter 15; see the comment on p. 295. More than that, for $\alpha = 0$

the equilibrium payoffs to both players given in Corollary 15.1 are the same as those given in Theorem 17.1. Whereas at first sight this is surprising, this is no longer so if one reflects the assumptions for the generalized Thomas-Nisgav game: Because it is not important in which night the Smuggler is caught or not, and because he decides at the beginning whether to behave legally during the next L nights or not, the No-No inspection game with $\alpha = 0$ analysed in Corollary 15.1 is basically equivalent to the Se-Se inspection game considered here in the sense that the equilibrium strategies can be transformed into each other and that they lead to the same payoffs to both players.

Combining this insight with the results for the game $\Gamma(4,2)$ as given by Lemma 17.2 we are led to the following

Conjecture 17.1. *Given the generalized Thomas-Nisgav inspection game with $L > k$ nights, k controls, and with errors of the second kind, i.e., $\Gamma(L,k)$. Let $\tilde{p}_i$, $i = L,\dots,1$, be the Smuggler's probability to smuggle in the $L-i+1$-th night, i.e., $i = L$ refers to the illegal strategy $\bar{\ell}_L$, $i = L-1,\dots,1$ to the illegal strategy $\ell_L\ell_{L-1}\dots\ell_{i+1}\bar{\ell}_i$, and $i = 0$ to legal behaviour $\ell_L\ell_{L-1}\dots\ell_1$, and let $\tilde{q}_j$, $j = L,\dots,1$, be Customs's probability to control in the $L-i+1$-th night.*

In case of illegal behaviour of the Smuggler – case (i) of Theorem 17.1 – an equilibrium strategy of the Smuggler is given by

$$\tilde{p}_i^* = \frac{1}{L}\,, \qquad i = L,\dots,1 \qquad \textit{and} \qquad \tilde{p}_{le}^* = 0\,,$$

and that of Customs by

$$\tilde{q}_j^* = \frac{k}{L}\,, \qquad j = L,\dots,1\,.$$

The equilibrium payoffs to the Smuggler and to Customs are given by (17.33).

In case of legal behaviour of the Smuggler – case (ii) of Theorem 17.1 – an equilibrium strategy of Customs is given by

$$\frac{1}{1-\beta}\,\frac{1}{1+b/d} \le \tilde{q}_j^*\,, \quad j = L,\dots,1 \quad \textit{with} \quad \sum_{j=1}^{L} \tilde{q}_j^* = k\,.$$

The equilibrium payoffs to the Smuggler and to Customs are $Op_{L,k}^* = In_{L,k}^* = 0$.

Note that there are $\binom{L}{k}$ possibilities of Customs for distributing its k controls on the L nights; only in the special cases $k = 1$ and $k = L-1$ control(s) the probabilities for these possibilities are in case of illegal behaviour of the Smuggler uniquely determined by the q_j^*. This was the reason why we considered the game $\Gamma(4,2)$ instead of the simpler $\Gamma(3,2)$: In the latter one we would not have detected the non-uniqueness of the solution of the general game.

So far, we could have interpreted this game, like the Dresher-Höpfinger inspection game, in terms of inspections and critical times. But then, Thomas and Nisgav (1976) proceed to consider a situation where Customs has two patrol boats, which can be used during the same night, and here, there is no longer a reasonable interpretation in terms of inspections. They distinguish two cases: 1) identical boats of Customs, i.e., boats which are characterized by the same detection probability, and 2) two types of boats, i.e., having different detection probabilities. For both cases, Thomas and Nisgav formulate the respective linear programming problems without solving them. It was the achievement of Baston and Bostock (1991) to present the optimal payoff to Customs resp. Smuggler explicitly for case 2); see Theorem 17.2 below. However, they did not provide any optimal strategies.

For case 2) of two non-identical boats the assumptions on p. 355 are further specified as follows:

(iii") The Smuggler performs an illegal activity once at one of the steps $L, \ldots, 1$.

(iv") During a control Customs may commit an error of the second kind. If only patrol boat i, $i = 1, 2$, is used during the night in which the Smuggler attempts to cross the strait, then $w_i \in (0, 1]$ is the probability that the Smuggler is detected and therefore caught.

If both patrol boats are used during the night in which the Smuggler attempts to cross the strait, then w with $w > max(w_1, w_2)$ is the probability that the Smuggler is detected and therefore caught.

The detection probabilities w_1, w_2 and w are the same for all controls.

(vi") The Smuggler decides at the beginning, i.e., at step L, whether to behave illegally at that step. If the Smuggler behaves legally at steps $L, \ldots, \ell + 1$ $(1 \leq \ell \leq L - 1)$, then he decides whether to behave illegally at step ℓ; and so on. Because of assumption (iii"), the Smuggler behaves illegally latest at step 1.

Customs decides at the beginning whether to control at step L. If it has still controls at its disposal, then Customs decides at step $L - 1$ whether to control at that step; and so on.

(viii") The payoffs to the two players (Smuggler, Customs) are given by

$$\begin{array}{ll} (1, -1) & \text{for undetected smuggling} \\ (-1, 1) & \text{for detected smuggling}\,. \end{array} \tag{17.37}$$

(x") The game ends either at the step at which the Smuggler behaves illegally, or at that step at which the number of controls for each boat left is zero, or at that step at which the number of controls left for each boat is equal to the number of steps left, or at step 1.

Three comments on these assumptions: First, note that while so far in this monograph the detection probabilities are denoted by $1 - \beta$, we use in assumption (iv") and in the remainder of this section the notation w, w_1 and w_2 in order to make the following analysis easier to be comparable to the original work by Baston and Bostock (1991), and to keep the equations as simple as possible. Also note that we exclude the cases $w = max(w_1, w_2)$ and $w = w_1 + w_2$, because w_1, w_2, and w cannot be estimated precisely.

Second, (17.37) indicates that Baston and Bostock (1991) model this situation as a zero-sum game with the same payoffs as used by Thomas and Nisgav (1976); see p. 356. If the Smuggler behaves illegally during the same night in which Customs controls, then the (expected) payoff to the Smuggler is given by

$$\begin{array}{ll} 1 - 2\,w_1 & \text{if the first boat is used} \\ 1 - 2\,w_2 & \text{if the second boat is used} \\ 1 - 2\,w & \text{if both boats are used.} \end{array}$$

Third, again we note with respect to assumption (x') that if the Smuggler behaves illegally at step i, $i = L, ..., 1$, then the game ends at step i regardless whether the illegal behaviour is detected at that step or not. In the latter case, the Smuggler has successfully performed his illegal activity and thus, the game ends as well.

Let $\Gamma(L, k_1, k_2)$ denote the inspection game in which the Smuggler makes the attempt to cross the strait in one of the L nights, and in which Customs has $k_i(> 0)$ patrols available for patrol boat i, $i = 1, 2$. It is also assumed that if the Smuggler does not attempt to cross the strait during a given night then he learns what patrol boats Customs used during that night.

Suppose there are ℓ nights left ($2 \leq \ell \leq L$), and Customs has $1 \leq k_1' \leq \min(\ell - 1, k_1)$ and $1 \leq k_2' \leq \min(\ell - 1, k_2)$ patrols left for each boat. If the Smuggler has not crossed the strait during the nights $L, \ldots, \ell + 1$, then the subgame $\Gamma(\ell, k_1', k_2')$ is reached, in which the (expected) payoffs to the Smuggler is denoted by $Op_{\ell,k_1',k_2'}$. Again, for brevity, we suppress in the expression $Op_{\ell-1,\cdot,\cdot}$ Smuggler's and Customs strategies in the remaining ℓ nights. The recursive normal form of the Baston-Bostock inspection game is given in Table 17.2.

Table 17.2 Recursive normal form of the subgame $\Gamma(\ell, k_1', k_2')$ of the Baston-Bostock inspection game $\Gamma(L, k_1, k_2)$, if ℓ nights and k_1' resp. k_2' controls are left, and the Smuggler behaves legally during the nights $L, \ldots, \ell+1$ ($2 \leq \ell \leq L$, $1 \leq k_1' \leq \min(\ell-1, k_1)$, $1 \leq k_2' \leq \min(\ell-1, k_2)$).

	patrol with 1st boat	patrol with 2nd boat	patrol with 1st and 2nd boat	no patrol
$\bar{\ell}_\ell$	$1 - 2\,w_1$	$1 - 2\,w_2$	$1 - 2\,w$	1
ℓ_ℓ	$Op_{\ell-1,k_1'-1,k_2'}$	$Op_{\ell-1,k_1',k_2'-1}$	$Op_{\ell-1,k_1'-1,k_2'-1}$	$Op_{\ell-1,k_1',k_2'}$

We mentioned on p. 368 that a formal proof for cutting the information sets in the generalized Thomas-Nisgav inspection game remains an open task. Note that from here on and especially in the next section only the recursive normal form of the inspection games are considered and not the respective extensive form games which would be appropriate to illustrate the information situation of the players during the course of the game. A potential trap when starting right away with the recursive normal form is that crucial information sets are not recognized. What would be the right approach from our point of view? There are two: First, one starts with modelling the inspection game in extensive form and proves that information sets can be cut in a way that leads to a recursive extensive form game and then to a recursive normal form game. The only inspection game in this Chapter for which these steps were performed consequently is von Stengel's inspection game in Section 17.2. Second, one assumes that after each step/night both players have full information, which can lead to strange modelling assumptions; see Hohzaki's inspection game in Section 17.2. However, also in this case the recursive approach can be justified.

Let $\bar{p}_{L,k_1,k_2}$ denote the probability to smuggle in the first night, and p_{L,k_1,k_2} denote the probability to postpone the smuggling. Note that because the Smuggler has to cross the strait, the probability p_{le} which had to be considered in the generalized Thomas-Nisgav inspection game, is zero. For any ℓ with $2 \leq \ell < L$, suppose that the Smuggler behaves legally during the nights $L, \ldots, \ell + 1$, and Customs has $1 \leq k_1' \leq \min(\ell - 1, k_1)$ resp. $1 \leq k_2' \leq \min(\ell - 1, k_2)$ resources left for boat 1 resp. 2, i.e., the game has reached the subgame $\Gamma(\ell, k_1', k_2')$. Then $\bar{p}_{\ell,k_1',k_2'}$ denotes the probability to smuggle in the $(L - \ell + 1)$-th night, and $p_{\ell,k_1',k_2'}$ the probability to postpone the smuggling again. Because the Smuggler has to cross the strait, we have $\bar{p}_{1,k_1',k_2'} = 1$.

In the generalized Thomas-Nisgav inspection game only the probability $q_{\ell,k'}$ for Customs to control in the next night had to be considered. In the Baston-Bostock inspection game, however, it also has to be distinguished which boat is used. In order to keep the notation simple, let q_0 denote the probability that no patrol is scheduled for the next night, let q_i denote the probability that patrol boat i, $i = 1, 2$, is going to patrol next night, and finally let q_{12} denote the probability that patrol boats 1 and 2 are going to patrol next night. It is clear that $q_0 + q_1 + q_2 + q_{12} = 1$. Note that we ignore ℓ, k_1' and k_2' in the $q.$ for the sake of brevity.

Again, in order to solve the recursive inspection game, boundary conditions for the optimal payoff to the Smuggler have to be met. For all $1 \le \ell \le L$ we have

$$Op^*_{\ell,k_1',k_2'} = \begin{cases} 1 & \text{for} \quad k_1' = k_2' = 0 \\ 1 - \dfrac{2\,k_1'\,w_1}{\ell} & \text{for} \quad 1 \le k_1' \le \min(\ell, k_1), k_2' = 0 \\ 1 - \dfrac{2\,k_2'\,w_2}{\ell} & \text{for} \quad k_1' = 0, 1 \le k_2' \le \min(\ell, k_2) \\ 1 - 2\,w & \text{for} \quad k_1' = k_2' = \ell \end{cases} \tag{17.38}$$

which can be justified as follows: If no control is left, then the payoff to the Smuggler is one. If only resources for one boat are left, i.e., $k_1' > 0$ and $k_2' = 0$ or vice versa, then the original Thomas-Nisgav inspection situation is met, and the boundary conditions are given by (17.33) with $b = d = -1$. In case $k_1' = k_2' = \ell$, i.e., the number of patrols left for each boat is the same as the number of nights left, then Customs controls in any of the remaining ℓ nights with both boats, and the payoff to the Smuggler is $1 - 2\,w$.

Although Baston and Bostock (1991) have only derived the optimal payoff to the Smuggler/Customs, we provide in addition the corresponding optimal strategies of both players, which were found using M. Canty's Mathematica® programs; see Canty (2003).

The game theoretical solution of this inspection game is presented in

Theorem 17.2. *Given the Baston-Bostock inspection game with L nights, $1 \le k_1 \le k_2 \le L$ patrols for boat 1 and 2, and with detection probabilities w_1, w_2 and w, i.e., $\Gamma(L, k_1, k_2)$, the recursive normal form of which is given in Table 17.2. The payoff to the Smuggler is defined recursively using the recursive normal form representation in Table 17.2, and the optimal payoff to the Smuggler fulfils the boundary conditions (17.38).*

Suppose ℓ nights, $2 \le \ell \le L$, are left for smuggling, Customs has k_1' and k_2' resources for boat 1 resp. 2 at its disposal, $0 \le k_1' \le \min(\ell - 1, k_1)$ and $0 \le k_2' \le \min(\ell - 1, k_2))$, and the Smuggler behaves legally during the nights $L, \ldots, \ell + 1$, i.e., the subgame $\Gamma(\ell, k_1', k_2')$ is reached.

Then optimal strategies and the optimal payoff $Op^*_{\ell,k_1',k_2'}$ to the Smuggler in the subgame $\Gamma(\ell, k_1', k_2')$ are given as follows: An optimal strategy of the Smuggler is given by

$$\bar{p}^*_{\ell,k_1',k_2'} = \frac{1}{\ell}\,. \tag{17.39}$$

In case $k_1' = 0$ and $k_2' \ge 1$ an optimal strategy of Customs is given by $q_2^* = k_2'/\ell$ and in case of $k_1' \ge 1$ and $k_2' = 0$ by $q_1^* = k_1'/\ell$. If $k_1'\,k_2' > 0$ an optimal strategy of Customs is given by

1) For $w > w_1 + w_2$ by

$$q_{12}^* = \frac{1}{w}\,\frac{1}{\ell}\,\left(k_1'\,w + (k_2' - k_1')\,w_2\right)\,, \qquad q_0^* = 1 - q_{12}^*\,, \qquad q_1^* = q_2^* = 0\,. \tag{17.40}$$

2) For $w < w_1 + w_2$ and

- *for $k_1' + k_2' \le \ell$ and*
  - $w_1 < w_2$ *by*

$$q_2^* = \frac{1}{w_2}\,\frac{1}{\ell}\,\left(k_1'\,w_1 + k_2'\,w_2\right)\,, \qquad q_0^* = 1 - q_2^*\,, \qquad q_1^* = q_{12}^* = 0\,. \tag{17.41}$$

  - $w_1 > w_2$ *by*

$$q_1^* = \frac{1}{w_1}\,\frac{1}{\ell}\,\left(k_1'\,w_1 + k_2'\,w_2\right)\,, \qquad q_0^* = 1 - q_1^*\,, \qquad q_2^* = q_{12}^* = 0\,. \tag{17.42}$$

- *for $k_1' + k_2' \ge \ell$ and*
  - $w_1 < w_2$ *by*

$$\begin{aligned} q_2^* &= \frac{1}{w - w_2}\,\frac{1}{\ell}\,\left((2\,\ell - k_1' - k_2')\,w - (\ell - k_2')\,w_1 - (\ell - k_1')\,w_2\right)\,, \\ q_{12}^* &= 1 - q_2^*\,, \qquad q_0^* = q_1^* = 0\,. \end{aligned} \tag{17.43}$$

  - $w_1 > w_2$ *by*

$$\begin{aligned} q_1^* &= \frac{1}{w - w_1}\,\frac{1}{\ell}\,\left((2\,\ell - k_1' - k_2')\,w - (\ell - k_2')\,w_1 - (\ell - k_1')\,w_2\right)\,, \\ q_{12}^* &= 1 - q_1^*\,, \qquad q_0^* = q_2^* = 0\,. \end{aligned} \tag{17.44}$$

The optimal payoff to the Smuggler in the subgame $\Gamma(\ell, k_1', k_2')$ is

1) for $w > w_1 + w_2$

$$Op_{\ell,k_1',k_2'}^* = 1 - \frac{2}{\ell}\left(k_1'\,w + (k_2' - k_1')\,w_2\right)\,, \tag{17.45}$$

2) for $w < w_1 + w_2$

$$Op_{\ell,k_1',k_2'}^* = \begin{cases} 1 - \dfrac{2}{\ell}\left(k_1'\,w_1 + k_2'\,w_2\right) & \text{for} \quad k_1' + k_2' \le \ell \\ 1 - \dfrac{2}{\ell}\Big((k_1' + k_2' - \ell)\,w & \\ \qquad\qquad + (\ell - k_2')\,w_1 + (\ell - k_1')\,w_2\Big) & \text{for} \quad k_1' + k_2' \ge \ell \end{cases}\,, \tag{17.46}$$

which – for $\ell = L$, $k_1' = k_1$ and $k_2' = k_2$ – is the optimal payoff to the Smuggler of the entire game $\Gamma(L, k_1, k_2)$.

Proof. According to Figure 17.2 we deal with a 2×4 zero-sum matrix game. In the following we will only prove solution (17.40); the other ones can be shown in the same way.

Mixing the 3rd and 4th column we get with $q_1^* = q_2^* = 0$, $q^* = q_{12}^*$ and $1 - q^* = q_0^*$ the following condition for the optimal payoff $Op^*_{\ell,k_1',k_2'}$, which makes the Smuggler indifferent with regard to the choice of his pure strategies, see Theorem 19.1:

$$\begin{aligned} Op^*_{\ell,k_1',k_2'} &= q^*\,(1-2\,w) + (1-q^*) \\ &= q^*\,Op^*_{\ell-1,k_1'-1,k_2'-1} + (1-q^*)\,Op^*_{\ell-1,k_1',k_2'}\,, \end{aligned} \tag{17.47}$$

which leads to

$$q^* = \frac{1 - Op^*_{\ell-1,k_1',k_2'}}{2\,w + Op^*_{\ell-1,k_1'-1,k_2'-1} - Op^*_{\ell-1,k_1',k_2'}}\,. \tag{17.48}$$

Inserting (17.48) into the first equation of (17.47) leads to the recursive relation for the optimal payoff

$$Op^*_{\ell,k_1',k_2'} - 1 = -2\,w\,\frac{1 - Op^*_{\ell-1,k_1',k_2'}}{2\,w + Op^*_{\ell-1,k_1'-1,k_2'-1} - Op^*_{\ell-1,k_1',k_2'}}\,. \tag{17.49}$$

It can be seen immediately that (17.45) fulfils (17.49) as well as the 1st, 3rd and 4th boundary condition in (17.38). In the comment below on the asymmetry of (17.40) and (17.45) it is shown that $k_1 \leq k_2$ implies $k_1' \leq k_2'$, and thus, the 2nd boundary condition in (17.38) is not applied. Inserting (17.45) into (17.48) leads to (17.40).

Using (17.39) and (17.45), it can be seen immediately that

$$Op^*_{\ell,k_1',k_2'} = (1-2\,w)\,\frac{1}{\ell} + Op^*_{\ell-1,k_1'-1,k_2'-1}\,\left(1 - \frac{1}{\ell}\right)$$

$$Op^*_{\ell,k_1',k_2'} = \frac{1}{\ell} + Op^*_{\ell-1,k_1',k_2'}\,\left(1 - \frac{1}{\ell}\right)\,.$$

In order to prove that the saddle point criterion for Customs is satisfied, we also need to consider the 1st and 2nd column; we have to show that

$$Op^*_{\ell,k_1',k_2'} \leq (1-2\,w_1)\,\frac{1}{\ell} + Op^*_{\ell-1,k_1'-1,k_2'}\,\left(1 - \frac{1}{\ell}\right) \tag{17.50}$$

$$Op^*_{\ell,k_1',k_2'} \leq (1-2\,w_2)\,\frac{1}{\ell} + Op^*_{\ell-1,k_1',k_2'-1}\,\left(1 - \frac{1}{\ell}\right)\,. \tag{17.51}$$

Using (17.45), it can be seen that (17.50) holds only because of the condition $w > w_1 + w_2$, whereas (17.51) is fulfilled as equality.

As mentioned above, (17.46) can be obtained in the same way as demonstrated for (17.45): (17.41) and (17.42) are obtained by mixing the 2nd and 4th, respectively the 1st and 4th columns, and (17.43) and (17.44) are obtained by mixing the 2nd and 3rd, respectively the 1st and 3rd columns. □

Let us comment the results of Theorem 17.2: First, as in the generalized Thomas-Nisgav inspection game, see case (i) in Theorem 17.1, it is remarkable that the optimal strategy

(17.39) of the Smuggler is independent of Customs' detection probabilities w_1, w_2 and w, and independent of the inspections resources k_1' and k_2' available. The Smuggler's probability $\bar{p}^*_{\ell,k_1',k_2'}$ as given by (17.39) only depends on the number of steps left, and thus, they form a harmonic progression; see also Table 4.1 on p. 72 for an overview of inspection games with this property.

Second, the optimal payoff in the Thomas-Nisgav inspection game can be obtained from (17.45) as follows: Let us assume that no patrol boat 1 is available, i.e., $(k_1' \leq) k_1 = 0$. Then (17.45) simplifies to

$$Op^*_{L,0,k_2} = 1 - \frac{2}{L}\, k_2\, w_2\,,$$

regardless whether $w > w_1 + w_2$ or $w < w_1 + w_2$, i.e., the result obtained by Thomas and Nisgav (1976); see (17.36).

Third, we see that the expressions (17.41) – (17.44) and (17.46) are somehow symmetric in k_1' and k_2'. This is not true for (17.40) and (17.45). Why? Because we have assumed w.l.o.g. that $k_1 \leq k_2$ and because (17.40) implies $q^*_{12} > 0$ and $q^*_0 > 0$, we know that before the second night either $k_1' = k_1$ and $k_2' = k_2$ or $k_1' = k_1 - 1$ and $k_2' = k_2 - 1$ must hold, i.e., $k_1' \leq k_2'$, and so on. Thus, the asymmetry in (17.40) and (17.45) is due to the assumption $k_1 \leq k_2$.

Fourth, although only one optimal strategy of Customs is given in Theorem 17.2, in case 1) there are two additional optimal strategies of Customs which are obtained by mixing the 2nd and 4th, respectively the 2nd and 3rd columns, which require, however, additional conditions to k_1', k_2', ℓ, w_2 and w; see (17.51) which is fulfilled as equality in case 1). The same effect occurs in case 2).

Fifth, Customs might switch between the optimal probabilities (17.41) and (17.43) or (17.42) and (17.44), respectively. For example: Consider the game $\Gamma(6,2,3)$ and $w_1 < w_2$. Then the following subgames can be reached during the course of the game:

$$\Gamma(6,2,3) \overset{(17.41)}{\longrightarrow} \Gamma(5,2,3) \overset{(17.43)}{\longrightarrow} \Gamma(4,2,2) \overset{(17.43)}{\longrightarrow} \Gamma(3,1,1) \overset{(17.41)}{\longrightarrow} \dots .$$

Sixth, in Baston and Bostock (1991) it is mentioned, that because "nowhere [it is] assumed that $w_1 \neq w_2$, it is clear that the case when the two patrol boats are identical can be obtained by taking $w_1 = w_2$ in the above." The word *above* refers here to (17.45) and (17.46). It is true that formally $w_1 = w_2$ can be chosen. But what does that mean in practice? According to the recursive normal form of the Baston-Bostock inspection game in Table 17.2, they distinguish from our point of view even in the case $w_1 = w_2$ two types of patrol boats (maybe having different related costs, different boats need specialized staff for operation, etc.), but with the same detection probabilities $w_1 = w_2$. From our point of view they do *not* treat the case of identical patrol boats in the sense of Thomas and Nisgav (1976), i.e., case 1 on p. 369, because the recursive normal form of that inspection game is given in Table 17.3, and it is clearly different from that of the Baston-Bostock inspection game in Table 17.2.

Seventh, note that the optimal payoff depends on the relation between the probabilities w and $w_1 + w_2$. In case the two boats operate stochastically independently, then the non-detection probabilities are multiplied, i.e.

$$1 - w = (1 - w_1)\,(1 - w_2)$$

Table 17.3 Recursive normal form of the subgame $\Gamma(\ell, k')$ of the Thomas-Nisgav inspection game with two identical boats $\Gamma(L, k)$, if ℓ nights and k' controls for both patrol boats together are left, and the Smuggler behaves legally during the nights $L, \dots, \ell+1$ ($2 \leq \ell \leq L$, $2 \leq k' \leq \min(2\,\ell - 1, k)$).

	patrol with one boat	patrol with two boats	no patrol
$\bar{\ell}_\ell$	$1 - 2\,w_1$	$1 - 2\,w$	1
ℓ_ℓ	$Op_{\ell-1,k'-1}$	$Op_{\ell-1,k'-2}$	$Op_{\ell-1,k'-1}$

which implies, because $w_i \in (0, 1]$, that, beside $\max(w_1, w_2) < w$, we have

$$w < w_1 + w_2\,,$$

i.e., they are subadditive. Thus, (17.45) holds only if the two boats' detection activities are positively correlated.

Eight, what has been said on p. 368 regarding the generalized Thomas-Nisgav inspection game applies here as well: It remains an open problem whether the game theoretical solution of the Baston-Bostock inspection game in *extensive form representation* has the same solution as given by Theorem 17.2.

Finally, let us repeat that Baston and Bostock (1991) did not present optimal strategies of Customs. This is somewhat surprising since these optimal strategies are advices to Customs, in other words, important for those practitioners who are not able to derive these advices themselves.

But nevertheless, there are further interesting questions to the Baston-Bostock inspection game and its game theoretical solution which still wait for an answer. Why, for example, is the 1st column never used in an optimal strategy of Customs under the condition $w > w_1 + w_2$; see $q_1^* = 0$ in (17.40)? Or, why is the 3rd column never used in an optimal strategy of Customs under the conditions $w < w_1 + w_2$ and $k_1' + k_2' \leq \ell$; see $q_{12}^* = 0$ in (17.41) and (17.42)? Or, why is the 4th column never used in an optimal strategy of Customs under the conditions $w < w_1 + w_2$ and $k_1' + k_2' \geq \ell$; see q_0^* in (17.43) and (17.44)?

Garnaev (1991) and Garnaev (2000) extended the work by Baston and Bostock (1991) such that he considered three different types of control boats which are characterized by different detection probabilities. He used linear programming in order to find optimal strategies for some cases. He showed that also in the three boat variant of the inspection game an optimal strategy of the Smuggler is given by (17.39). He also noted that by its structure the Customs and Smuggler game is similar to the Searcher-Evader game with a time lag; see Gal (1980).

17.2 Multiple illegal activities: Models by von Stengel, by Sakaguchi, by Ferguson and Melolidakis, and by Hohzaki

This section is devoted to models which deal with multiple illegal activities. We will see that they do not only generalize the inspection models with exactly one and at most one illegal activity, but show also interesting and surprising structural relations to models considered in Chapter 16 and Section 17.1. It should be mentioned, however, that according to our best knowledge these models are not the results of studies of real world inspection problems; we will return to this issue on p. 392.

The players in the inspection games treated in this section are differently named by the authors. While the Operator or Smuggler is sometimes called inspectee or attacker, the Inspectorate or Customs is referred to as inspector or defender. In the following, due to the different possible applications, we call synonymously the two players Operator or Smuggler and Inspectorate or Customs, and we use the general terms step(s) and control(s).

Let us start with the inspection game by von Stengel (1991), for which assumptions (iii), (iv), (vi), (viii) and (x) of Chapter 14 are specified as follows:

(iii') The Operator may perform at most m illegal activities at the steps $L, \dots, 1$, i.e., m is the number of *intended* illegal activities, where he can behave illegally at most once per step.

(iv') During a control the Inspectorate does not commit any errors of the first and second kind, i.e., if the illegal activity is carried through at the same step at which a control is performed, then it is detected with certainty at that step.

(vi') The Operator decides at the beginning, i.e., at step L, whether to behave illegally at that step. At a step ℓ $(1 \leq \ell \leq L-1)$ and in case the Operator *can* still perform an illegal activity, see assumption (iii'), he decides whether to behave illegally at that step; and so on.

The Inspectorate decides at the beginning whether to control at step L. If it has still controls at its disposal, then the Inspectorate decides at step $L-1$ whether to control at that step; and so on.

(viii') The payoffs to the two players (Operator, Inspectorate) are given by

$$\begin{array}{rl} (1,-1) & \text{for an untimely control and illegal behaviour} \\ (-b,b) & \text{for a timely control and illegal behaviour} \\ (0,0) & \text{for legal behaviour}\,, \end{array}$$

with $b > 0$.

(x') The game ends either at the step at which the Smuggler behaves illegally and a control is performed, or at that step at which the number of controls left is zero, or at that step at which the number of controls left is equal to the number of steps left, or at that step at which the number of intended illegal activities left is zero, or at step 1. In case of a timely control, the Operator does not need to return the payoff he received for previous successful illegal activities.

The remaining assumptions of Chapter 14 except (ix) hold throughout this section.

As before, L and k denote the number of steps and controls, respectively. Note that like in Dresher (1962) the inspection game is a zero-sum game. Also note that von Stengel (1991) uses instead of L, k and m a different notation; see also Table 18.1 on p. 389.

Suppose ℓ steps are left ($2 \leq \ell \leq L$), the Inspectorate has still k' controls at its disposal ($1 \leq k' \leq \min(\ell-1,k)$), and the Operator has m' intended illegal activities left ($1 \leq m' \leq \min(\ell,m)$). Then the subgame $\Gamma(\ell,k',m')$ is reached, in which the (expected) payoff to the Operator is denoted by $Op_{\ell,k',m'}$. We suppress again Operator's and Inspectorate's strategies in the expression $Op_{\ell-1,\cdot,\cdot}$. The recursive normal form of von Stengel's inspection game is presented in Table 17.4.

Table 17.4 Recursive normal form of the subgame $\Gamma(\ell,k',m')$ of the von-Stengel inspection game $\Gamma(L,k,m)$, if ℓ steps, k' controls and m' intended illegal activities are left ($2 \leq \ell \leq L$, $1 \leq k' \leq \min(\ell-1,k)$, $1 \leq m' \leq \min(\ell,m)$).

	c_ℓ	$\bar{c}_\ell$
$\bar{\ell}_\ell$	$-b$	$1 + Op_{\ell-1,k',m'-1}$
ℓ_ℓ	$Op_{\ell-1,k'-1,m'}$	$Op_{\ell-1,k',m'}$

Table 17.4 can be explained as follows: Let us assume that the Operator behaves illegally ($\bar{\ell}_\ell$) at step ℓ. If the Inspectorate controls (c_ℓ), then the game terminates, and the Operator has to pay $-b\,(<0)$ for being detected, but, according to assumption (x'), the Operator need not return the payoff he gets for previous successful illegal activities. If the Inspectorate does not control at step ℓ ($\bar{c}_\ell$), a payoff of 1 is credited to the Operator and m' reduces to $m'-1$. If the Operator behaves legally (ℓ_ℓ) in that step, then m' leaves unchanged. Furthermore, it is *assumed* that even after a step without a control, it becomes common knowledge whether the Operator behaved legally or not. At first sight this assumption seems to be absurd. It will turn out on p. 380, however, that the optimal strategies of this inspection game do not depend on this assumption.

Suppose ℓ steps, k' controls and m' intended illegal activities are left with $2 \leq \ell \leq L$, $1 \leq k' \leq \min(\ell-1,k)$, and $1 \leq m' \leq \min(\ell,m)$). Let $\bar{p}_{\ell,k',m'}$ denote the probability to behave illegally at step ℓ ($2 \leq \ell \leq L$). Then $1-\bar{p}_{\ell,k',m'}$ is the probability to postpone the m' intended illegal activities. Note that $\bar{p}^*_{\ell,\ell,m'} = 0$, especially $\bar{p}^*_{1,1,m'} = 0$, and $\bar{p}^*_{1,0,1} = 1$. The probability for the Inspectorate to perform a control at step ℓ is denoted by $q_{\ell,k',m'}$. The optimal payoff to the Operator is denoted by $Op^*_{\ell,k',m'}$.

In order to solve this recursive inspection game, the boundary conditions for the optimal payoff to the Operator

$$Op^*_{\ell,k',m'} = \begin{cases} 0 & \text{for} \quad k' = \ell \\ m' & \text{for} \quad k' = 0, 1 \leq m' \leq \min(\ell,m) \\ 0 & \text{for} \quad m' = 0, 1 \leq k' \leq \min(\ell-1,k) \end{cases} \tag{17.52}$$

for all $1 \leq \ell \leq L$ have to be met, which can be justified as follows: If the Inspectorate must control in every remaining step ($k' = \ell$), then the Operator will not gain anything by behaving illegally. Thus, he will behave legally and $Op^*_{\ell,\ell,m'} = 0$. If the Inspectorate has no controls left ($k' = 0$), then the Operator will behave illegally as often as he intended to (but – by assumption – at most once in every remaining step), each time increasing the Operator's payoff by one, i.e., $Op^*_{\ell,0,m'} = m'$. If there is no intended illegal activity left ($m' = 0$), then the Operator behaves legally in all remaining steps, and his optimal payoff is zero.

The game theoretical solution of this inspection game, see von Stengel (1991), is presented in

Theorem 17.3. *Given the von-Stengel inspection game with $L > k$ steps, k controls, and at most m intended illegal activities ($m \leq L$), i.e., $\Gamma(L, k, m)$, the recursive normal form of which is represented in Table 17.4. The payoff to the Operator is defined recursively using the recursive normal form representation in Table 17.4, and the optimal payoff to the Operator fulfils the boundary conditions (17.52).*

Suppose ℓ steps, $2 \leq \ell \leq L$, are left, the Inspectorate has k' controls at its disposal, $1 \leq k' \leq \min(\ell - 1, k)$, and the Operator has m' intended illegal activities left, $1 \leq m' \leq \min(\ell, m)$, i.e., the subgame $\Gamma(\ell, k', m')$ is reached.

Define

$$s(\ell, k') = \sum_{i=0}^{k'} \binom{\ell}{i} b^{k'-i} .$$

Then optimal strategies and the optimal payoff $Op^*_{\ell,k',m'}$ to the Operator in the subgame $\Gamma(\ell, k', m')$ are given as follows: An optimal strategy of the Operator is given by, using (17.55),

$$\bar{p}^*_{\ell,k',m'} = \frac{Op^*_{\ell-1,k'-1,m'} - Op^*_{\ell,k',m'}}{b + Op^*_{\ell-1,k'-1,m'}} , \tag{17.53}$$

and an optimal strategy of the Inspectorate by

$$q^*_{\ell,k',m'} = \frac{s(\ell - 1, k' - 1)}{s(\ell, k')} . \tag{17.54}$$

The optimal payoff to the Operator in the subgame $\Gamma(\ell, k', m')$ is given by

$$Op^*_{\ell,k',m'} = \frac{\binom{\ell}{k'+1} - \binom{\ell-m'}{k'+1}}{s(\ell, k')} , \tag{17.55}$$

which – for $\ell = L$, $k' = k$ and $m' = m$ – is the optimal payoff to the Operator of the entire game $\Gamma(L, k, m)$.

Proof. Like in earlier cases, we apply the principle of making the adversary indifferent; see Theorem 19.1. Given that the payoff matrix in Table 17.4 is cyclic, we get with $q^* = q^*_{\ell,k',m'}$

$$\begin{aligned} Op^*_{\ell,k',m'} &= q^* \, (-b) + (1 - q^*) \, (1 + Op^*_{\ell-1,k',m'-1}) \\ &= q^* \, Op^*_{\ell-1,k'-1,m'} + (1 - q^*) \, Op^*_{\ell-1,k',m'} . \end{aligned} \tag{17.56}$$

This leads to

$$q^* = \frac{-1 - Op^*_{\ell-1,k',m'-1} + Op^*_{\ell-1,k',m'}}{-b - 1 - Op^*_{\ell-1,k',m'-1} - Op^*_{\ell-1,k'-1,m'} + Op^*_{\ell-1,k',m'}} . \tag{17.57}$$

Inserting q^* from (17.57) into the first equation of (17.56) we get the following recursive relation for $Op^*_{\ell,k',m'}$:

$$Op^*_{\ell,k',m'} = (-b) \, \frac{-1 - Op^*_{\ell-1,k',m'-1} + Op^*_{\ell-1,k',m'}}{-b - 1 - Op^*_{\ell-1,k',m'-1} - Op^*_{\ell-1,k'-1,m'} + Op^*_{\ell-1,k',m'}} \tag{17.58}$$

$$+ (1 - Op^*_{\ell-1,k',m'-1}) \frac{-b - Op^*_{\ell-1,k'-1,m'}}{-b - 1 - Op^*_{\ell-1,k',m'-1} - Op^*_{\ell-1,k'-1,m'} + Op^*_{\ell-1,k',m'}} .$$

With the help of rather cumbersome algebraic manipulations, see von Stengel (1991) for details, one can show that (17.55) indeed fulfils the recursive relation (17.58) as well as the boundary conditions (17.52). Inserting (17.55) into (17.57) one obtains (17.54).

Using the payoff matrix in Table 17.4, the Operator's optimal strategy $\bar{p}^* = \bar{p}^*_{\ell,k',m'}$ can be determined from

$$Op^*_{\ell,k',m'} = \bar{p}^* (-b) + (1 - \bar{p}^*) Op^*_{\ell-1,k'-1,m'}$$

which leads to (17.53). Note that $\bar{p}^*_{\ell,k',m'}$ can not be expressed in terms on $s(\ell, k')$ as simple as $q^*_{\ell,k',m'}$; see also (16.11) in Theorem 16.1.

Finally, it is shown by von Stengel that the payoffs in Table 17.4 are cyclic, which completes the proof. □

Let us comment the results of Theorem 17.3: First, note that the Inspectorate's optimal payoff $-Op^*_{L,k,m}$ decreases with the number m of intended illegal activities. However, that payoff is constant for all $m \geq L - k$ since then $\binom{L-m}{k+1} = 0$. Indeed, the Operator will not perform more than $L - k$ illegal activities since otherwise he would be caught with certainty.

Second, (17.54) illustrates that the Inspectorate's optimal probability $q^*_{\ell,k',m'}$ of a control at step ℓ *does not depend on* m'. This means that the Inspectorate *need not know* the value of m' in order to play optimally. Hence, the assumption in this game about the knowledge of the Inspectorate after a step without a control can be removed, see p. 378: The solution of the recursive game, see Theorem 17.3, is also the solution of the inspection game *without* recursive structure, where – more realistically – the Inspectorate does not know what happened at a step without a control. This is analysed by von Stengel (1991) using the extensive form of the game. As mentioned on p. 371, von Stengel's inspection game is the only one in this chapter – except the generalized Thomas-Nisgav inspection games $\Gamma(2,1)$ and $\Gamma(4,2)$ of Section 17.1 – for which the reduction of the extensive form representation to the recursive form has been shown.

Third, Theorem 17.3 generalizes the zero-sum version of the Dresher-Höpfinger inspection game, i.e., $d = c = 1$ and $a = -b$ in (16.1), see Figure 16.3 and Theorem 16.1, to any $m \geq 1$. Note, that in a later work, von Stengel (2016) generalizes this model insofar as varying rewards to the Operator for successful illegal behaviour are introduced. Also, non-zero-sum models as well as an inspector leadership version of the original game are considered.

Sakaguchi (1994) has analysed an inspection problem which is very similar to that of von Stengel (1991). The basic difference is that Sakaguchi assumes that the game is not terminated after the detection of an illegal activity, but rather continues throughout the L steps. A minor difference is that Sakaguchi assumes $b = 1$ in assumption (viii'). Thus, only assumption (x') from p. 377 is specified as follows:

(x') The game ends either at the step at which the number of controls left is zero, or at that step at which the number of controls left is equal to the number of steps left, or at that step at which the number of intended illegal activities left is zero, or at step 1.

Note that Sakaguchi (1994) uses a slightly different notation from ours; see Table 18.1 on p. 389. The recursive normal form of this inspection game is given in Table 17.5. We see, if we compare Tables 17.4 and 17.5, that in case of detected illegal activity the game continues (indicated by $Op_{\ell-1,k'-1,m'-1}$) and that $-b$ is replaced by -1. The boundary conditions coincide with (17.52). The probabilities $\bar{p}_{\ell,k',m'}$ and $q_{\ell,k',m'}$ are defined as in von Stengel's inspection game.

Table 17.5 Recursive normal form of the subgame $\Gamma(\ell,k',m')$ of the Sakaguchi inspection game $\Gamma(L,k,m)$, if ℓ steps, k' controls and m' intended illegal activities are left ($2 \le \ell \le L$, $1 \le k' \le \min(\ell-1,k)$, $1 \le m' \le min(\ell,m)$).

	c_ℓ	$\bar{c}_\ell$
$\bar{\ell}_\ell$	$-1 + Op_{\ell-1,k'-1,m'-1}$	$1 + Op_{\ell-1,k',m'-1}$
ℓ_ℓ	$Op_{\ell-1,k'-1,m'}$	$Op_{\ell-1,k',m'}$

In contrast to von Stengel (1991), Sakaguchi was not able to prove the cyclic nature of the payoff matrix in Table 17.5 although numerical calculations led to the conjecture that it holds indeed.

The game theoretical solution of this inspection game, see Sakaguchi (1994), is presented in

Theorem 17.4. *Given the Sakaguchi inspection game with $L > k$ steps, k controls, and at most m intended illegal activities ($m \le L$), i.e., $\Gamma(L,k,m)$, the recursive normal form of which is represented in Table 17.5. The payoff to the Operator is defined recursively using the recursive normal form representation in Table 17.5, and the optimal payoff to the Operator fulfils the boundary conditions (17.52).*

Suppose ℓ steps, $2 \le \ell \le L$, are left, the Inspectorate has k' controls at its disposal, $1 \le k' \le \min(\ell-1,k)$, and the Operator has m' intended illegal activities left, $1 \le m' \le \min(\ell,m)$, i.e., the subgame $\Gamma(\ell,k',m')$ is reached.

Define

$$\mu_{\ell,k'} = \frac{-\binom{\ell-1}{k'}}{\sum_{i=0}^{k'} \binom{\ell}{i}}$$

and let us assume that

$$\frac{\mu_{\ell-2,k'-1}\,\mu_{\ell,k'}}{\mu_{\ell-1,k'-1}\,\mu_{\ell-1,k'}} > 1\,. \tag{17.59}$$

Then optimal strategies and the optimal payoff $Op^*_{\ell,k',m'}$ to the Operator in the subgame $\Gamma(\ell,k',m')$ are given as follows: An optimal strategy of the Operator is given by

$$\bar{p}^*_{\ell,k',m'} = \frac{m'\,(\mu_{\ell-1,k'} - \mu_{\ell-1,k'-1})}{2 + \mu_{\ell-1,k'} - \mu_{\ell-1,k'-1}}\,,$$

and optimal strategy of the Inspectorate by

$$q^*_{\ell,k',m'} = \frac{1 + \mu_{\ell-1,k'}}{2 + \mu_{\ell-1,k'} - \mu_{\ell-1,k'-1}} = \frac{\sum_{i=0}^{k'-1} \binom{\ell-1}{i}}{\sum_{i=0}^{k'} \binom{\ell}{i}}\,.$$

The optimal payoff to the Operator in the subgame $\Gamma(\ell, k', m')$ is given by

$$Op^*_{\ell,k',m'} = (-m')\,\mu_{\ell,k'} = m' \,\frac{\binom{\ell-1}{k'}}{\sum_{i=0}^{k'} \binom{\ell}{i}}\,, \tag{17.60}$$

which – for $\ell = L$, $k' = k$ and $m' = m$ – is the optimal payoff to the Operator of the entire game $\Gamma(L, k, m)$.

Proof. Once more, the principle of making the adversary indifferent is applied; see Theorem 19.1. Thus, the proof goes along the same lines as that of Theorem 17.3: It is shown that (17.60) fulfils the recursive relation

$$Op^*_{\ell,k',m'} = \frac{(1 + Op^*_{\ell-1,k'-1,m'-1})\,Op^*_{\ell-1,k',m'} - (-1 + Op^*_{\ell-1,k',m'-1})\,Op^*_{\ell-1,k'-1,m'}}{2 + Op^*_{\ell-1,k'-1,m'-1} + Op^*_{\ell-1,k',m'} - Op^*_{\ell-1,k',m'-1} - Op^*_{\ell-1,k'-1,m'}}\,.$$

as well as the boundary conditions (17.52). The details of this – again – cumbersome calculation can be found in Sakaguchi's original work. Let us note that the assumption (17.59) guarantees the cyclic structure of the payoff matrix in Table 17.5, and that $0 < \bar{p}^*_{\ell,k',m'} < 1$. □

We see that the optimal payoff to the Operator according to (17.60) has a similar structure as that according to (17.55). In case of $m = 1$ intended illegal activity, the game theoretical solution of Sakaguchi's inspection game coincides with the game theoretical solution of Dresher-Höpfinger's inspection game for $b = d = 1$, see Theorem 16.1, which coincides with the game theoretical solution of von-Stengel's inspection game for $b = 1$, see Theorem 17.3, as expected.

There is, however, more to say, about recursive inspection games with multiple illegal activities: We now shortly discuss the work by Sakaguchi (1977), by Ferguson and Melolidakis (1998) and by Sakaguchi (2003), and will conclude this section with the inspection game by Hohzaki (2011) .

In an earlier paper Sakaguchi (1977) considered an inspection model with the assumptions (iv') and (viii') with $b = 1$ from p. 377, assumptions (x') from p. 380, and assumptions (iii') and (vi') are specified as follows:

(iii') The Operator performs m illegal activities at the steps $L, \ldots, 1$, where he can behave illegally at most once per step.

(vi') The Operator decides at the beginning, i.e., at step L, whether to behave illegally at that step. At a step ℓ $(1 \le \ell \le L-1)$ and in case the Operator *must* still perform an illegal activity, see assumption (iii"), he decides whether to behave illegally at that step; and so on.

The Inspectorate decides at the beginning whether to control at step L. If it has still controls at its disposal, then the Inspectorate decides at step $L-1$ whether to control at that step; and so on.

Whereas, of course, the recursive normal form is the same as that given in Table 17.5, the boundary conditions for the optimal payoff to the Operator are now, only the first one is different from them in (17.52):

$$Op^*_{\ell,k',m'} = \begin{cases} -m' & \text{for} \quad k' = \ell \\ m' & \text{for} \quad k' = 0, 1 \le m' \le \min(\ell, m) \\ 0 & \text{for} \quad m' = 0, 1 \le k' \le \min(\ell - 1, k) \end{cases} \tag{17.61}$$

for all $1 \leq \ell \leq L$: Because the Operator has to perform the m' illegal activities, he will be caught with certainty in case the Inspectorate performs a control at every step ($k' = \ell$). Thus, the Operator looses m'.

The probabilities $\bar{p}_{\ell,k',m'}$ and $q_{\ell,k',m'}$ are defined as in von Stengel's inspection game. Note, however, that $\bar{p}_{\ell,\ell,\ell} = 1$.

The game theoretical solution of this inspection game, see Sakaguchi (1977), is presented in

Theorem 17.5. *Given the Sakaguchi inspection game with $L > k$ steps, k controls, and m illegal activities ($m < L$), i.e., $\Gamma(L,k,m)$, the recursive normal form of which is represented in Table 17.5. The payoff to the Operator is defined recursively using the recursive normal form representation in Table 17.5, and the optimal payoff to the Operator fulfils the boundary conditions (17.61).*

Suppose ℓ steps, $2 \leq \ell \leq L$, are left, the Inspectorate has k' controls at its disposal, $1 \leq k' \leq \min(\ell-1,k)$, and the Operator has m' illegal activities left, $1 \leq m' \leq \min(\ell,m)$, i.e., the subgame $\Gamma(\ell,k',m')$ is reached.

Then optimal strategies and the optimal payoff $Op^*_{\ell,k',m'}$ to the Operator in the subgame $\Gamma(\ell,k',m')$ are given as follows: An optimal strategy of the Operator is given by

$$\bar{p}^*_{\ell,k',m'} = \frac{m'}{\ell},$$

and optimal strategy of the Inspectorate by

$$q^*_{\ell,k',m'} = \frac{k'}{\ell}.$$

The optimal payoff to the Operator in the subgame $\Gamma(\ell,k',m')$ is given by

$$Op^*_{\ell,k',m'} = m'\left(1 - \frac{2\,k'}{\ell}\right),$$

which – for $\ell = L$, $k' = k$ and $m' = m$ – is the optimal payoff to the Operator of the entire game $\Gamma(L,k,m)$.

Proof. The proof can be found in Sakaguchi (1977). □

Note that the optimal strategies as well as the optimal payoffs to the Operator in the two versions of Sakaguchi's inspection game, see Theorems 17.4 and 17.5, are totally different! This demonstrates the sensitivity of the optimal strategies and payoffs on the model assumptions, an effect that can also be observed in the Dresher-Höpfinger's inspection game with errors of the second kind; see Theorem 16.2.

Ferguson and Melolidakis (1998) consider the same recursive structure as Sakaguchi (1994), see Table 17.5, but they use different payoffs which result in different boundary conditions which then, of course, result in different solutions. Since, however, the presented models of v. Stengel and Sakuguchi – more than one illegal activity without and with the termination of the game after the first detection of an illegal activity – which we still call fundamental, Ferguson and Melolidakis analyse variations of Sakaguchi's model. Therefore, we do not present their work here, but refer the interested reader to the original work. We will come back to the issue of fundamental models at the end of this section on p. 388.

Sakaguchi (2003) analysed in a later work the case that the illegal activity (expressed as cargo to be smuggled during one night) is considered to be a uniformly distributed random variable which then leads back to a game the recursive structure of which has only two variables.

Let us conclude our collection of models with multiple illegal activities with an inspection game presented by Hohzaki (2011), see also Hohzaki et al. (2006), which deals with multiple smuggling as well, the difference to the notation in this monograph can be found in Table 18.1 on p. 389: During L steps – Hohzaki considers days – the Smuggler tries to smuggle the contraband $x \in \mathbb{N}$, which may be divided in portions $1, 2, \ldots, x-1$, which then are smuggled in separate steps. If portion y, $y = 1, \ldots, x$, is smuggled, and if a patrol boat is under way, then the smuggling is detected with probability $q_1(y)$, and it is not detected with probability $q_2(y)$, where $q_1(y) + q_2(y) \leq 1$. According to Hohzaki, the probability of no capture and no successful smuggling (no event) is $1 - q_1(y) - q_2(y)$. We do not understand what no event means, but this is not important for the subsequent analysis.

Customs can patrol at most k times. Before each step, Customs decides to patrol or not. Unless Customs captures the Smuggler, the game transfers to the next step. Before each step, both players, Smuggler and Customs, get information about the strategies their adversaries took before the previous step. Especially the portion y smuggled in the last step – if at all – is known to Customs. Upon capture of the Smuggler or the expiration of the preplanned number of steps, the inspection game ends.

The payoff to both players is zero-sum. Successful smuggling yields the Smuggler a reward of 1 per unit of smuggled contraband while Customs looses the same. If Customs captures the Smuggler, it gets a reward $\alpha > 0$ (α should not be confused with the false alarm probability used so far in this monograph), which is an amount relative to the value of the contraband.

Therefore, the assumptions from p. 377 are specified as follows:

(iii') The Smuggler smuggles at steps $L, \ldots, 1$ as much as possible from the contraband x which may be divided in portions $1, 2, \ldots, x-1$.

(iv') During a control Customs may commit an errors of the second kind, i.e., if portion y, $y = 1, \ldots, x$, is smuggled at the same step at which a control is performed, then the smuggling is detected with probability $q_1(y)$, and not detected with probability $q_2(y)$ where $q_1(y) + q_2(y) \leq 1$.

(vi') The Smuggler decides at the beginning, i.e., at step L, which portion $0, 1, \ldots, x$ to smuggle at that step, where portion 0 refers to "no smuggling". At a step ℓ ($1 \leq \ell \leq L-1$) and in case the Smuggler can still smuggle, see assumption (iii'), he decides which portion $0, 1, \ldots$ to smuggle from the remaining contraband at that step; and so on.

Customs decides at the beginning whether to control at step L. If it has still controls at its disposal, then Customs decides at step $L-1$ whether to control at that step; and so on.

(viii') The payoffs to the two players (Smuggler, Customs) are given by

$$\begin{array}{rl} (1, -1) & \text{for an untimely control and illegal behaviour} \\ (-\alpha, \alpha) & \text{for a timely control and illegal behaviour} \\ (0, 0) & \text{for legal behaviour}, \end{array}$$

with $\alpha > 0$.

(x') The game ends either at the control at which the smuggle is detected, or at that step at which the portion of contraband left is zero, or at that step at which the number of controls left is zero, or at step 1.

Suppose ℓ steps are left ($2 \le \ell \le L$), Customs has k' patrols at its disposal ($1 \le k' \le \min(\ell-1, k)$), and the Smuggler has the contraband x' with $1 \le x' \le x$ left. Let $Op^*_{\ell,k'}(x'-y)$ be the optimal payoff to the Smuggler, if ℓ steps and k' patrols remain, and when the contraband $x' - y$ still has to be smuggled. The recursive normal form of this inspection game has $x' + 1$ rows and 2 columns; it is presented in Table 17.6.

Table 17.6 Recursive normal form of the subgame $\Gamma(\ell, k', x')$ of the Hohzaki inspection game $\Gamma(L, k, x)$, if ℓ steps, k' controls and the contraband x' are left ($2 \le \ell \le L$, $1 \le k' \le \min(\ell - 1, k)$, $1 \le x' \le x$).

	c_ℓ	$\bar{c}_\ell$
$0\,(y = x')$	$-\alpha\, q_1(x') + x'\, q_2(x')$ $+\,(1 - q_1(x'))\, Op_{\ell-1,k'-1}(0)$	$x' + Op_{\ell-1,k'}(0)$
$1\,(y = x' - 1)$	$-\alpha\, q_1(x'-1) + (x'-1)\, q_2(x'-1)$ $+(1-q_1(x'-1))\, Op_{\ell-1,k'-1}(1)$	$x' - 1 + Op_{\ell-1,k'}(1)$
$\vdots$	$\vdots$	$\vdots$
$x' - y\,(y)$	$-\alpha\, q_1(y) + y\, q_2(y)$ $+(1-q_1(y))\, Op_{\ell-1,k'-1}(x'-y)$	$y + Op_{\ell-1,k'}(x' - y)$
$\vdots$	$\vdots$	$\vdots$
$x' - 1\,(y = 1)$	$-\alpha\, q_1(1) + y\, q_2(1)$ $+(1-q_1(1))\, Op_{\ell-1,k'-1}(x'-1)$	$1 + Op_{\ell-1,k'}(x' - 1)$
$x'\,(y = 0)$	$Op_{\ell-1,k'-1}(x')$	$Op_{\ell-1,k'}(x')$

Boundary conditions for the optimal payoff to the Smuggler are given by

$$Op^*_{\ell,k'}(x') = \begin{cases} 0 & \text{for} \quad x' = 0, 1 \le k' \le \min(\ell - 1, k) \\ x' & \text{for} \quad k' = 0, 1 \le x' \le x \end{cases} \tag{17.62}$$

for all $1 \le \ell \le L$: If nothing is left to be smuggled ($x' = 0$), then the Smuggler's payoff is zero. If no control is left ($k' = 0$), then he can smuggle the remaining amount whenever he wants, i.e., he gains x'.

In order to find closed expressions for the optimal strategies and optimal payoff, Hohzaki further assumes a constant detection probability $q_1 := q_1(y)$, and a constant non-detection probability $q_2 := q_2(y)$.

Now Hohzaki proceeds as follows: He assumes the *no-partially smuggling assumption*, i.e., the Smuggler considers only two pure strategies in each step: Either not to smuggle at all during the next step ($y = 0$), or to smuggle the whole contraband that is left ($y = x'$). Thus, the recursive normal form game presented in Table 17.6 is reduced to the 2×2-game presented in Table 17.7, where (17.62) and the constant detection and non-detection probabilities have already been taken into account.

Table 17.7 Reduced recursive normal form of the subgame $\Gamma(\ell, k', x')$ of the Hohzaki inspection game $\Gamma(L, k, x)$ of Table 17.6.

	c_ℓ	$\bar{c}_\ell$
0	$-\alpha\, q_1 + x\, q_2$	x
x	$Op_{\ell-1,k'-1}(x)$	$Op_{\ell-1,k'}(x)$

Hohzaki solves the inspection game in Table 17.7 and shows that the "no-partially smuggling assumption" is valid for the original game in Table 17.6, i.e., it is indeed optimal for the Smuggler to smuggle the total contraband x' at the next step or not.

In the next Theorem we only present the optimal payoff to the Smuggler. Statements on the optimal strategies can be found in Hohzaki (2011).

Theorem 17.6. *Given the Hohzaki inspection game with $L > k$ steps, k controls, and a contraband x, the recursive normal form of which is represented in Table 17.6. The payoff to the Smuggler is defined recursively using the recursive normal form representation in Table 17.7, and the optimal payoff to the Smuggler fulfils the boundary conditions (17.62).*

Then the optimal payoff to the Smuggler is given by

$$Op^*_{L,k}(x) = \begin{cases} x - \gamma(x)\,\dfrac{k}{L} & \text{for} \quad \alpha\, q_1 - x\, q_2 < 0 \\ \dfrac{\binom{L-1}{k} x^{k+1}}{\sum\limits_{i=0}^{k} \binom{L-k+i-1}{i} x^i\,(\gamma(x))^{k-i}} & \text{for} \quad \alpha\, q_1 - x\, q_2 > 0 \end{cases}, \tag{17.63}$$

where $\gamma(x) = \alpha\, q_1 - x\, q_2 + x$.

Proof. The proof can be found in Hohzaki (2011). □

Let us look at the recursive normal form of the Dresher-Höpfinger inspection game, given in Figure 16.3 with a and b replaced by $\tilde{a}$ and $\tilde{b}$, and at Theorem 16.2. If we define $\tilde{b} := \alpha\, q_1 - x\, q_2$ and $d := x$, then we get $\gamma(x) = \tilde{b} + d = (1-\beta)\,(b+d)$, and the optimal payoff (17.63) coincides with the equilibrium payoffs to the Operator (16.42) in case of $\alpha\, q_1 - x\, q_2 < 0$, and with (16.13) in case of $\alpha\, q_1 - x\, q_2 > 0$: Whereas for the case $\alpha\, q_1 - x\, q_2 < 0$ this can be seen immediately, the other case takes a little more effort: Here we have according to (17.63)

$$Op^*_{L,k}(x) = d\, \frac{\binom{L-1}{k}}{\sum\limits_{i=0}^{k} \binom{L-k+i-1}{i} \left(1 + \dfrac{\tilde{b}}{d}\right)^{k-i}}\,, \tag{17.64}$$

and according to Theorem 16.2, i.e., using (16.13),

$$Op^*_{L,k} = d\,\frac{\binom{L-1}{k}}{f(L,k)} = d\,\frac{\binom{L-1}{k}}{\sum_{i=0}^{k}\binom{L}{i}\left(\frac{\tilde{b}}{d}\right)^{k-i}} = d\,\frac{\binom{L-1}{k}}{\sum_{j=0}^{k}\binom{L}{k-j}\left(\frac{\tilde{b}}{d}\right)^{j}}\,. \tag{17.65}$$

Let us write the denominator of Hohzaki's form (17.64) as

$$\sum_{i=0}^{k}\binom{L-k+i-1}{i}\left(1+\frac{\tilde{b}}{d}\right)^{k-i} = \sum_{i=0}^{k}\binom{L-k+i-1}{i}\sum_{j=0}^{k-i}\binom{k-i}{j}\left(\frac{\tilde{b}}{d}\right)^{j}.$$

Changing the order of summation leads to

$$\sum_{j=0}^{k}\sum_{i=0}^{k-j}\binom{L-1-(k-i)}{i}\binom{k-i}{j}\left(\frac{\tilde{b}}{d}\right)^{j},$$

thus, both the denominators of (17.64) and (17.65) are equal if

$$\sum_{i=0}^{k-j}\binom{L-1-(k-i)}{i}\binom{k-i}{j} = \binom{L}{k-j},$$

or, substituting $k-i \to i$ and $\binom{L-1-i}{k-i} = \binom{L-1-i}{L-1-k}$, if

$$\sum_{i=j}^{k}\binom{L-1-i}{L-1-k}\binom{i}{j} = \binom{L}{k-j} \tag{17.66}$$

for all $L = 2, 3, \ldots$, $k = 1, \ldots, L-1$ and $j = 0, \ldots, k$.

Indeed, this can be shown by induction with respect to L. For $L = 2$ this can be seen directly. Let us assume that (17.66) is fulfilled for $L-1$, all $k = 1, \ldots, L-1$, and all $j = 0, \ldots, k$. We have to show that

$$\sum_{i=j}^{k}\binom{L-i}{L-k}\binom{i}{j} = \binom{L+1}{k-j} \tag{17.67}$$

for all $k = 1, \ldots, L$ and all $j = 0, \ldots, k$. For $L-1$ towards L we have for all $k = 1, \ldots, L-1$, using elementary properties of binomial coefficients,

$$\begin{aligned}\sum_{i=j}^{k}\binom{L-i}{L-k}\binom{i}{j} &= \sum_{i=j}^{k}\left[\binom{L-1-i}{L-1-k} + \binom{L-1-i}{L-k}\right]\binom{i}{j}\\ &= \sum_{i=j}^{k}\binom{L-1-i}{L-1-k}\binom{i}{j} + \sum_{i=j}^{k}\binom{L-1-i}{L-1-(k-1)}\binom{i}{j}\\ &= \binom{L}{k-j} + \sum_{i=j}^{k-1}\binom{L-1-i}{L-1-(k-1)}\binom{i}{j}\end{aligned}$$

$$= \binom{L}{k-j} + \binom{L}{k-1-j} = \binom{L+1}{k-j},$$

i.e., the right hand side of (17.67). Using (4.50) with $a = j$ and $b = L$, we obtain for the left hand side of (17.67) in case of $k = L$

$$\sum_{i=j}^{L} \binom{L-i}{L-L} \binom{i}{j} = \sum_{i=j}^{L} \binom{i}{j} = \binom{L+1}{j+1} = \binom{L+1}{L-j},$$

i.e., the right hand side of (17.67) for $k = L$. Thus, (17.66) is proven, and it is shown that the optimal payoff (17.63) coincides with the equilibrium payoffs to the Operator (17.65) resp. (16.42).

The interesting point of this identity is the fact that in the original Dresher-Höpfinger inspection game, i.e., (16.13), the first solution in (17.63) – identical to that of the generalized Thomas-Nisgav inspection game as given by Theorem 17.1! – could not be seen since b was assumed to be positive; only $\tilde{b}$ can be positive or negative.

Note that it would be interesting to find out, whether and like in von Stengel's inspection game, the result of Theorem 17.6 can also be obtained if one does not assume that Customs is always fully informed, i.e., also after those steps where no patrol boat was under way or no smuggling was detected.

Let us conclude this chapter with a remark concerning the subtitle of this monograph: Today, it may be a question of taste whether one considers the inspection games presented in this section still as fundamental or not. We have taken them into account here in order to demonstrate that some of the approaches developed for the analysis of simpler games, e.g., recursive ones with only two variables, still can be applied successfully and furthermore, in order to give some idea of already existing extensions of assumptions and models. Once the inspection games considered in this section are better understood than now and above all can be supported by convincing practical applications, we think that they can also be called fundamental models without hesitation.

Chapter 18

Classification of models in Part III

Even for those who have already studied inspection games of the kind described in Chapters 16 and 17, it is not so easy to maintain control over all the different assumptions underlying these games. They all satisfy, of course, the general assumptions listed in Chapter 14. But it was also pointed out there that many more assumptions are required to specify completely an inspection game over time dealing with Customs and Smugglers and related problems. Figure 14.2 extends Figure 1.1 by the two levels *Operator's illegal activity* and *End of Inspectorate's activities*.

Before continuing let us keep in mind that all games considered in Chapters 16 and 17 are Se-Se inspection games in the sense of Table 2.1. More than that, they are step-by-step games in contrast to the games in Chapters 5, except for Section 5.3, and 12 which may be called event-by-event games: Both players take their decisions only after an inspection/control.

For the inspection games presented in Section 17.2 we recommend to study the original papers in order to fully understand the models and their analysis. Since in these papers the notations differs considerably and may lead to confusion, in Table 18.1 an overview on the notation for the most important quantities in the publications dealing with multiple illegal activities is given.

Table 18.1 Overview of the notation used in this monograph and by different authors.

	This monograph	von Stengel (1991)	Sakaguchi (1994), Sakaguchi (1977)	Ferguson and Melolidakis (1998)	Hohzaki (2011), Hohzaki et al. (2006)
number of steps/nights	L	n	n	n	n
number of inspections/ controls	k	m	k	k	k
number of illegal activities/ smuggling	m	k	l	l	
optimal payoff to Operator/ Smuggler	$Op_{L,k,m}$	$-v(n,m,k)$	$-v_{k,l}(n)$	$-V(n,k,l)$	$-v(n,k,x)$

In Table 18.2 an attempt has been made to present a quick overview on the Se-Se inspection games of Chapters 16 and 17. Basically they are classified according to the scheme given in Figure 14.2: The rows describe the Operator's legal or illegal behaviour, and the columns the termination of the game. The columns are subdivided according to the sampling procedure used by the Inspectorate resp. Customs. Obviously not all assumptions can be covered by such an overview, and not all variants are taken into account, e.g., the Dresher-Höpfinger inspection game with errors first and second kind and one inspection. Also, Table 18.2 does not reflect the fact that small changes in the assumptions may lead to totally different optimal strategies and payoffs, compare, e.g., the model by von Stengel (1991) and Sakaguchi (1994) on one hand and those by Sakaguchi (1977) and Ferguson and Melolidakis (1998) on the other. Thus, for all these details one has to look at the models themselves.

Table 18.2 Classification according to Figure 14.2: Se-Se inspection games of Chapters 16 and 17.

<table>
<tr><th></th><th colspan="2">game ends after a detection of an illegal activity or after L resp. n steps</th><th>game ends only after L resp. n steps</th></tr>
<tr><td></td><td>$\beta = 0$</td><td>$\beta \geq 0$</td><td>$\beta = 0$</td></tr>
<tr><td rowspan="2">at most one illegal activity</td><td>Dresher (1962), Höpfinger (1971)</td><td></td><td></td></tr>
<tr><td colspan="2">Generalized Dresher-Höpfinger[1]
Generalized Thomas-Nisgav[1,2]</td><td></td></tr>
<tr><td>(exactly) one illegal activity</td><td colspan="2">Thomas and Nisgav (1976), one boat
Baston and Bostock (1991), two boats
Garnaev (1991), three boats</td><td></td></tr>
<tr><td rowspan="2">at most m (≥ 1) illegal activities</td><td>von Stengel (1991), von Stengel (2016)</td><td></td><td rowspan="2">Sakaguchi (1994)</td></tr>
<tr><td colspan="2">Hohzaki (2011)</td></tr>
<tr><td>(exactly) m (≥ 1) illegal activities</td><td></td><td></td><td>Sakaguchi (1977), Ferguson and Melolidakis (1998)</td></tr>
</table>

[1] Published in this monograph for the first time.
[2] Smuggler decides at the beginning of the game whether to behave illegally or not.

One aspect in Table 18.2 which is hidden behind the terms *at most* and *exactly* deserves special attention, and it will be paid in the next paragraph. Before let us remember that in all inspection problems analysed in Parts I and II which were described as zero-sum games and in which only the expected detection time was the payoff to the Operator, it was assumed that the Operator behaves illegally with certainty. Furthermore, it was pointed out in Chapters 7 and 12 that under this assumption and for the case that false alarms can be excluded, the games under consideration are equivalent to zero-sum games with the expected detection time as payoff to the Operator.

Now, in Part III we consider inspection problems where it is either assumed that at the beginning of the game the Operator decides to behave legally or not – Chapter 15 and the generalized Thomas-Nisgav inspection game in of Section 17.1 – or he decides this in the course of the game – Chapter 16, the Baston-Bostock inspection game in of Section 17.1 and Section 17.2. More than that, in case the Operator decides to behave illegally during the course of the game, part (ii) of Theorem 16.2 shows that in equilibrium the Operator behaves illegally with certainty even though he has not decided this at the beginning of the game. Note that in this case the equilibrium payoffs to the players are the same as if the Operator would have taken this decision at the beginning of the game.

Let us consider those cases where the Operator decides at the beginning of the game whether to behave legally or not, i.e., the No-No inspection game of Chapter 15 and the generalized Thomas-Nisgav inspection game treated in Section 17.1. In case he behaves illegally, and if we denote the timely detection probability by $w_{L,k}$, then quite generally and according to (14.1), the (expected) payoffs to the two players are given by

$$Op_{L,k} = d\,(1 - w_{L,k}) - b\,w_{L,k} \qquad \text{and} \qquad In_{L,k} = -c\,(1 - w_{L,k}) - a\,w_{L,k}\,, \tag{18.1}$$

or, slightly changed,

$$Op_{L,k} = d - (b + d)\,w_{L,k} \qquad \text{and} \qquad In_{L,k} = -c + (c - a)\,w_{L,k}\,.$$

In other words, these payoffs are linear functions of $w_{L,k}$, and the games under consideration are strategically equivalent to zero-sum games with the timely detection probability as payoff to the Inspectorate, independently of how $w_{L,k}$ is composed of the strategies of both players. This implies that the equilibrium strategies of both players do not depend on the payoff parameters a, b, c and d. We have observed this property in Corollary 15.1, in Theorem 16.2 (ii) as well as in Theorem 17.1, see (15.78), (16.42) and (17.33), but typically not in Theorem 16.1. This may also explain why the equilibrium payoffs in all three cases are the same and therefore, why the equilibrium probability of timely detection

$$w^*_{L,k} = (1 - \beta)\,\frac{k}{L}$$

is the same in all three cases: As mentioned on p. 273, it does not interest either party at which point of time an illegal activity is detected or not, and furthermore, the equilibrium payoff to the Operator in case of illegal behaviour is positive. Therefore, the Operator behaves illegally in any case, and the equilibrium strategies of both parties lead to the same results.

Of course, if we consider a priori zero-sum games with the timely detection probability as payoff to the Operator, then we do not get any more a condition for legal behaviour of the Operator being his equilibrium strategy. In the original game and according to (18.1), this condition is given by

$$Op^*_{L,k} = 0 > d\,(1 - w^*_{L,k}) - b\,w^*_{L,k}$$

or equivalently,

$$w^*_{L,k} > \frac{1}{1 + b/d}\,,$$

which means that the larger the ratio b/d is, the smaller needs the timely detection probability to be in equilibrium. But even if this ratio b/d of the payoff parameters can hardly be estimated it helps, as we have seen, that the equilibrium strategy of the Inspectorate in case of illegal behaviour of the Operator is also an equilibrium strategy in case of legal behaviour.

This situation corresponds to that in Parts I and II where we use the detection time – the time between the start of an illegal activity and its detection – as payoff to the Operator. This holds also for the case $\alpha = 0$ of the Se-Se inspection game with continuous time which is analysed in Chapter 12: If the Operator behave illegally in equilibrium then we see again, part (i) of Theorem 12.1, that the equilibrium payoffs are again proportional to the optimized expected detection time.

If false alarms cannot be avoided, then the situation is much more complicated since the game is no longer strategically equivalent to a zero-sum game. The same holds if the Operator does not decide at the beginning of the game whether to behave legally or not, and if certain illegal behaviour is not his equilibrium strategy. In this case neither the detection time nor the non-detection probability can be defined therefore, general payoffs have to be introduced in the way we did it or some other – openendedness has its prize.

These considerations apply primarily to practitioners: If expected detection times in Parts I and II and probabilities of detection in Part III can be used as payoffs to the Operator in zero-sum games, then only purely technical parameters describe the inspection problem, see Section 1.4, and thus, in most real world inspection problems the use of optimal inspection strategies becomes much more attractive.

These findings lead us to a final remark: One quickly realizes that there are several white areas in Table 18.2. They indicate plenty of material for further studies in this exciting area. However, even though not all of the models in this Table have been motivated by real world problems, we maintain our view expressed earlier that it is wise not just to try to fill gaps in the literature and change or generalize assumptions, but rather to study interesting applications and try to model them as carefully as possible.

Part IV
Appendices

Chapter 19

Non-cooperative two-person games

Before starting, some friendly warning: For those readers of this monograph, who do not know game theory, the study of the following pages does not replace that of a textbook on the subject like that by Morris (1994) or more elaborate, by Myerson (1991). Rather, the purpose of this description is twofold, namely first, to let the reader know which game theoretical concepts and tools are used throughout this monograph, and second, to present the notation for these concepts and tools which in game theory, as mentioned on p. 10, is not yet so unified as it should be.

Throughout this monograph we consider games in *normal* and in *extensive* form. Both forms are equivalent in the sense that each extensive form game can be represented as a normal form game and vice versa – we make abundantly use of this fact – however, it is necessary to make a reservation: If one addresses the problem of equilibrium selection then one has to realize that there are subtle differences between these two forms; see van Damme (1987).

In the following we consider only *finite* games, i.e., games with finite sets of pure strategies of the players. In Part II we consider also infinite games, but they are explained and analysed in detail at the hand of the concrete problems treated in that part such that we need not discuss them here again in general.

Also, we consider only *two-person* games according to the basic inspection problem considered in this monograph where there is only an Operator and an Inspectorate. Of course, there exist inspection problems with more than two players which then have to be described with the help of n-person games, $n > 2$, but again, we do not deal with them in this monograph.

19.1 Normal form games

For the purpose of illustration we propose to keep in mind the games described in Sections 3.1 and 7.2 even though we will consider now more general ones. Let us start with the assumption that the first player – the Operator – has $|I|$ pure strategies[1] and that the second player – the Inspectorate – has $|J|$ pure strategies. The i-th resp. the j-th pure strategy of the Operator

[1] For a finite set A the number of elements is denoted by $|A|$.

resp. the Inspectorate represents the i-th resp. the j-th unit vector. In order to avoid problems with the enumeration we will write – although mathematically slightly incorrect – i instead of e_i and j instead of e_j. If both the Operator and the Inspectorate decide to play the pair of strategies (i, j) then the Operator receives the payoff $Op(i, j)$ and the Inspectorate the payoff $In(i, j)$. Since these payoffs can be arranged in the form of two $|I| \times |J|$ matrices, we call this type of games also *bimatrix* games.

Note that in game theory it is assumed that both players know the pure strategy sets (the own and that of the other player) and also the payoffs to both players. This condition is called *common knowledge* without which game theory – as used in this monograph – would not work.

In the first example in Section 3.1 we see that there is no pure strategy combination (i^*, j^*) with the property that any unilateral deviation of one player from his strategy i^* resp j^* does not improve his payoff. Thus, a new idea is needed in order to get closer to a satisfactory concept of a solution of a bimatrix game, if there is no such (i^*, j^*): Each player should choose a pure strategy at *random*. In this way, the other player has no way of predicting which pure strategy will be used. The *probabilities* with which the various pure strategies are chosen will probably be known to both opponents, since they both can make game theoretical consideration and because of the common knowledge assumption; the particular pure strategy chosen for a game, i.e., the realization of the random experiment, however, will only be known to the adversary when the game is actually played.

Formally, one has to introduce – following a general procedure, see Peters and Vrieze (1992), Morris (1994) or Myerson (1991) – the concept of *mixed* strategies. A mixed strategy of a player is a probability distribution over his set of pure strategies. Thus, the Operator's set of mixed strategies is given by

$$P := \left\{ \mathbf{p} := (p_1, \ldots, p_{|I|})^T \in [0,1]^{|I|} : \sum_{i=1}^{|I|} p_i = 1 \right\} \tag{19.1}$$

and that for the Inspectorate by

$$Q := \left\{ \mathbf{q} := (q_1, \ldots, q_{|J|})^T \in [0,1]^{|J|} : \sum_{j=1}^{|J|} q_j = 1 \right\} . \tag{19.2}$$

If the players decide to play the mixed strategy combination $(\mathbf{p}, \mathbf{q})$, then the (expected) payoff to the Operator is given by

$$Op(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{|I|} \sum_{j=1}^{|J|} p_i \, q_j \, Op(i, j) \tag{19.3}$$

and that to the Inspectorate by

$$In(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{|I|} \sum_{j=1}^{|J|} p_i \, q_j \, In(i, j) \, . \tag{19.4}$$

Thus, we define

Definition 19.1. *The quadruple (P, Q, Op, In) is called a two-person game in normal form.*

Having in mind inspection games where both players do not cooperate in the sense that they do not take into account mutually binding agreements, we consider only so-called *non-cooperative* two-person games. The solution concept for these games is the so-called *Nash equilibrium*, see Nash (1951), which says that any unilateral deviation from this equilibrium does not improve the deviator's payoff:

Definition 19.2. *Given the two-person normal form game (P, Q, Op, In) as defined by (19.1) – (19.4). Then the pair of strategies $(\mathbf{p}^*, \mathbf{q}^*) \in P \times Q$ consitutes a Nash equilibrium if and only if*

$$Op^* := Op(\mathbf{p}^*, \mathbf{q}^*) \geq Op(\mathbf{p}, \mathbf{q}^*) \qquad \textit{and} \qquad In^* := In(\mathbf{p}^*, \mathbf{q}^*) \geq In(\mathbf{p}^*, \mathbf{q}) \tag{19.5}$$

for any $\mathbf{p} \in P$ and any $\mathbf{q} \in Q$,

It can be shown, see Nash (1951), that every non-cooperative two-person game with finite pure strategy sets possesses at least one Nash equilibrium in mixed strategies, but of course – see the example in Section 3.1 – not always in pure strategies.

How can equilibrium strategies be found? Let us just mention three techniques; a comprehensive overview can be found in McKelvey and McLennan (1996): First, Nash's proof relies on the fact that a Nash equilibrium can be seen as a fix point of an appropriate continuous vector function that maps a closed, bounded and convex set onto itself. Applying Brouwer's fix point theorem provides the existence of a fix point and thus, of a Nash equilibrium. An algorithm for finding a Nash equilibrium that uses fix point computations has been developed by Scarf; see Scarf (1967) and Scarf and Hansen (1973).

Second, it can be shown that a Nash equilibrium of a bimatrix game is equivalent to the determination of a solution of an appropriate Linear Complementary Problem (LCP); see Peters and Vrieze (1992) or Canty (2003). The LCP formulation has the advantages that – in principle – *all* Nash equilibria of a bimatrix game can be found, and that there exist efficient algorithms for finding at least one solution of an LCP, a well-known one being the algorithm of Lemke and Howson (1964) that finds one Nash equilibrium, but usually, not all. It should be mentioned that Canty's Mathematica® programs, see Canty (2003), allow the determination of symbolic Nash equilibria, a possibility – in particular for the explicit solution of large size games as they occur in practical applications – that has, at least from our point of view, not yet received the attention it deserves. His programs has been used to find Nash equilibria resp. optimal strategies (see below) for the games in Sections 6.1, 6.3 and 17.1, and in Chapters 12 and 15, which confirms in a convincing way the programs' usefulness.

Third, Nash equilibria can be found using the indifference principle that is explicitly applied in Section 5.2 and Chapters 16 and 17.[2]

Theorem 19.1 (Indifference Principle). *Given the two-person normal form game (P, Q, Op, In) as defined by (19.1) – (19.4). Then the two statements are equivalent:*

- *(i) $(\mathbf{p}^*, \mathbf{q}^*) \in P \times Q$ constitutes a Nash equilibrium with the payoffs Op^* and In^*.*
- *(ii) If $Op^* > Op(i, \mathbf{q}^*)$ for a pure strategy $i \in I$, then $p_i^* = 0$, and if $In^* > In(\mathbf{p}^*, j)$ for a pure strategy $j \in J$, then $q_j^* = 0$.*

[2] The ominous *explicitly* refers to the fact that many other Nash equilibria resp. optimal strategies in this monograph have been determined using the indifference principle, for which the determination, however, is not presented here but only the prove that they constitute a Nash equilibrium resp. saddle point.

Proof. The proof can be found in Myerson (1991), pp. 93, or – for zero-sum matrix games – in Morris (1994), pp. 49. □

The name indifference principle can be motivated from (ii) of Theorem 19.1: The equilibrium strategy of the Inspectorate $\mathbf{q}^*$ is determined such that the Operator is rendered indifferent regarding his pure strategies which he plays with positive probability, and the Operator's equilibrium strategy $\mathbf{p}^*$ is determined such that the Inspectorate is rendered indifferent regarding its pure strategies which it plays with positive probability.

As mentioned, Definition 19.2 does not give any advise how to find equilibria. But even in case one wants to prove that a pair of strategies $(\mathbf{p}^*, \mathbf{q}^*)$ – found in some way or other – is a pair of equilibrium strategies, one may encounter technical difficulties. Therefore, it is helpful to know that $(\mathbf{p}^*, \mathbf{q}^*)$ constitutes a Nash equilibrium if and only if

$$\begin{aligned} Op^* &\geq Op(i, \mathbf{q}^*) \quad \text{for all} \quad i = 1, \ldots, |I| \quad \text{and} \\ In^* &\geq In(\mathbf{p}^*, j) \quad \text{for all} \quad j = 1, \ldots, |J|, \end{aligned} \tag{19.6}$$

in other words, it is sufficient to consider only the pure strategies of the players. If one compares the complexity to find a Nash equilibrium (see the techniques above) to the relatively simple task of showing that the $|I| + |J|$ inequalities in (19.6) are fulfilled, it becomes clear why it is much easier to prove that proposed equilibrium strategies fulfil the Nash equilibrium conditions; see also the comment on p. 25 for zero-sum games.

Nash equilibria are not unique in general, neither the equilibrium strategies nor the equilibrium payoffs. In the case that there are multiple equilibria, the selection of one equilibrium, which is then considered the solution of the game, represents a major problem; see van Damme (1987). In this monograph this problem is not addressed systematically, i.e., no attempt is made either to determine all equilibria of a game or else, to prove that the presented equilibria are unique. There are, however, exceptions: For special reasons, multiple equilibrium strategies are discussed in Sections 15.2, 15.3 and 17.1. More than that, sets of equilibrium strategies of the Inspectorate are presented in all cases in which the Operator behaves legally.

At some places in this monograph we use the concept of strategically equivalent games, see Maschler et al. (2013): Two normal form games (P, Q, Op, In) and $(P, Q, \widetilde{Op}, \widetilde{In})$ are called *strategically equivalent*, if and only if $\widetilde{Op}$ resp. $\widetilde{In}$ is a positive affine transformation of Op resp. In, i.e., there exist $a, b > 0$ and $c, d \in \mathbb{R}$ with $\widetilde{Op}(i,j) = a\,Op(i,j) + c$ resp. $\widetilde{In}(i,j) = b\,Op(i,j) + d$ for any $(i,j) \in I \times J$. It follows directly from this definition, that each Nash equilibrium in (P, Q, Op, In) remains a Nash equilibrium in $(P, Q, \widetilde{Op}, \widetilde{In})$ with the equilibrium payoffs $\widetilde{Op}^* = a\,Op^* + c$ and $\widetilde{In}^* = b\,In^* + d$.

If we define for any $a, b > 0$, any $c_j \in \mathbb{R}$, all $j = 1, \ldots, |J|$, any $d_i \in \mathbb{R}$, and all $i = 1, \ldots, |I|$,

$$\widetilde{Op}(i,j) := a\,Op(i,j) + c_j \qquad \text{and} \qquad \widetilde{In}(i,j) := b\,In(i,j) + d_i\,, \tag{19.7}$$

then the game (P, Q, Op, In) is strategically equivalent to the game $(P, Q, \widetilde{Op}, \widetilde{In})$.

So far, the Inspectorate's strategy set (19.2) consists of probabilities which add up to one. In Chapters 15 and 24 and Section 17.1, however, the Inspectorate/Customs uses another type of strategy set; see (15.51), (17.18) and (24.21). Thus, for demonstration, let us assume that the Inspectorate's/Customs strategy set is now given by

$$\tilde{Q} := \left\{ \tilde{\mathbf{q}} := (\tilde{q}_1, \ldots, \tilde{q}_{|J|})^T \in [0,1]^{|J|} : \sum_{j=1}^{|J|} \tilde{q}_j = k \right\},$$

where k is a given integer with $k \le |J|$. Thus, a strategy set is considered in which the probabilities $\tilde{q}_j$ do no longer add up to one. The payoffs to both players in Chapters 15 and 24 and Section 17.1 have essentially the same structure as (19.3) and (19.4).

The Nash equilibrium of the normal form game $(P, \tilde{Q}, Op, In)$ is still defined by (19.5), the inequalities (19.6), however, need to be modified: $(\mathbf{p}^*, \tilde{\mathbf{q}}^*)$ constitutes a Nash equilibrium of $(P, \tilde{Q}, Op, In)$ if and only if

$$Op(\mathbf{p}^*, \tilde{\mathbf{q}}^*) \ge Op(i, \tilde{\mathbf{q}}^*) \qquad \text{and} \qquad In(\mathbf{p}^*, \tilde{\mathbf{q}}^*) \ge In(\mathbf{p}^*, \tilde{\mathbf{q}}) \tag{19.8}$$

for all $i = 1, \ldots, |I|$ and any $\tilde{\mathbf{q}} \in \tilde{Q}$.

An important special class of non-cooperative games is the class of *zero-sum* games. Here, the payoffs to both players add up to zero for any (pure) strategy combination, i.e., $In(i,j) = -Op(i,j)$ for all $i = 1, \ldots, |I|$ and for all $j = 1, \ldots, |J|$, and thus,

$$In(\mathbf{p}, \mathbf{q}) = -Op(\mathbf{p}, \mathbf{q}), \tag{19.9}$$

for any $\mathbf{p} \in P$ and any $\mathbf{q} \in Q$, which means that only the payoff to the Operator needs to be specified. Accordingly we also call these games *matrix* games.

Because for zero-sum games the Nash conditions (19.5) reduce to simple forms,

$$Op(\mathbf{p}^*, \mathbf{q}^*) \ge Op(\mathbf{p}, \mathbf{q}^*) \quad \text{and} \quad -Op(\mathbf{p}^*, \mathbf{q}^*) \ge -Op(\mathbf{p}^*, \mathbf{q})$$

for any $\mathbf{p} \in P$ and any $\mathbf{q} \in Q$, we call the Nash equilibrium in this case a *saddle point*, and the Nash conditions *saddle point condition*:

Definition 19.3. *Given the two-person normal form game (P, Q, Op) as defined by (19.1), (19.2) and (19.9). Then the pair of strategies $(\mathbf{p}^*, \mathbf{q}^*) \in P \times Q$ constitutes a saddle point if and only if*

$$Op(\mathbf{p}, \mathbf{q}^*) \le Op(\mathbf{p}^*, \mathbf{q}^*) \le Op(\mathbf{p}^*, \mathbf{q}) \tag{19.10}$$

for any $\mathbf{p} \in P$ and any $\mathbf{q} \in Q$.

Saddle point strategies are also called *optimal strategies*. In analogy to the property in (19.6), $\mathbf{p}^*$ and $\mathbf{q}^*$ are optimal strategies if and only if

$$Op(i, \mathbf{q}^*) \le Op(\mathbf{p}^*, \mathbf{q}^*) \le Op(\mathbf{p}^*, j), \tag{19.11}$$

for all $i = 1, \ldots, |I|$ and for all $j = 1, \ldots, |J|$, i.e., both inequalities have only to be proven for the pure strategies of the players.

An important property of zero-sum games for practical applications is the following: If a zero-sum game has the saddle points $(\mathbf{p}^*, \mathbf{q}^*)$ and $(\mathbf{p}_1^*, \mathbf{q}_1^*)$, then $(\mathbf{p}^*, \mathbf{q}_1^*)$ and $(\mathbf{p}_1^*, \mathbf{q}^*)$ are also saddle points of the game with the property

$$Op(\mathbf{p}^*, \mathbf{q}^*) = Op(\mathbf{p}^*, \mathbf{q}_1^*) = Op(\mathbf{p}_1^*, \mathbf{q}^*) = Op(\mathbf{p}_1^*, \mathbf{q}_1^*),$$

i.e., all saddle points are interchangeable and lead to the same optimal payoff to the Operator. For this reason finding all saddle points is more a mathematical challenge than necessary for applications. There is, however, an exception: In Section 4.2 it is demonstrated that a selection of optimal strategies based on practical considerations may be useful.

Again, there are different techniques for finding saddle points. Quite generally, saddle points of a matrix games can be determined with the help of Linear Programming methods; see von Neumann and Morgenstern (1947), Karlin (1959b) or Morris (1994). Symbolic saddle points can again be found using the Mathematica® programs in Canty (2003).

19.2 Extensive form games

Normal form games are deceptively simple. The concept of a strategy, however, comprises many different aspects, for example sequencing, information, chance and others. These aspects, which are so important for the description and analysis of real life conflicts, are much better expressed in extensive form games.

A non-cooperative game in extensive form is a graphical representation of the possible moves of all players from the beginning of the game until its end. It has the form of the tree – growing from the top to the bottom – where a set of branches starting at some point indicate a player's alternative at that point.

A precise mathematical definition of extensive form games has been given, for example, by Hart (1992) and Myerson (1991) and goes as follows. Let us mention in passing that we present the general definition for n-person extensive form games, even though we consider only the case $n = 2$, since it is not more complicated then the special one.

Definition 19.4. *An n-person game non-cooperative extensive form game is a rooted tree – usually growing from the top to the bottom – together with labels at every decision point or node and decision alternative or branch, defined as follows:*

- *Each non-terminal node has a player label that is taken from the set $\{0, 1, \dots, n\}$. Nodes that are assigned a player label 0 are called chance nodes. The set $\{1, 2, \dots, n\}$ represents the set of players in the game, and for each individual player i in this set, the nodes with the player label i are decision nodes that are controlled by that player.*
- *Every alternative at a chance node has a label that specifies its probability. At each chance node, these chance probabilities of the alternatives are non-negative numbers that sum to one.*
- *Every decision point or node that is controlled by a player has a second label that specifies the information state that the player would have if the path of the play reached this node. When the path of the play reaches a node controlled by a player, the player knows only the information state on the current node. Thus, two nodes that belong to the same player should have the same information state only if the player would be unable to distinguish between the situations represented by these nodes when either occurs in the play of the game.*
- *Each alternative or branch at a node that is controlled by a player has an alternative or move label. Furthermore, for any two nodes x and y that have the same player label and*

the same information label, there must be one alternative or move at both nodes that has the same move label.

- *Each terminal or outcome node has a payoff label for each player, such that for each player i, there is a payoff u_i, measured on some utility scale.*

As mentioned above, we consider only two-person games. Chance nodes play a major role in the inspection games of this monograph: Once the Inspectorate performs an inspection after the beginning of an illegal activity, a chance node describes whether it will be detected with probability $1-\beta$, or not with probability β. Or, once the inspection is performed before the beginning of an illegal activity, a chance node describes whether the legal behaviour will be confirmed with probability $1-\alpha$ or a false alarm will be raised with probability α.

A *pure* strategy in an extensive form game is any rule for determining a move at every possible information state in the game. Mathematically, a strategy is a function that maps information states into moves. For each player i let S_i denote the set of possible information states of player i in the game. For each information state s in S_i let D_s denote the set of moves that would be available to player i when he moved at a node with information state s. Then the set of pure strategies for player i in the extensive form game is the cartesian product $\times_{s \in S_i} D_s$. In other words, a pure strategy of a player is a complete plan for his choices at all his information sets.

A *mixed* strategy means that the player chooses, before the beginning of the game, one such comprehensive plan at random according to a certain probability distribution.

An alternative method of randomization for the player is to make an independent random choice at each one of his information states. That is, rather than selecting for every information set, one definite choice – as in a pure strategy – he specifies instead a probability distribution over the set of choices there; moreover, the choices at different information sets are stochastically independent. These randomization procedures are called *behaviour(al)* strategies.

Without going into details of *games with perfect recall* – which we are considering exclusively in our applications – we assert that mixed strategies and behavioural strategies of these games are equivalent to each other in the sense that they lead to the same (expected) payoffs; see Hart (1992).

The concept of the Nash equilibrium and the saddle point in a zero-sum game is defined in the same way as in normal form games. It can be determined in different ways, let us just mentioned three of them: In extensive form games with *perfect information*, i.e., in extensive form games where all information states of all players consists of exactly one decision node, a backward induction procedure is used which means that non-optimal moves are eliminated from the bottom to the top. In this monograph such games do not occur, even though we apply this method to some branches of extensive games. Or one uses behavioural strategies and tries to find a Nash equilibrium with the help of the Nash conditions. Or one transforms the extensive form game into a normal form game and applies the solution techniques available for this type of games. In our application we use both methods.

It should be mentioned in passing that normal form games, which are derived from an extensive form game, may have more Nash equilibria than the latter ones, but in our applications we do not encounter this difficulty.

Chapter 20

Receiver Operating Characteristic of a binary classifier system

In Section 9.5 we have introduced the Receiver Operating Characteristic (ROC) to illustrate the performance of a binary classifier system, and we have assumed that the requirements (9.69) are fulfilled. In this chapter two test procedures namely an intuitive test and a Neyman-Pearson test are considered for two test problems. It will be shown, that all tests meet the requirements (9.69), but that the ROC curve is not always a concave function. Note that the concavity of the ROC curve is usual needed to show that game theoretical solutions are unique; see, e.g., Avenhaus and Canty (1996) and Avenhaus and Krieger (2014). It is, however, not utilized in this monograph.

We introduce the requirements (9.69) here again for easy reference, namely

$$\beta(0) = 1 \quad \text{and} \quad \beta(1) = 0, \tag{20.1}$$

$$\alpha + \beta(\alpha) \leq 1, \tag{20.2}$$

$$\beta'(\alpha) < 0\,. \tag{20.3}$$

Requirement (20.1) can be motivated as follows: If one wants to avoid any false alarm, i.e., $\alpha = 0$, then one must never raise an alarm, which means that an illegal activity will never be detected, i.e., $\beta = 1$. Conversely, if one wants to detect the illegal activity with certainty, i.e., $\beta = 0$, then one always has to raise an alarm, i.e., $\alpha = 1$. In condition (20.2) the false alarm probability α has to be smaller than the detection probability $1 - \beta(\alpha)$, i.e., the probability to raise an alarm, if there is no illegal activity, must not be smaller than the probability to detect the illegal activity. A decision procedure with this property is called *unbiased*; see Rohatgi (1976). Finally, (20.3) means that the non-detection probability decreases with increasing false alarm probability. This is a reasonable requirement to any detection system, consider, e.g., a simple fire alarm device consisting of a bi-metal strip.

In statistical terms we deal here with a test problem: Let us call H_0 the null hypothesis which is in our case the hypothesis that the Operator behaves legally, and H_1 the alternative hypothesis, in our case the hypothesis that the Operator behaves illegally. Then, α is the probability to reject H_0 if it is true, and β the probability to reject H_1 if it is true.

For the purpose of illustration a sample of size one is taken to test the two simple hypotheses

H_0 and H_1.[1] Consider a Gaussian (or Normally) distributed random variable X with expected value μ and variance σ^2 and assume

$$H_0 : X \sim \mathcal{N}(\mu_0, \sigma_0^2) \qquad \text{and} \qquad H_1 : X \sim \mathcal{N}(\mu_1, \sigma_1^2)\,, \tag{20.4}$$

where $\mu_0 < \mu_1$ is assumed and the both cases $\sigma_0^2 = \sigma_1^2$ and $\sigma_0^2 \neq \sigma_1^2$ are considered. Because $\mathbb{E}_{H_0}(X) = \mu_0 < \mu_1 = \mathbb{E}_{H_1}(X)$ an *intuitive test* is given by the critical region C

$$C := \{x \in \mathbb{R} : x > c\} \tag{20.5}$$

for some c, i.e., for a realization x larger than a threshold c, H_0 is rejected and H_1 accepted, and for a realization x smaller than the threshold, H_1 is rejected and H_0 accepted. Because we are interested in a size α test, i.e., $\mathbb{P}_{H_0}(C) = \alpha$, we get for the threshold c

$$c = \mu_0 + \sigma_0\, \Phi^{-1}(1-\alpha)\,, \tag{20.6}$$

where $\Phi(.)$ denotes the cumulative distribution function of a Gaussian distributed random variable with expected value 0 and variance 1 (standard normal cumulative distribution function)

$$\Phi(z) := \int_{-\infty}^{z} \frac{1}{\sqrt{2\,\pi}}\, e^{-x^2/2}\, dx$$

and where $\Phi^{-1}(.)$ is the inverse function of $\Phi(.)$. Thus, using (20.6), the error second kind probability, i.e., the non-detection probability $\beta = \beta(\alpha)$, is for any σ_0 and σ_1 given by

$$\beta(\alpha) = \mathbb{P}_{H_1}(\bar{C}) = \mathbb{P}_{H_1}(X < c) = \Phi\left(\frac{c-\mu_1}{\sigma_1}\right) = \Phi\left(\frac{\sigma_0}{\sigma_1}\,\Phi^{-1}(1-\alpha) - \frac{\mu_1-\mu_0}{\sigma_1}\right)\,. \tag{20.7}$$

This function, i.e., one minus the ROC curve for the intuitive test, is represented in Figure 20.1 for $\mu_0 = 0$ and $\mu_1 = 1$, and for the cases $\sigma_0^2 = \sigma_1^2 = 1$ and $\sigma_0^2 = 1, \sigma_1^2 = 4$. A random guess would give a point along the diagonal line which is also called the line of no-discrimination.[2]

Before discussing Figure 20.1 in more detail, we present the *Neyman-Pearson test* for two simple hypotheses, in our case for the test problem (20.4), i.e., a size α test which minimizes the second kind error probability β; see Rohatgi (1976) or Casella and Berger (2002). Let $f_{H_0}(x)$ and $f_{H_1}(x)$ be the density functions of the random variable X under H_0 and H_1. Then according to the Neyman-Pearson Lemma the critical region of this test is given by

$$C_k := \left\{x \in \mathbb{R} : \frac{f_{H_1}(x)}{f_{H_0}(x)} \geq k\right\} \tag{20.8}$$

for some $k \in [0, \infty)$. Using the density functions

$$f_{H_0}(x) = \frac{1}{\sqrt{2\,\pi}\,\sigma_0}\, e^{-x^2/(2\,\sigma_0^2)} \qquad \text{and} \qquad f_{H_1}(x) = \frac{1}{\sqrt{2\,\pi}\,\sigma_1}\, e^{-(x-1)^2/(2\,\sigma_1^2)}\,,$$

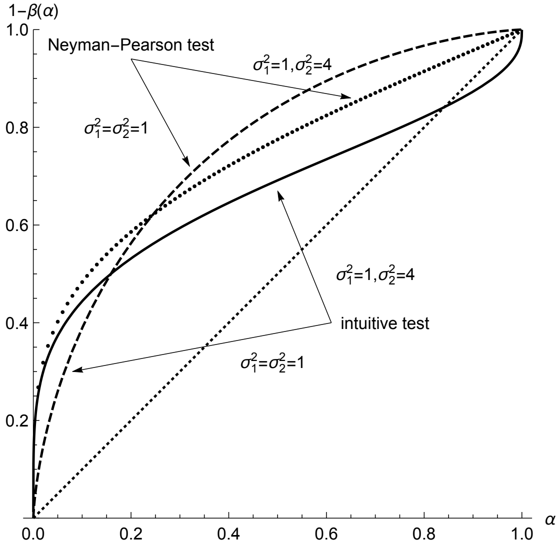
we obtain for the test problem (20.4) with $\mu_0 = 0$ and $\mu_1 = 1$ the critical region

$$C_{k'} := \left\{x \in \mathbb{R} : -\frac{(x-1)^2}{\sigma_1^2} + \frac{x^2}{\sigma_0^2} \geq k'\right\} \tag{20.9}$$

[1] In Statistics, a hypothesis is called simple if all parameters of the distribution of the random variable under consideration are specified, in contrast to composite hypothesis, in which not all of the parameters are specified.

[2] The 45° ROC curve is attained using a detector that bases its decision on flipping a coin, ignoring all the data; see Kay (1998).

Figure 20.1 The Receiver Operating Characteristic (ROC) curves for the four tests.



for some k'. Note that if $\sigma_0^2 = \sigma_1^2$, then the critical region (20.9) simplifies to the critical region (20.5) of the intuitive test. This is the reason why in Figure 20.1 the case $\sigma_0^2 = \sigma_1^2 = 1$ is also referred to as Neyman-Pearson test.

For $\sigma_0^2 = 1$ and $\sigma_1^2 = 4$, (20.9) simplifies to

$$C_{k''} = \left\{x \in \mathbb{R} : 3\,x^2 + 2\,x - 1 \geq k''\right\}$$
$$= \left(-\infty, \frac{1}{3}\left(-1 - \sqrt{4 + 3\,k''}\right)\right] \cup \left[\frac{1}{3}\left(-1 + \sqrt{4 + 3\,k''}\right), \infty\right)$$

for some k''. Because $\beta(\alpha)$ cannot be given explicitly as in (20.7), we first determine numerically for any $\alpha = i/100$, $i = 1, \ldots, 99$, the threshold k'' that fulfils

$$\mathbb{P}_{H_0}(C_{k''}) = 1 - \left(\Phi\left(\frac{1}{3}\left(-1 + \sqrt{4 + 3\,k''}\right)\right) - \Phi\left(\frac{1}{3}\left(-1 - \sqrt{4 + 3\,k''}\right)\right)\right) = \frac{i}{100}\,,$$

and determine then the error second kind probability

$$\beta\left(\frac{i}{100}\right) = \mathbb{P}_{H_1}(\bar{C}_{k''})$$
$$= \Phi\left(\frac{1}{6}\left(-1 + \sqrt{4 + 3\,k''}\right) - \frac{1}{2}\right) - \Phi\left(\frac{1}{6}\left(-1 - \sqrt{4 + 3\,k''}\right) - \frac{1}{2}\right).$$

The pairs $(i/100, \beta(i/100))$, $i = 1, \ldots, 99$, lead to the curve referred to as "Neyman-Pearson tests, $\sigma_0^2 = 1, \sigma_1^2 = 4$" in Figure 20.1.

Let us comment Figure 20.1: First, (20.1) is fulfilled for all four tests which lead only to three different ROC curves.

Second, only the Neyman-Pearson tests fulfil (20.2).

Third, for $\sigma_0^2 = 1$ and $\sigma_1^2 = 4$ the ROC curve of the Neyman-Pearson test lies well above that of the intuitive test for all α, which illustrates that the Neyman-Pearson test maximizes the detection probability $1 - \beta(\alpha)$ for all size α tests, i.e., especially for the intuitive test.

Fourth, while all three tests have a monotone increasing ROC curve, i.e., (20.3) is valid, only the Neyman-Pearson tests lead to concave ROC curves. The intuitive test is biased for all $0.8414 < \alpha < 1$.

Finally, and in view of further studies, note that according to van Trees (1968) and Pepe (2004), the following statements regarding the ROC curve hold for all simple hypothesis testing problems: Let $f_{H_0}(x)$ and $f_{H_1}(x)$ be the density functions of the random variable X under H_0 and H_1. If the so-called *likelihood ratio* $f_{H_1}(x)/f_{H_0}(x)$ is a continuous function, then

(1) the points $(0, 0)$ and $(1, 1)$ belong to the ROC curve;

(2) the ROC curve is a concave function;

(3) the ROC curve is above the line of no-discrimination, i.e., the test/classifier is unbiased. This property follows from (1) and (2).

Chapter 21

Proof of Lemma 10.3

We have to prove that the Nash equilibrium conditions

$$Op_2^*(t_3) \geq Op_2(t_3; (g_3, g_2(t_2^*), g_1(t_1^*)), (t_2^*, t_1^*)) \qquad \text{and} \tag{21.1}$$

$$In_2^*(t_3) \geq In_2(t_3; (g_3^*, g_2^*(t_2), g_1^*(t_1)), (t_2, t_1)) \tag{21.2}$$

are fulfilled for any $\mathbf{g} = (g_3, g_2(t_2^*), g_1(t_1^*)) \in G_2$ and any $\mathbf{t} = (t_2, t_1) \in \mathcal{T}_2$, where $Op_2(t_3; \mathbf{g}, \mathbf{t})$ and $In_2(t_3; \mathbf{g}, \mathbf{t})$ are defined by (10.39) and (10.40), respectively.

Ad (i): We first prove that the Operator's Nash equilibrium condition (21.1) is satisfied. Using (10.43), we get

$$\begin{aligned} t_1^* - t_2^* &= \frac{1-\beta}{2-\beta}\,(t_0 - t_3 - (t_2^* - t_3)) - \frac{f}{d}\,\frac{\alpha}{2-\beta} \\ &= \frac{1-\beta}{3-2\,\beta}\,(t_0 - t_3) - \frac{f}{d}\,\alpha\,\frac{1}{2-\beta}\,\left(1 - \frac{(1-\beta)\,(3-3\,\beta+\beta^2)}{3-2\,\beta}\right) \end{aligned} \tag{21.3}$$

and therewith

$$t_1^* - t_3 = t_1^* - t_2^* + t_2^* - t_3 = 2\,\frac{1-\beta}{3-2\,\beta}\,(t_0 - t_3) - \frac{f}{d}\,\alpha\,\frac{3-\beta}{3-2\,\beta} \tag{21.4}$$

and further

$$t_0 - t_1^* = t_0 - t_3 - (t_1^* - t_3) = \frac{1}{3-2\,\beta}\,(t_0 - t_3) + \frac{f}{d}\,\alpha\,\frac{3-\beta}{3-2\,\beta}\,. \tag{21.5}$$

Therefore, the coefficient of $g_1(t_1^*)$ in $Op_2(t_3; (g_3, g_2(t_2^*), g_1(t_1^*)), (t_2^*, t_1^*))$ according to (10.39) is

$$-d\,(t_0 - t_1^*) + b = -d\,\left(\frac{1}{3-2\,\beta}\,(t_0 - t_3) + \frac{f}{d}\,\alpha\,\frac{3-\beta}{3-2\,\beta}\right) + b < 0$$

because of the left hand inequality of (10.41). Thus, the right hand side of (21.1) is maximized by $g_1^*(t_1^*) = 0$, and we get

$$Op_2(t_3; (g_3, g_2(t_2^*), g_1(t_1^*)), (t_2^*, t_1^*)) \leq Op_2(t_3; (g_3, g_2(t_2^*), 0), (t_2^*, t_1^*))$$

for any $g_1(t_1^*) \in [0,1]$. Furthermore, we obtain, using (21.3) and (21.5), for the coefficient of $g_2(t_2^*)$ in $Op_2(t_3; (g_3, g_2(t_2^*), 0), (t_2^*, t_1^*))$

$$-f\,\alpha - d\,[t_1^* - t_2^* - (1-\beta)\,(t_0 - t_1^*)] = 0$$

i.e., the coefficient of $g_2(t_2^*)$ vanishes. Using this fact and (10.43), (21.4) and (21.5), we get for the coefficient of g_3 in $Op_2(t_3; (g_3, 0, 0), (t_2^*, t_1^*))$

$$-d\,[(1-\beta)\,(t_2^* - t_3) + \beta\,(1-\beta)\,(t_1^* - t_3) + \beta^2\,(t_0 - t_3)] + d\,(t_0 - t_1^*) - 2\,f\,\alpha = 0$$

after some lengthy calculations. Therefore, we finally have by (10.39), (10.44) and (21.5),

$$\begin{aligned} Op_2(t_3; (g_3, g_2(t_2^*), g_1(t_1^*)), (t_2^*, t_1^*)) &\le Op_2(t_3; (0,0,0), (t_2^*, t_1^*)) \\ &= d\,(t_0 - t_1^*) - b - 2\,f\,\alpha = Op_2^*(t_3) \end{aligned}$$

for any $g_3, g_2(t_2^*), g_1(t_1^*) \in [0,1]$, i.e., the Operator's Nash equilibrium condition (21.1) is fulfilled.

In order to prove that the Inspectorate's Nash equilibrium condition (21.2) we obtain from (10.40) for any $\mathbf{t} = (t_2, t_1) \in \mathcal{T}_2$, using (10.42),

$$\begin{aligned} &In_2(t_3; (g_3^*, g_2^*(t_2), g_1^*(t_1)), (t_2, t_1)) \\ &= \frac{1}{3-2\,\beta}\Big(-a\,[(1-\beta)\,(t_2 - t_3) + \beta\,(1-\beta)\,(t_1 - t_3) + \beta^2\,(t_0 - t_3)]\Big) \\ &+ \frac{1-\beta}{3-2\,\beta}\Big(-a\,[(1-\beta)\,(t_1 - t_2) + \beta\,(t_0 - t_2) + (t_0 - t_1)] - 3\,g\,\alpha\Big)\,. \end{aligned}$$

Collecting the terms with t_1, t_2 and t_3 yields

$$\begin{aligned} &In_2(t_3; (g_3^*, g_2^*(t_2), g_1^*(t_1)), (t_2, t_1)) \\ &= \frac{-a}{3-2\,\beta}\Big(t_3\,[-(1-\beta) - \beta\,(1-\beta) - \beta^2] + t_2\,[1 - \beta - (1-\beta)^2 - \beta\,(1-\beta)] \\ &+ t_1\,[\beta\,(1-\beta) + (1-\beta)^2 - (1-\beta)] + t_0\,[\beta^2 + \beta\,(1-\beta) + (1-\beta)]\Big) - 3\,g\,\alpha\,\frac{1-\beta}{3-2\,\beta} \\ &= \frac{-a}{3-2\,\beta}\,(t_0 - t_3) - 3\,g\,\alpha\,\frac{1-\beta}{3-2\,\beta}\,, \end{aligned}$$

i.e., $In_2^*(t_3)$ according to (10.44) for any $\mathbf{t} = (t_2, t_1) \in \mathcal{T}_2$. Thus, (21.2) is fulfilled as equality.

We still have to show $t_3 < t_2^* < t_1^* < t_0$. The inequality $t_3 < t_2^*$ is, using (10.43), equivalent to

$$(1-\beta)\,(t_0 - t_3) - \frac{f}{d}\,\alpha\,(3 - 3\,\beta + \beta^2) > 0$$

which is equivalent to the right hand inequality of (10.41). Because of

$$1 - \frac{(1-\beta)\,(3 - 3\,\beta + \beta^2)}{3 - 2\,\beta} = \frac{\beta\,(2-\beta)^2}{3 - 2\,\beta}\,,$$

(21.3) is equivalent to

$$t_1^* - t_2^* = \frac{1-\beta}{3-2\,\beta}\,(t_0 - t_3) - \frac{f}{d}\,\alpha\,\frac{\beta\,(2-\beta)}{3-2\,\beta}\,.$$

Thus, $t_2^* < t_1^*$ is equivalent to

$$(1-\beta)\,(t_0-t_3) - \frac{f}{d}\,\alpha\,\beta\,(2-\beta) > 0\,. \tag{21.6}$$

Because of

$$3 - 3\,\beta + \beta^2 - \beta\,(2-\beta) = 3 - 5\,\beta + 2\,\beta^2 = (3-2\,\beta)\,(1-\beta) > 0$$

we obtain

$$3 - 3\,\beta + \beta^2 > \beta\,(2-\beta)$$

and with the right hand inequality of (10.41)

$$\frac{f}{d}\,\frac{\alpha}{1-\beta} < \frac{t_0 - t_3}{3 - 3\,\beta + \beta^2} < \frac{t_0 - t_3}{\beta\,(2-\beta)}\,,$$

which is equivalent to (21.6). Finally, $t_1^* < t_0$ is by (21.5) equivalent to

$$t_0 - t_3 + \frac{f}{d}\,\alpha\,(3-\beta) > 0\,,$$

which is fulfilled anyhow.

Ad (ii): Because the payoff to the Operator in case of legal behaviour is $-2\,f\,\alpha$, he will choose this strategy in equilibrium, using (10.44), if

$$-2\,f\,\alpha \geq d\,\frac{1}{3-2\,\beta}\,(t_0-t_3) - f\,\alpha\,\frac{3\,(1-\beta)}{3-2\,\beta} - b$$

which follows from (10.45). Whereas the Nash equilibrium condition (21.2) for the Inspectorate is fulfilled as equality, the proof of (21.1) is more complicated. We first note that (21.1) is equivalent to

$$Op_2^*(t_3) \geq Op_2(t_3; (0, g_2(t_2^*), g_1(t_1^*)), (t_2^*, t_1^*)) \tag{21.7}$$

$$Op_2^*(t_3) \geq Op_2(t_3; (1, 0, g_1(t_1^*)), (t_2^*, t_1^*)) \tag{21.8}$$

$$Op_2^*(t_3) \geq Op_2(t_3; (1, 1, 0), (t_2^*, t_1^*)) \tag{21.9}$$

$$Op_2^*(t_3) \geq Op_2(t_3; (1, 1, 1), (t_2^*, t_1^*))\,. \tag{21.10}$$

This can be seen as follows:

$\Longrightarrow$: Because (21.1) holds for any $g_3, g_2(t_2), g_1(t_1) \in [0,1]$, we get (21.7) by successively choosing $\{g_3 = 0, g_2(t_2^*) \in [0,1], g_1(t_1^*) \in [0,1]\}$, $\{g_3 = 1, g_2(t_2^*) = 0, g_1(t_1^*) \in [0,1]\}$, $\{g_3 = g_2(t_2^*) = 1, g_1(t_1^*) = 0\}$ and finally $\{g_3 = g_2(t_2^*) = g_1(t_1^*) = 1\}$.

$\Longleftarrow$: We multiply (21.10) by $g_1(t_1^*)$ and (21.9) by $1 - g_1(t_1^*)$ and add the two inequalities. This gives

$$-2\, f\, \alpha \geq (1 - g_1(t_1^*))\, Op_2(t_3; (1,1,0), (t_2^*, t_1^*)) + \, g_1(t_1^*)\, Op_2(t_3; (1,1,1), (t_2^*, t_1^*))\,.$$

Now let us multiply this inequality by $g_2(t_2^*)$ and (21.8) by $1 - g_2(t_2^*)$ and add these two inequalities. This gives

$$\begin{aligned} -2\, f\, \alpha \geq\; & (1 - g_2(t_2^*))\, Op_2(t_3; (1,0,g_1(t_1^*)), (t_2^*, t_1^*)) \\ & + g_2(t_2^*) \Big[(1 - g_1(t_1^*))\, Op_2(t_3; (1,1,0), (t_2^*, t_1^*)) + \, g_1(t_1^*)\, Op_2(t_3; (1,1,1), (t_2^*, t_1^*))\Big]\,. \end{aligned}$$

If we finally multiply this inequality by g_3 and (21.7) by $(1 - g_3)$ and add these two inequalities, we get (21.1). Thus the equivalence is shown.

Whereas (21.10) is fulfilled as equality, the inequalities (21.7) – (21.9) are explicitly given by

$$\begin{aligned} -2\, f\, \alpha &\geq d\,[(1-\beta)\,(t_2^* - t_3) + \beta\,(1-\beta)\,(t_1^* - t_3) + \beta^2\,(t_0 - t_3)] - b \\ -2\, f\, \alpha &\geq d\,[(1-\beta)\,(t_1^* - t_2^*) + \beta\,(t_0 - t_2^*)] - b - f\,\alpha \\ -2\, f\, \alpha &\geq d\,(t_0 - t_1^*) - b - 2\, f\, \alpha\,, \end{aligned}$$

which are equivalent to (10.46).

This completes the proof of Lemma 10.3. □

Chapter 22

Recurrence relations used in Chapters 10, 11 and 12

In this chapter we prove the equivalence of three recursive relations.

Lemma 22.1. *Let $\beta \in [0, 1)$. For $k > 1$ and arbitrary $t_0, t_1, \ldots, t_k, t_{k+1} \in \mathbb{R}$ the three recursive relations*

$$t_n - t_{k+1} = (1-\beta)\,\frac{k+1-n}{1+k\,(1-\beta)}\,(t_0 - t_{k+1}) \tag{22.1}$$

$$t_n - t_{n+1} = \frac{1-\beta}{1+n\,(1-\beta)}\,(t_0 - t_{n+1}) \qquad \text{and} \tag{22.2}$$

$$t_n - t_{n+1} = \frac{1-\beta}{1+k\,(1-\beta)}\,(t_0 - t_{k+1})\,, \tag{22.3}$$

where $n = 1, \ldots, k$, are equivalent. If $t_{k+1} < t_0$ then $t_k < \ldots < t_1$.

Proof. The proof consists of three parts:

1: (22.1) $\Longleftrightarrow$ (22.3): For all $n = 1, \ldots, k-1$ we subtract (22.1) with $n \to n+1$ from (22.1) and get

$$t_n - t_{n+1} = \frac{1-\beta}{1+k\,(1-\beta)}\Big(k+1-n-k+n\Big)\,(t_0 - t_{k+1}) = \frac{1-\beta}{1+k\,(1-\beta)}\,(t_0 - t_{k+1})\,,$$

i.e., (22.3) for $n = 1, \ldots, k-1$. In case $n = k$, (22.1) and (22.3) are equivalent by definition.

Starting with (22.3) we get

$$\begin{aligned} t_n - t_{k+1} &= (t_n - t_{n+1}) + (t_{n+1} - t_{n+2}) + \ldots + (t_k - t_{k+1}) \\ &= (k+1-n)\,\frac{1-\beta}{1+k\,(1-\beta)}\,(t_0 - t_{k+1})\,, \end{aligned}$$

i.e., (22.1) for $n = 1, \ldots, k-1$. Again, for $k = n$, (22.3) and (22.1) are equivalent by definition.

2: (22.1) and (22.3) $\Longrightarrow$ (22.2): From (22.1) with $n \to n+1$ we get

$$t_{n+1} = (1-\beta)\,\frac{k-n}{1+k\,(1-\beta)}\,t_0 + \left(-(1-\beta)\,\frac{k-n}{1+k\,(1-\beta)} + 1\right)\,t_{k+1}$$

$$= (1-\beta)\,\frac{k-n}{1+k\,(1-\beta)}\,t_0 + \frac{1+n\,(1-\beta)}{1+k\,(1-\beta)}\,t_{k+1}\,,$$

or, equivalently

$$\frac{t_{k+1}}{1+k\,(1-\beta)} = \frac{1}{1+n\,(1-\beta)}\,\left(t_{n+1} - \frac{(1-\beta)\,(k-n)}{1+k\,(1-\beta)}\,t_0\right)\,.$$

Inserting this into (22.3) we get

$$t_n - t_{n+1} = \frac{1-\beta}{1+k\,(1-\beta)}\,t_0 - \frac{1-\beta}{1+n\,(1-\beta)}\,\left(t_{n+1} - \frac{(1-\beta)\,(k-n)}{1+k\,(1-\beta)}\,t_0\right)$$

$$= \frac{1-\beta}{1+k\,(1-\beta)}\,\left(1 + \frac{(1-\beta)\,(k-n)}{1+n\,(1-\beta)}\right)\,t_0 - \frac{1-\beta}{1+n\,(1-\beta)}\,t_{n+1}$$

$$= \frac{1-\beta}{1+n\,(1-\beta)}\,(t_0 - t_{n+1})\,,$$

i.e., (22.2).

3: (22.2) $\Longrightarrow$ (22.1): We show by induction with respect to k that (22.1) follows from (22.2). For $k=1$ both equations are identical. Thus, we have to show that

$$t_n - t_{n+1} = \frac{1-\beta}{1+n\,(1-\beta)}\,(t_0 - t_{n+1})\,, \qquad n = 1,\ldots,k$$

implies

$$t_n - t_{k+1} = (1-\beta)\,\frac{k+1-n}{1+k\,(1-\beta)}\,(t_0 - t_{k+1})\,, \qquad n = 1,\ldots,k \tag{22.4}$$

also holds for $k+1$, i.e., we have to prove that

$$t_n - t_{n+1} = \frac{1-\beta}{1+n\,(1-\beta)}\,(t_0 - t_{n+1})\,, \qquad n = 1,\ldots,k,$$

and

$$t_{k+1} - t_{k+2} = \frac{1-\beta}{1+(k+1)\,(1-\beta)}\,(t_0 - t_{k+2}) \tag{22.5}$$

imply

$$t_n - t_{k+2} = (1-\beta)\,\frac{k+1+1-n}{1+(k+1)\,(1-\beta)}\,(t_0 - t_{k+2})\,, \qquad n = 1,\ldots,k+1\,. \tag{22.6}$$

We have for $n = 1, \ldots, k$ and (22.4)

$$\begin{aligned}
& t_n - t_{k+2} \\
&= t_n - t_{k+1} + t_{k+1} - t_{k+2} \\
&= (1-\beta)\,\frac{k+1-n}{1+k\,(1-\beta)}\,(t_0 - t_{k+1}) + t_{k+1} - t_{k+2} \\
&= (1-\beta)\,\frac{k+1-n}{1+k\,(1-\beta)}\,(t_0 - t_{k+2} - (t_{k+1} - t_{k+2})) + t_{k+1} - t_{k+2} \\
&= (1-\beta)\,\frac{k+1-n}{1+k\,(1-\beta)}\,(t_0 - t_{k+2}) - \left((1-\beta)\,\frac{k+1-n}{1+k\,(1-\beta)} - 1\right)\,(t_{k+1} - t_{k+2})\,.
\end{aligned}$$

Using (22.5) it follows that

$$\begin{aligned}
& t_n - t_{k+2} \\
&= \left[(1-\beta)\,\frac{k+1-n}{1+k\,(1-\beta)} - \left((1-\beta)\,\frac{k+1-n}{1+k\,(1-\beta)} - 1\right)\,\frac{1-\beta}{1+(k+1)\,(1-\beta)}\right]\,(t_0 - t_{k+2}) \\
&= \frac{1-\beta}{1+k\,(1-\beta)}\,\left[k+1-n - \frac{(1-\beta)\,(k+1-n) - (1+k\,(1-\beta))}{1+(k+1)\,(1-\beta)}\right]\,(t_0 - t_{k+2})\,,
\end{aligned}$$

which simplifies to

$$t_n - t_{k+2} = (1-\beta)\,\frac{k+1+1-n}{1+(k+1)\,(1-\beta)}\,(t_0 - t_{k+2})\,,$$

which was to be shown for $n = 1, \ldots, k$. For $n = k+1$, (22.5) and (22.6) are equivalent by definition.

If $t_{k+1} < t_0$ then $t_k < \ldots < t_1$ follows directly from (22.3), which completes the proof. □

All three variants play their roles in Section 10.1. Note that it follows from (22.3) that the time differences $t_n - t_{n+1}$, $n = 1, \ldots, k$, between two subsequent interim inspections are the same up to the last one which is by (22.1)

$$t_0 - t_1 = t_0 - t_{k+1} - k\,\frac{1-\beta}{1+k\,(1-\beta)}\,(t_0 - t_{k+1}) = \frac{t_0 - t_{k+1}}{1+k\,(1-\beta)}\,, \qquad (22.7)$$

and which is the optimal expected detection time in the inspection game considered in Section 10.1. (22.7), however, also plays – in a modified form – an important role in Chapters 11 and 12. (22.2) and (22.3) for instance can be generalized to any number $N \geq 2$ of facilities; see (11.53) and (11.64).

Chapter 23

Supplementary considerations to Sections 11.2 and 11.3

In Section 23.1 it is proven that the heuristically derived strategies on p. 221 do not satisfy the saddle point criterion. Thereafter, in Section 23.2 two Lemmata are provided the results of which are essential for the proof of Theorem 11.2.

23.1 Proof of (11.41)

With $\tilde{\mathbf{g}}^* = (\mathbf{g}_3^*, \tilde{\mathbf{g}}_2^*)$, where $\mathbf{g}_3^*$ resp. $\tilde{\mathbf{g}}_2^*$ are given by (11.25) resp. (11.40), we get by (11.24)

$$\begin{aligned}
&(N + 2\,(1-\beta))\, Op_{N,2}(\tilde{\mathbf{g}}^*, (\mathbf{q}, \mathbf{t})) \\
&= \sum_{(i_2,i_1)\in\{1,\ldots,N\}^2} q_{(i_2,i_1)} \Big[(t_0 - t_1)\,(1-\beta) \\
&+ (t_0 - t_2)\,\frac{1-\beta}{N} \sum_{r=1}^{N} \beta^{\mathbb{1}_{i_1}(r)} + (t_0 - t_3) \sum_{r=1}^{N} \beta^{\sum_{j=1}^{2} \mathbb{1}_{i_j}(r)} \\
&+ (t_1 - t_2)\,\frac{(1-\beta)^2}{N} + (t_1 - t_3)\,(1-\beta)\,\beta^{\mathbb{1}_{i_2}(i_1)} + (t_2 - t_3)\,(1-\beta)\Big]\,. \qquad (23.1)
\end{aligned}$$

Because we have for all $i_1 = 1, \ldots, N$

$$\sum_{r=1}^{N} \beta^{\mathbb{1}_{i_1}(r)} = N - 1 + \beta$$

and, by (11.39), for any $(i_2, i_1) \in \{1, \ldots, N\}^2$

$$\sum_{r=1}^{N} \beta^{\sum_{j=1}^{2} \mathbb{1}_{i_j}(r)} = \begin{cases} N - 2 + 2\,\beta & \text{for} \quad i_2 \neq i_1 \\ N - 1 + \beta^2 & \text{for} \quad i_2 = i_1 \end{cases} ,$$

the coefficients $A_0, \ldots, A_3$ of $t_0, \ldots, t_3$ are, using (23.1) and

$$S := \frac{1}{N} - \sum_{i_2=1}^{N} q_{(i_2,i_2)}\,,$$

given by

$$A_0 = \sum_{(i_2,i_1)\in\{1,\dots,N\}^2} q_{(i_2,i_1)} \left[(1-\beta) + \frac{1-\beta}{N}(N-1+\beta) + \sum_{r=1}^{N} \beta^{\sum_{j=1}^{2} \mathbb{1}_{i_j}(r)}\right]$$

$$= (1-\beta) + \frac{1-\beta}{N}(N-1+\beta) + \sum_{i_2=1}^{N} q_{(i_2,i_2)}\,(N-1+\beta^2)$$

$$+ \sum_{i_2=1}^{N} \sum_{i_1=1, i_1\neq i_2}^{N} q_{(i_2,i_1)}\,(N-2+2\,\beta)$$

$$= N - (1-\beta)^2\,S\,,$$

by

$$A_1 = -(1-\beta) + \frac{(1-\beta)^2}{N} + (1-\beta) \sum_{(i_2,i_1)\in\{1,\dots,N\}^2} q_{(i_2,i_1)}\,\beta^{\mathbb{1}_{i_2}(i_1)}$$

$$= -(1-\beta) + \frac{(1-\beta)^2}{N} + (1-\beta)\left(\sum_{i_2=1}^{N} q_{(i_2,i_2)}\,\beta + 1 - \sum_{i_2=1}^{N} q_{(i_2,i_2)}\right)$$

$$= (1-\beta)^2\,S\,,$$

by

$$A_2 = \sum_{(i_2,i_1)\in\{1,\dots,N\}^2} q_{(i_2,i_1)} \left[-\frac{1-\beta}{N}(N-1+\beta) - \frac{(1-\beta)^2}{N} + (1-\beta)\right] = 0\,,$$

and by

$$A_3 = \sum_{(i_2,i_1)\in\{1,\dots,N\}^2} q_{(i_2,i_1)} \left[-\sum_{r=1}^{N} \beta^{\sum_{j=1}^{2} \mathbb{1}_{i_j}(r)} - (1-\beta)\,\beta^{\mathbb{1}_{i_2}(i_1)} - (1-\beta)\right]$$

$$= \sum_{i_2=1}^{N} q_{(i_2,i_1)} \left[-(N-1+\beta^2) - (1-\beta)\,\beta - (1-\beta)\right]$$

$$+ \sum_{i_2=1}^{N} \sum_{\substack{i_1=1\\ i_1\neq i_2}}^{N} q_{(i_2,i_1)} \left[-(N-2+2\,\beta) - (1-\beta) - (1-\beta)\right] = -N\,.$$

Thus, (23.1) yields

$$(N + 2\,(1-\beta))\,Op_{N,2}(\tilde{\mathbf{g}}^*, (\mathbf{q},\mathbf{t})) = t_0\left[N - (1-\beta)^2\,S\right] + t_1\,(1-\beta)^2\,S - t_3\,N\,,$$

i.e., (11.41), which completes the proof. □

23.2 Lemmata for the proof of Theorem 11.2

The following Lemma 23.1 is used for proving the right hand inequality of (11.55).

Lemma 23.1. *Consider for any $k \in \mathbb{N}$ with $k \geq 2$, any $N \in \mathbb{N}$, any $(i_k, \ldots, i_1) \in \{1, \ldots, N\}^k$ and arbitrary $t_{k+1}, \ldots, t_0 \in \mathbb{R}$ the function R defined by (11.57).*

Then we have for any $(i_k, \ldots, i_1) \in \{1, \ldots, N\}^k$

$$R(i_k, \ldots, i_1) = \frac{N}{1-\beta} \left(t_0 - t_{k+1}\right). \tag{23.2}$$

Proof. Let $(i_k, \ldots, i_1) \in \{1, \ldots, N\}^k$ be a fixed but arbitrary combination of facilities to be inspected. We collect the coefficient A_h of $t_h, h = 1, \ldots, k$, and get by (11.57)

$$\begin{aligned} A_h := & -\mathbb{1}_{i_h}(i_{h-1})\,(1-\beta) - R(h) - \beta^{\sum_{j=1}^{h-1} \mathbb{1}_{i_j}(i_h)} \\ & + \mathbb{1}_{i_{h+1}}(i_h)\,(1-\beta) + L(h) + \beta^{\sum_{j=h+1}^{k} \mathbb{1}_{i_j}(i_h)} \end{aligned} \tag{23.3}$$

with $\mathbb{1}_{i_1}(i_0) := 0$, $\mathbb{1}_{i_{k+1}}(i_k) := 0$, $\sum_{j=k+1}^{k} \mathbb{1}_{i_j}(i_h) := 0$ and $\sum_{j=1}^{0} \mathbb{1}_{i_j}(i_h) := 0$, and where

$$R(h) = \begin{cases} \sum_{\ell=1}^{h-2} \mathbb{1}_{i_h}(i_\ell)\,\beta^{\sum_{j=\ell+1}^{h-1} \mathbb{1}_{i_j}(i_\ell)}\,(1-\beta) & : \quad h = 3, \ldots, k \\ 0 & : \quad h = 1, 2 \end{cases} \quad \text{and} \tag{23.4}$$

$$L(h) = \begin{cases} \sum_{m=h+2}^{k} \mathbb{1}_{i_m}(i_h)\,\beta^{\sum_{j=h+1}^{m-1} \mathbb{1}_{i_j}(i_h)}\,(1-\beta) & : \quad h = 1, \ldots, k-2 \\ 0 & : \quad h = k-1, k \end{cases}. \tag{23.5}$$

We start by evaluating $R(h)$ for $h = 3, \ldots, k$. Case 1: If $\sum_{\ell=1}^{h-2} \mathbb{1}_{i_h}(i_\ell) = 0$, then $\mathbb{1}_{i_h}(i_\ell) = 0$ for all $\ell = 1, \ldots, h-2$, i.e., $R(h) = 0$. Case 2: If $\sum_{\ell=1}^{h-2} \mathbb{1}_{i_h}(i_\ell) \geq 1$ then $i_h = i_\ell$ for at least one $\ell = 1, \ldots, h-2$. As a consequence there exist a largest index $\tilde{\ell} \in \{1, \ldots, h-2\}$ and a smallest index $\hat{\ell} \in \{1, \ldots, h-2\}$ with $i_{\tilde{\ell}} = i_{\hat{\ell}} = i_h$ and $\tilde{\ell} \geq \hat{\ell}$. With (23.4) we get

$$\begin{aligned} \frac{R(h)}{1-\beta} &= \sum_{\ell=1}^{h-2} \mathbb{1}_{i_h}(i_\ell)\,\beta^{\sum_{j=\ell+1}^{h-1} \mathbb{1}_{i_j}(i_\ell)} = \sum_{\substack{1 \leq \ell \leq h-2: \\ i_\ell = i_h}} \beta^{\sum_{j=\ell+1}^{h-1} \mathbb{1}_{i_j}(i_h)} \\ &= \beta^{\sum_{j=\hat{\ell}+1}^{h-1} \mathbb{1}_{i_j}(i_h)} + \beta^{\sum_{j=\hat{\ell}+1}^{h-1} \mathbb{1}_{i_j}(i_h)+1} + \ldots + \beta^{\sum_{j=\tilde{\ell}+1}^{h-1} \mathbb{1}_{i_j}(i_h)}. \end{aligned} \tag{23.6}$$

By definition of $\tilde{\ell}$ we know that $i_{\tilde{\ell}+1} \neq i_h, i_{\tilde{\ell}+2} \neq i_h, \ldots, i_{h-2} \neq i_h$. Therefore, we have $\sum_{j=\tilde{\ell}+1}^{h-1} \mathbb{1}_{i_\ell}(i_h) = \mathbb{1}_{i_{h-1}}(i_h)$. Also we obtain by the definition of $\hat{\ell}$ that $i_{\hat{\ell}} = i_h$ but $i_{\hat{\ell}-1} \neq i_h, \ldots, i_1 \neq i_h$ which implies $\sum_{j=\hat{\ell}+1}^{h-1} \mathbb{1}_{i_j}(i_h) = \sum_{j=1}^{h-1} \mathbb{1}_{i_j}(i_h) - 1$. Now (23.6) yields

$$\begin{aligned} \frac{R(h)}{1-\beta} &= \beta^{\sum_{j=\hat{\ell}+1}^{h-1} \mathbb{1}_{i_j}(i_h)} + \beta^{\sum_{j=\hat{\ell}+1}^{h-1} \mathbb{1}_{i_j}(i_h)+1} + \ldots + \beta^{\mathbb{1}_{i_{h-1}}(i_h)} \\ &= \begin{cases} \left(1 - \beta^{\sum_{j=\hat{\ell}+1}^{h-1} \mathbb{1}_{i_j}(i_h)+1}\right) / (1-\beta) & \text{for} \quad i_{h-1} \neq i_h \\ \beta \left(1 - \beta^{\sum_{j=\hat{\ell}+1}^{h-1} \mathbb{1}_{i_j}(i_h)}\right) / (1-\beta) & \text{for} \quad i_{h-1} = i_h \end{cases} \end{aligned}$$

$$= \begin{cases} \left(1-\beta^{\sum_{j=1}^{h-1} \mathbb{1}_{i_j}(i_h)}\right)/(1-\beta) & \text{for} \quad i_{h-1} \neq i_h \\ \left(\beta-\beta^{\sum_{j=1}^{h-1} \mathbb{1}_{i_j}(i_h)}\right)/(1-\beta) & \text{for} \quad i_{h-1} = i_h \end{cases}. \tag{23.7}$$

Case 1 together with (23.7) of case 2 lead for all $h = 3, \ldots, k$ to

$$R(h) = \beta^{\mathbb{1}_{i_{h-1}}(i_h)} - \beta^{\sum_{j=1}^{h-1} \mathbb{1}_{i_j}(i_h)}. \tag{23.8}$$

Note that (23.8) even holds for $h = 2$: $R(2) = \beta^{\mathbb{1}_{i_1}(i_2)} - \beta^{\mathbb{1}_{i_1}(i_2)} = 0$ in accordance with (23.4).

Evaluating $L(h)$ for $h = 1, \ldots, k-2$ we proceed in a similar way. Case 1: If $\sum_{j=h+2}^{k} \mathbb{1}_{i_j}(i_h) = 0$ then $\mathbb{1}_{i_m}(i_h) = 0$ for all $m = h+2, \ldots, k$, i.e., $L(h) = 0$. Case 2: If $\sum_{j=h+2}^{k} \mathbb{1}_{i_j}(i_h) \geq 1$ then $i_j = i_h$ for at least one $j = h+2, \ldots, k$. As a consequence there exist a largest index $\tilde{\ell} \in \{h+2, \ldots, k\}$ and a smallest index $\hat{\ell} \in \{h+2, \ldots, k\}$ with $i_{\tilde{\ell}} = i_{\hat{\ell}} = i_h$ and $\tilde{\ell} \geq \hat{\ell}$. It follows that $\sum_{j=h+1}^{\hat{\ell}-1} \mathbb{1}_{i_j}(i_h) = \mathbb{1}_{i_{h+1}}(i_h)$. With (23.5) we get

$$\frac{L(h)}{1-\beta} = \sum_{m=h+2}^{k} \mathbb{1}_{i_m}(i_h)\, \beta^{\sum_{j=h+1}^{m-1} \mathbb{1}_{i_j}(i_h)} = \sum_{\substack{h+2 \leq m \leq k: \\ i_m = i_h}} \beta^{\sum_{j=h+1}^{m-1} \mathbb{1}_{i_j}(i_h)}$$

$$= \beta^{\sum_{j=h+1}^{\hat{\ell}-1} \mathbb{1}_{i_j}(i_h)} + \beta^{\sum_{j=h+1}^{\hat{\ell}-1} \mathbb{1}_{i_j}(i_h)+1} + \ldots + \beta^{\sum_{j=h+1}^{\tilde{\ell}-1} \mathbb{1}_{i_j}(i_h)}. \tag{23.9}$$

With a similar argument as above we have $\sum_{j=h+1}^{\hat{\ell}-1} \mathbb{1}_{i_j}(i_h) = \mathbb{1}_{i_{h+1}}(i_h)$ and $\sum_{j=h+1}^{\tilde{\ell}-1} \mathbb{1}_{i_j}(i_h) = \sum_{j=h+1}^{k} \mathbb{1}_{i_j}(i_h) - 1$, and thus get by (23.9)

$$\frac{L(h)}{1-\beta} = \beta^{\mathbb{1}_{i_{h+1}}(i_h)} + \beta^{\sum_{j=h+1}^{\hat{\ell}-1} \mathbb{1}_{i_j}(i_h)+1} + \ldots + \beta^{\sum_{j=h+1}^{\tilde{\ell}-1} \mathbb{1}_{i_j}(i_h)}$$

$$= \begin{cases} \left(1-\beta^{\sum_{j=h+1}^{\tilde{\ell}-1} \mathbb{1}_{i_j}(i_h)+1}\right)/(1-\beta) & \text{for} \quad i_{h+1} \neq i_h \\ \beta\left(1-\beta^{\sum_{j=h+1}^{\tilde{\ell}-1} \mathbb{1}_{i_j}(i_h)}\right)/(1-\beta) & \text{for} \quad i_{h+1} = i_h \end{cases}$$

$$= \begin{cases} \left(1-\beta^{\sum_{j=h+1}^{k} \mathbb{1}_{i_j}(i_h)}\right)/(1-\beta) & \text{for} \quad i_{h+1} \neq i_h \\ \left(\beta-\beta^{\sum_{j=h+1}^{k} \mathbb{1}_{i_j}(i_h)}\right)/(1-\beta) & \text{for} \quad i_{h+1} = i_h \end{cases}. \tag{23.10}$$

Case 1 together with (23.10) of case 2 lead for all $h = 1, \ldots, k-2$ to

$$L(h) = \beta^{\mathbb{1}_{i_{h+1}}(i_h)} - \beta^{\sum_{j=h+1}^{k} \mathbb{1}_{i_j}(i_h)}. \tag{23.11}$$

In analogy to $R(2)$, (23.11) holds also for $h = k-1$: We have $L(k-1) = \beta^{\mathbb{1}_{i_k}(i_{k-1})} - \beta^{\mathbb{1}_{i_k}(i_{k-1})} = 0$, in accordance with (23.5).

Using (23.8) and (23.11), (23.3) simplifies for all $h = 2, \ldots, k-1$ (only applicable in case of $k \geq 3$) to

$$A_h = -\mathbb{1}_{i_h}(i_{h-1})\,(1-\beta) - \beta^{\mathbb{1}_{i_{h-1}}(i_h)} + \mathbb{1}_{i_{h+1}}(i_h)\,(1-\beta) + \beta^{\mathbb{1}_{i_{h+1}}(i_h)}, \tag{23.12}$$

for $h=1$ with $\mathbb{1}_{i_1}(i_0):=0$ and $\sum_{j=1}^{0}\mathbb{1}_{i_j}(i_h):=0$ to

$$\begin{aligned} A_1 &= -\mathbb{1}_{i_1}(i_0)\,(1-\beta) - \beta^{\sum_{j=1}^{0}\mathbb{1}_{i_j}(i_1)} + \mathbb{1}_{i_2}(i_1)\,(1-\beta) + \beta^{\mathbb{1}_{i_2}(i_1)} \\ &= -1 + \mathbb{1}_{i_2}(i_1)\,(1-\beta) + \beta^{\mathbb{1}_{i_2}(i_1)}\,, \end{aligned} \tag{23.13}$$

and for $h=k$ with $\mathbb{1}_{i_{k+1}}(i_k):=0$ and $\sum_{j=k+1}^{k}\mathbb{1}_{i_j}(i_h):=0$ to

$$\begin{aligned} A_k &= -\mathbb{1}_{i_k}(i_{k-1})\,(1-\beta) - \beta^{\mathbb{1}_{i_{k-1}}(i_k)} + \mathbb{1}_{i_{k+1}}(i_k)\,(1-\beta) + \beta^{\sum_{j=k+1}^{k}\mathbb{1}_{i_j}(i_k)} \\ &= -\mathbb{1}_{i_k}(i_{k-1})\,(1-\beta) - \beta^{\mathbb{1}_{i_{k-1}}(i_k)} + 1\,. \end{aligned} \tag{23.14}$$

Evaluating (23.12) – (23.14) for the four cases $i_{h-1}=i_h=i_{h+1}$, $i_{h-1}=i_h\neq i_{h+1}$, $i_{h-1}\neq i_h=i_{h+1}$ and $i_{h-1}\neq i_h, i_h\neq i_{h+1}$ leads to $A_h=0$ for all $h=1,\ldots,k$.

Because $(i_k,\ldots,i_1)\in\{1,\ldots,N\}^k$ was a fixed but arbitrary combination of facilities to be inspected, we have $A_1=\ldots=A_k=0$ for any $(i_k,\ldots,i_1)\in\{1,\ldots,N\}^k$, and therefore by (11.57)

$$\begin{aligned} R(i_k,\ldots,i_1) &= t_0 - t_{k+1} + \sum_{m=2}^{k} t_0\,\beta^{\sum_{j=1}^{m-1}\mathbb{1}_{i_j}(i_m)} \\ &\quad + (t_0-t_{k+1})\,\frac{1}{1-\beta}\sum_{r=1}^{N}\beta^{\sum_{j=1}^{k}\mathbb{1}_{i_j}(r)} - \sum_{\ell=1}^{k-1} t_{k+1}\,\beta^{\sum_{j=\ell+1}^{k}\mathbb{1}_{i_j}(i_\ell)} \end{aligned} \tag{23.15}$$

for any $(i_k,\ldots,i_1)\in\{1,\ldots,N\}^k$. From (23.15) we get for the coefficient A_0 of t_0 – we write for the sake of clarity $A_0(i_k,\ldots,i_1)$ –

$$A_0(i_k,\ldots,i_1) = 1 + \sum_{m=2}^{k}\beta^{\sum_{j=1}^{m-1}\mathbb{1}_{i_j}(i_m)} + \frac{1}{1-\beta}\sum_{r=1}^{N}\beta^{\sum_{j=1}^{k}\mathbb{1}_{i_j}(r)}\,, \tag{23.16}$$

and show by induction with respect to k that $A_0(i_k,\ldots,i_1)=N/(1-\beta)$. For $k=2$ we have for any $(i_2,i_1)\in\{1,\ldots,N\}^2$ by (23.16)

$$\begin{aligned} A_0(i_2,i_1) &= 1 + \beta^{\mathbb{1}_{i_1}(i_2)} + \frac{1}{1-\beta}\sum_{r=1}^{N}\beta^{\mathbb{1}_{i_1}(r)+\mathbb{1}_{i_2}(r)} \\ &= \begin{cases} 1+\beta^1+\dfrac{1}{1-\beta}\,\left(N-1+\beta^2\right) & \text{for } i_1=i_2 \\ 1+\beta^0+\dfrac{1}{1-\beta}\,(N-2+2\,\beta) & \text{for } i_1\neq i_2 \end{cases} \\ &= \frac{N}{1-\beta}\,. \end{aligned}$$

Let $A_0(i_k,\ldots,i_1)=N/(1-\beta)$ for an arbitrary $k\geq 3$ and any $(i_k,\ldots,i_1)\in\{1,\ldots,N\}^k$. Let $(i_{k+1},i_k,\ldots,i_1)\in\{1,\ldots,N\}^{k+1}$ be a fixed but arbitrary combination of facilities to be inspected in the game with $k+1$ interim inspections. Using $\mathbb{1}_{i_{k+1}}(r)=0$ for $r \neq i_{k+1}$ and $\mathbb{1}_{i_{k+1}}(i_{k+1})=1$, (23.16) yields for $k \to k+1$

$$\begin{aligned}
&A_0(i_{k+1}, i_k, \dots, i_1) \\
&= 1 + \sum_{m=2}^{k+1} \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(i_m)} + \frac{1}{1-\beta} \left(\sum_{\substack{r=1 \\ r\neq i_{k+1}}}^{N} \beta^{\sum_{j=1}^{k} \mathbb{1}_{i_j}(r)} + \beta^{\sum_{j=1}^{k} \mathbb{1}_{i_j}(i_{k+1})+1} \right) \\
&= 1 + \sum_{m=2}^{k} \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(i_m)} + \beta^{\sum_{j=1}^{k} \mathbb{1}_{i_j}(i_{k+1})} + \frac{1}{1-\beta}\, \beta^{\sum_{j=1}^{k} \mathbb{1}_{i_j}(i_{k+1})+1} \\
&\qquad + A_0(i_k, \dots, i_1) - 1 - \sum_{m=2}^{k} \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(i_m)} - \frac{1}{1-\beta}\, \beta^{\sum_{j=1}^{k} \mathbb{1}_{i_j}(i_{k+1})} \\
&= A_0(i_k, \dots, i_1)\,.
\end{aligned}$$

Therefore, we have $A_0(i_k, \dots, i_1) = N/(1-\beta)$ for all $k \geq 2$ and any $(i_k, \dots, i_1) \in \{1, \dots, N\}^k$.

Let $(i_k, \dots, i_1) \in \{1, \dots, N\}^k$ be again a fixed but arbitrary combination of facilities to be inspected. For the coefficient A_{k+1} of t_{k+1} we get again from (23.15)

$$A_{k+1}(i_k, \dots, i_1) = -1 - \frac{1}{1-\beta} \sum_{r=1}^{N} \beta^{\sum_{j=1}^{k} \mathbb{1}_{i_j}(r)} - \sum_{\ell=1}^{k-1} \beta^{\sum_{j=\ell+1}^{k} \mathbb{1}_{i_j}(i_\ell)}\,. \tag{23.17}$$

Replacing i_ℓ by $i_{k-\ell+1}$, $\ell = 1, \dots, k$, in (23.17), we get by (23.16)

$$A_{k+1}(i_k, \dots, i_1) = -A_0(i_k, \dots, i_1) = \frac{N}{1-\beta}$$

for any $(i_k, \dots, i_1) \in \{1, \dots, N\}^k$. Therefore, (23.15) simplifies to (23.2), which completes the proof. ☐

It should be noted that (23.2) holds for any $t_{k+1}, \dots, t_0 \in \mathbb{R}$ and not just for $(t_k, \dots, t_1) \in \mathcal{T}_{N,k}$, as required in the proof of Theorem 11.2.

The next Lemma is used for proving the left hand inequality of (11.55).

Lemma 23.2. *Consider for any $k \in \mathbb{N}$ with $k \geq 2$, any $N \in \mathbb{N}$ and any $\mathbf{g} \in G_{N,k}$ the function $L(\mathbf{g})$ defined by (11.62). Then we have for any $\mathbf{g} \in G_{N,k}$*

$$L(\mathbf{g}) = N^{k+1}\,. \tag{23.18}$$

Proof. We evaluate (11.62) separately for each g_m, $m = 2, \dots, k+1$, and start with g_2. For the sake of brevity we again suppress the arguments of g_m in the following equations. Because w_3 depends on $(i_k, \dots, i_3, t_k, \dots, t_3)$ and g_2 on $(i_k, \dots, i_2, t_k, \dots, t_2)$, we obtain for the terms of (11.62) containing g_2

$$w_3 \sum_{i_1=1}^{N} \Big[g_{2,i_1} (1-\beta)^2 + (N+1-\beta) \sum_{r=1}^{N} g_{2,r}\, \beta^{\mathbb{1}_{i_1}(r)} \Big]$$

$$= w_3 \left[(1-g_2)\,(1-\beta)^2 + (N+1-\beta) \sum_{r=1}^{N} g_{2,r} \sum_{i_1=1}^{N} \beta^{\mathbb{1}_{i_1}(r)} \right]$$

$$= w_3 \left[(1-g_2)\,(1-\beta)^2 + (N+1-\beta)\,(N-1+\beta)\,(1-g_2) \right]$$

$$= w_3\, N^2\,(1-g_2)\,. \tag{23.19}$$

Therefore, we get

$$S_2(i_k,\ldots,i_1)$$

$$:= \sum_{(i_k,\ldots,i_2)\in\{1,\ldots,N\}^{k-1}} w_3 \sum_{i_1=1}^{N} \left[g_{2,i_1}\,(1-\beta)^2 + (N+1-\beta) \sum_{r=1}^{N} g_{2,r}\, \beta^{\mathbb{1}_{i_1}(r)} \right]$$

$$= N^2 \sum_{(i_k,\ldots,i_2)\in\{1,\ldots,N\}^{k-1}} w_3\,(1-g_2)\,. \tag{23.20}$$

For g_m with $m \in \{3,\ldots,k+1\}$ we get from (11.62) for the terms containing g_m

$$S_m(i_k,\ldots,i_1) := \sum_{(i_k,\ldots,i_m)\in\{1,\ldots,N\}^{k-m+1}} \sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}} \Big[w_{m+1}\, g_{m,i_{m-1}}\,(1-\beta)^2$$

$$+ \left(N+(m-1)\,(1-\beta)\right) w_{m+1} \sum_{r=1}^{N} g_{m,r}\, \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r)}$$

$$+ \sum_{\ell=1}^{m-2} (m-\ell)\, w_{m+1}\, g_{m,i_\ell}\, \beta^{\sum_{j=\ell+1}^{m-1} \mathbb{1}_{i_j}(i_\ell)}\,(1-\beta)^2 \Big]\,. \tag{23.21}$$

To evaluate (23.21) we perform some preliminary calculations regarding the expressions in line 1, 2 and 3 in (23.21). First, we have

$$\sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}} g_{m,i_{m-1}} = \sum_{i_{m-1}=1}^{N} g_{m,i_{m-1}} \sum_{(i_{m-2},\ldots,i_1)\in\{1,\ldots,N\}^{m-2}} \mathbb{1}$$

$$= N^{m-2}\,(1-g_m)\,. \tag{23.22}$$

Second, in order to simplify the expression

$$\sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}} \sum_{r=1}^{N} g_{m,r}\, \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r)}$$

$$= \sum_{r=1}^{N} g_{m,r} \sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}} \beta^{\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r)}\,, \tag{23.23}$$

we count the tuples $(i_{m-1},\ldots,i_1) \in \{1,\ldots,N\}^{m-1}$ that lead to $\sum_{j=1}^{m-1} \mathbb{1}_{i_j}(r) = \ell$, $\ell \in \{0,\ldots,m-1\}$ and get the results in Table 23.1.

Table 23.1 Number of tuples $(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}$ that lead to $\sum_{j=1}^{m-1}\mathbb{1}_{i_j}(r)=\ell$, $\ell\in\{0,\ldots,m-1\}$.

ℓ	tuples $(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}$ that lead to $\sum_{j=1}^{m-1}\mathbb{1}_{i_j}(r)=\ell$
0	$\binom{m-1}{0}(N-1)^{m-1}$
1	$\binom{m-1}{1}(N-1)^{m-1-1}$
$\vdots$	$\vdots$
ℓ	$\binom{m-1}{\ell}(N-1)^{m-1-\ell}$
$\vdots$	$\vdots$
$m-1$	$\binom{m-1}{m-1}(N-1)^{m-1-(m-1)}$

Thus, we obtain using the binomial theorem and the results of Table 23.1

$$\sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}}\beta^{\sum_{j=1}^{m-1}\mathbb{1}_{i_j}(r)}=\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}\beta^{\ell}(N-1)^{m-1-\ell}=(N-1+\beta)^{m-1},$$

and therefore by (23.23)

$$\sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}}\sum_{r=1}^{N}g_{m,r}\,\beta^{\sum_{j=1}^{m-1}\mathbb{1}_{i_j}(r)}=(N-1+\beta)^{m-1}(1-g_m)\,. \tag{23.24}$$

Third, we evaluate the term

$$\sum_{\ell=1}^{m-2}(m-\ell)\sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}}g_{m,i_\ell}\,\beta^{\sum_{j=\ell+1}^{m-1}\mathbb{1}_{i_j}(i_\ell)}\,. \tag{23.25}$$

For a fixed ℓ, $\ell\in\{1,\ldots,m-2\}$, we get

$$\begin{aligned}&\sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}}g_{m,i_\ell}\,\beta^{\sum_{j=\ell+1}^{m-1}\mathbb{1}_{i_j}(i_\ell)}\\&=\sum_{(i_{\ell-1},\ldots,i_1)\in\{1,\ldots,N\}^{\ell}}\sum_{i_\ell=1}^{N}g_{m,i_\ell}\sum_{(i_{m-1},\ldots,i_{\ell+1})\in\{1,\ldots,N\}^{m-\ell-1}}\beta^{\sum_{j=\ell+1}^{m-1}\mathbb{1}_{i_j}(i_\ell)}\\&=N^{\ell-1}(1-g_m)(N-1+\beta)^{m-\ell-1}\,.\end{aligned} \tag{23.26}$$

Therefore, we get by (23.25), using (23.26) and $z:=(N-1+\beta)/N$,

$$\sum_{\ell=1}^{m-2}(m-\ell)\sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}}g_{m,i_\ell}\,\beta^{\sum_{j=\ell+1}^{m-1}\mathbb{1}_{i_j}(i_\ell)}$$

$$= N^{m-2}\,(1-g_m)\sum_{\ell=1}^{m-2}(m-\ell)\,z^{m-\ell-1}\,. \tag{23.27}$$

Because $z \in (0,1)$, we have

$$\sum_{\ell=1}^{m-2}(m-\ell)\,z^{m-\ell-1} = \sum_{\ell=1}^{m-2}(\ell+1)\,z^{\ell} = \sum_{\ell=1}^{m-2}\frac{d}{dz}\,z^{\ell+1} = \frac{d}{dz}\left(\sum_{\ell=1}^{m-2} z^{\ell+1}\right)$$

$$= \frac{d}{dz}\left(z^2\,\frac{1-z^{m-2}}{1-z}\right) = \frac{z\,(2-z)-z^{m-1}\,(m-(m-1)\,z)}{(1-z)^2}$$

and obtain, using the definition of z,

$$\sum_{\ell=1}^{m-2}(m-\ell)\left(\frac{N-1+\beta}{N}\right)^{m-1-\ell}$$

$$= \frac{N^2}{(1-\beta)^2}\left(1-\left(\frac{N-1+\beta}{N}\right)^{m-1}\frac{N+(m-1)\,(1-\beta)}{N}\right)-1\,.$$

and therewith by (23.27)

$$\sum_{\ell=1}^{m-2}(m-\ell)\sum_{(i_{m-1},\ldots,i_1)\in\{1,\ldots,N\}^{m-1}} g_{m,i_\ell}\,\beta^{\sum_{j=\ell+1}^{m-1}\mathbb{1}_{i_j}(i_\ell)}$$

$$= (1-g_m)\left(\frac{N^m}{(1-\beta)^2}-\frac{(N-1+\beta)^{m-1}\,(N+(m-1)\,(1-\beta))}{(1-\beta)^2}-N^{m-2}\right)\,. \tag{23.28}$$

Because (11.47) implies $w_{m+1}\,g_m = w_m$, (23.21) simplifies for $m = 3,\ldots,k$, using (23.22), (23.24) and (23.28), to

$$S_m(i_k,\ldots,i_1) = N^m \sum_{(i_k,\ldots,i_m)\in\{1,\ldots,N\}^{k-m+1}} w_{m+1}\,(1-g_m)$$

$$= N^m \sum_{(i_k,\ldots,i_m)\in\{1,\ldots,N\}^{k-m+1}} (w_{m+1}-w_m) \tag{23.29}$$

and for $m = k+1$, because of $w_{k+2} = 1$, see (11.47), to

$$S_{k+1}(i_k,\ldots,i_1) = N^{k+1}\,(1-g_{k+1})\,. \tag{23.30}$$

Thus, using (23.20), (23.29) and (23.30), (11.62) simplifies to

$$L(\mathbf{g}) = \sum_{(i_k,\ldots,i_1)\in\{1,\ldots,N\}^k} N\,w_2 + \sum_{m=2}^{k+1} S_m(i_k,\ldots,i_1)$$

$$= \sum_{(i_k,\ldots,i_1)\in\{1,\ldots,N\}^k} N\,w_2$$

$$+ N^2 \sum_{(i_k,\ldots,i_2)\in\{1,\ldots,N\}^{k-1}} (w_3 - w_2)$$

$$\vdots$$

$$+ N^k \sum_{i_k\in\{1,\ldots,N\}} (w_{k+1} - w_k)$$

$$+ N^{k+1} (1 - g_{k+1}) \,.$$

Keeping in mind that w_m, $m = 2, \ldots, k$, only depends on $(i_k, \ldots, i_m)$, we finally obtain (23.18), which completes the proof. □

Chapter 24

A Se-No inspection game with an expected number of inspections: Krieger-Avenhaus model

As mentioned on p. 18, there are two reasons why in this monograph so far only inspection games with a deterministically[1] fixed integer $1, 2, \ldots$ of inspections resp. controls are considered. In short, let us repeat that these inspection games were in the focus of the research interests from the very beginning, and also, that practitioners have only been interested in this type of inspection game; see Avenhaus et al. (2010). There exist, however, a priori no reasons why inspection games with a fixed number of inspections should be preferred to inspection games with an expected number of inspections.

We saw in Chapter 11 that for a fixed number k of interim inspections for all N facilities the optimal strategy of the Inspectorate means to randomize the number of interim inspections in each of the N facilities. Thus, for the sake of clarity we emphasize that in this chapter we consider just one facility for which the expected number of interim inspections for some time period is given and known to the Operator.

Recently the interest in inspection models with an expected number of inspections has increased. Where does this interest come from? We have observed two reasons: First, practitioners like to argue with the term *unpredictability*, and they usually mean the following: "Because the Operator only knows the expected number of inspections per year, he never knows how many inspections are truly carried out, and this should be somehow advantageous for the Inspectorate because there is always the possibility that another inspection is performed after all." One way to evaluate this statement is to compare the optimal payoff to the Operator for the inspection model with a fixed number k of inspections to that with an expected number of $\mu = k$ inspections. If the latter optimal payoff is smaller than the first one, the practitioner would prefer the inspection model with an expected number of inspections. Note, however, that planning considerations are neglected: The number of inspections can be larger or smaller than k, which can be seen as an disadvantage from the planning point of view.

Second, some practitioners argue "that in case the Operator plans sequentially, i.e., a Se-No

[1] In contrast to an expected or even random number of inspections resp. controls. In the following we use just the wording *fixed* and *expected* number.

or a Se-Se inspection game with a fixed number of inspections is considered, the Operator could wait until the last inspection is performed and then behave illegally immediately (in case of playing for time games) or else, behave illegally at one of the remaining steps (in case of critical time games)." This, however, is a faulty argument, because this strategy is in *none* of the inspection games discussed in this monograph an optimal resp. an equilibrium strategy of the Operator.

For purpose of illustration and to address the first reason given above we treat in this chapter a Se-No critical time inspection game with an expected number of inspections, where the analysis can be seen as a starting point for further research, see Section 1.5, in two directions: First, the game considered in this chapter could be solved taking payoff parameters into account as in the corresponding games in Part III. Second, any other inspection game of this monograph might be worth to be analysed under the assumption of an expected instead of a fixed number of inspections resp. controls.

In this chapter, assumptions (iii), (iv), (v), (vi), (viii) and (x) of Chapter 14 are specified as follows:

(iii') The Operator performs an illegal activity once at one of the steps $L, \ldots, 1$.

(iv') During an inspection the Inspectorate may commit an error of the second kind, i.e., if the Operator behaves illegally at the same step at which the Inspectorate performs an inspection, then the illegal activity is not detected with probability β. This non-detection probability is the same for all inspections.

(v') The Inspectorate performs an *expected* number $\mu \in \mathbb{Q}$ ($\mathbb{Q}$ is the set of rational numbers) of inspections at steps $L, L-1, \ldots, 1$ with $0 < \mu < L$. The number μ is known to the Operator.

(vi') The Operator decides at the beginning, i.e., at step L, whether to behave illegally at that step. If he behaves legally at steps $L, \ldots, \ell+1$ ($1 \leq \ell \leq L-1$), then the Operator decides whether to behave illegally at step ℓ; and so on. The Operator behaves illegally latest at step 1; see assumption (iii').

The Inspectorate decides at the beginning when the inspections are performed; see assumption (v').

(viii') The payoffs to the two players (Operator, Inspectorate) are given by

$(1,-1)$	for an untimely inspection or	
	a timely inspection and no detection of the illegal behaviour	(24.1)
$(-1,1)$	for a timely inspection and detection of the illegal behaviour	

(x') The game ends either at the step at which the Operator behaves illegally, or at step 1.

The remaining assumptions of Chapter 14 except (ix) hold throughout this chapter. Because of the payoffs (24.1), the Se-No inspection game treated in this chapter is the analogon to the original Thomas-Nisgav game with a fixed number of controls; see p. 356. Regarding assumption (x') we note again that if the Operator behaves illegally at step i, $i = L, \ldots, 1$, then the game ends at step i regardless whether the illegal behaviour is detected at that step or not. In the latter case, the Operator has successfully performed his illegal activity and thus, the game ends as well.

In Section 24.1 the Se-No inspection game with $L = 3$ steps is analysed and special optimal strategies of the Inspectorate are discussed. Section 24.2 deals with a *simplified* Se-No inspection game with L steps which is used to formulate a conjecture about the optimal strategies of both players and the optimal payoff to the Operator of the *original* Se-No inspection game. Surprising relations to the generalized Thomas-Nisgav and the Canty-Rothenstein-Avenhaus inspection games are highlighted.

24.1 Three steps; errors of the second kind

For the purpose of illustration we start with the case of three steps, i.e., $L = 3$, the extensive form of which is presented in Figure 24.1. Because the Inspectorate behaves non-sequentially, the game starts with the Inspectorate's decision; see the comment on p. 50. Here, the chance moves are already incorporated into the Operator's payoff.

At step 3, i.e., at the top of the tree, the Inspectorate's decides when to perform its inspections: 0 resp. 1 indicates no resp. an inspection, where the first component of the triple refers to the Inspectorate's decision at step 3, the second component to its decision at step 2, and the third one to its decision at step 1. Note that in contrast to the inspection games with a fixed number of inspections here the expected number of inspections is not part of the game tree, but only appears in the boundary condition (24.3).

The Operator decides at step 3 – not knowing the Inspectorate's decision – to behave illegally immediately ($\bar{\ell}_3$) or not (ℓ_3). In the latter case he decides at step 2 to behave illegally immediately ($\bar{\ell}_{2,0}$ resp. $\bar{\ell}_{2,1}$) or not ($\ell_{2,0}$ resp. $\ell_{2,1}$). In the latter case he must behave illegally at step 1 because of assumption (iii'). The payoff to the Operator is given at the end nodes of the tree and are, using (24.1), self-explaining keeping in mind that in case of a timely inspection the payoff is $1\,\beta + (-1)\,(1-\beta) = 2\,\beta - 1$.

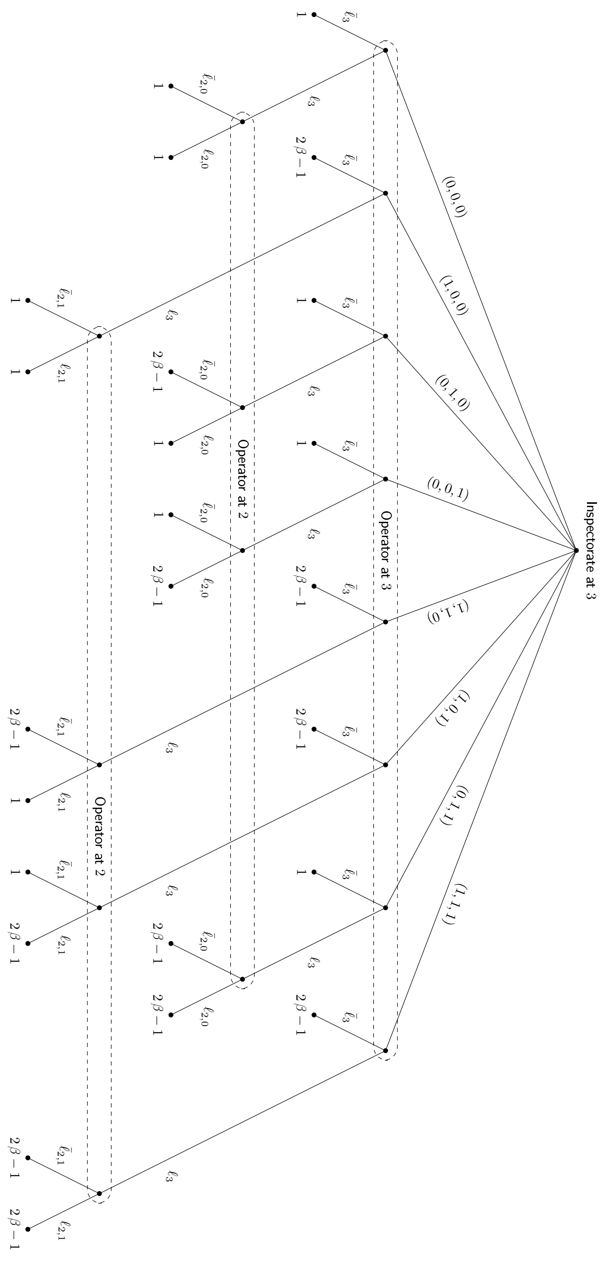
The Operator's two information sets can be explained as follows: At step 2 he only knows whether there was an inspection at step 3 or not. Thus, all decision nodes of the Operator at step 2 that belong to triples of the kind $(0,\cdot,\cdot)$ resp. $(1,\cdot,\cdot)$ need to be in the same in information set. To indicate the Operator's decisions at the information sets, they are denoted by $\bar{\ell}_{2,0}$ resp. $\bar{\ell}_{2,1}$ and $\ell_{2,0}$ resp. $\ell_{2,1}$ where the second index indicates whether there was an inspection at step 3 or not.

In accordance with the notation in Parts I and II, let the probability of behaving illegally at step 3 ($\bar{\ell}_3$) be $1 - g_3$, and behaving illegally at step 2 ($\bar{\ell}_{2,0}$ resp. $\bar{\ell}_{2,1}$) be $1 - g_{2,0}$ resp. $1 - g_{2,1}$. Because of assumption (iii'), the Operator must behave illegally at step 1. Thus, the Operator's set of behavioural strategies is given by

$$G_3 := \left\{ \mathbf{g} := (g_3, g_{2,0}, g_{2,1}) \in [0,1]^3 \right\}. \tag{24.2}$$

The Inspectorate chooses its triples $(j_3, j_2, j_1) \in \{0,1\}^3$ with probability $q_{(j_3,j_2,j_1)}$, where they do not only have to add up to one, but also have to lead to expected value μ. Therefore, the

Figure 24.1 Extensive form of the Se-No inspection game with $L = 3$ steps and with errors of the second kind.



Inspectorate's strategy set is given by

$$Q_3 := \Big\{ \mathbf{q} := (q_{(0,0,0)}, q_{(1,0,0)}, \ldots, q_{(0,1,1)}, q_{(1,1,1)}) \in [0,1]^8 : \sum_{(j_3,j_2,j_1)\in\{0,1\}^3} q_{(j_3,j_2,j_1)} = 1 \quad \text{and} \quad q_{(1,0,0)} + q_{(0,1,0)} + q_{(0,0,1)} + 2\,(q_{(1,1,0)} + q_{(1,0,1)} + q_{(0,1,1)}) + 3\,q_{(1,1,1)} = \mu \Big\}. \tag{24.3}$$

Using Figure 24.1, the (expected) payoff to the Operator is, for any $\mathbf{g} \in G_3$ and any $\mathbf{q} \in Q_3$, given by

$$\begin{aligned} Op_3(\mathbf{g},\mathbf{q}) := {} & q_{(0,0,0)} + q_{(1,0,0)} \Big((1-g_3)\,(2\,\beta-1) + g_3 \Big) \\ & + q_{(0,1,0)} \Big(1 - g_3 + g_3 \Big((1-g_{2,0})\,(2\,\beta-1) + g_{2,0} \Big) \Big) \\ & + q_{(0,0,1)} \Big(1 - g_3 + g_3 \Big(1 - g_{2,0} + g_{2,0}\,(2\,\beta-1) \Big) \Big) \\ & + q_{(1,1,0)} \Big((1-g_3)\,(2\,\beta-1) + g_3 \Big((1-g_{2,1})\,(2\,\beta-1) + g_{2,1} \Big) \Big) \\ & + q_{(1,0,1)} \Big((1-g_3)\,(2\,\beta-1) + g_3 \Big(1 - g_{2,1} + g_{2,1}\,(2\,\beta-1) \Big) \Big) \\ & + q_{(0,1,1)} \Big(1 - g_3 + g_3\,(2\,\beta-1) \Big) + q_{(1,1,1)}\,(2\,\beta-1)\,. \end{aligned} \tag{24.4}$$

The game theoretical solution of this inspection game, which is published in this monograph for the first time, is presented in

Lemma 24.1. *Given the Se-No inspection game with $L = 3$ steps, the expected number $\mu(> 0)$ of inspections, and with errors of the second kind. The Operator's set of behavioural strategies is given by (24.2), the Inspectorate's strategy set by (24.3), and the payoff to the Operator by (24.4).*

Then an optimal strategy of the Operator is given by

$$g_3^* = \frac{2}{3} \qquad \textit{and} \qquad g_{2,0}^* = g_{2,1}^* = \frac{1}{2}\,, \tag{24.5}$$

and an optimal strategy of the Inspectorate by

$$\begin{aligned} & q^*_{(0,0,0)} =: a_0 \qquad \textit{and} \qquad q^*_{(1,0,0)} = q^*_{(0,1,0)} = q^*_{(0,0,1)} =: a_1 \\ & q^*_{(1,1,0)} = q^*_{(1,0,1)} = q^*_{(0,1,1)} =: a_2 \qquad \textit{and} \qquad q^*_{(1,1,1)} =: a_3\,, \end{aligned} \tag{24.6}$$

where the a_i, $i = 0, \ldots, 3$, have to satisfy

$$a_0 + 3\,a_1 + 3\,a_2 + a_3 = 1 \qquad \textit{and} \qquad 3\,a_1 + 6\,a_2 + 3\,a_3 = \mu\,. \tag{24.7}$$

The optimal payoff to the Operator is

$$Op_3^* := Op_3(\mathbf{g}^*, \mathbf{q}^*) = 1 - 2\,(1-\beta)\,\frac{\mu}{3}\,. \tag{24.8}$$

Proof. Using (24.5) and (24.6) with (24.7), it is easily seen that $\mathbf{g}^* \in G_3$ and $\mathbf{q}^* \in Q_3$. We have to show that

$$Op_3(\mathbf{g}, \mathbf{q}^*) \le Op_3(\mathbf{g}^*, \mathbf{q}^*) \le Op_3(\mathbf{g}^*, \mathbf{q}) \tag{24.9}$$

for any $\mathbf{g} \in G_3$ and any $\mathbf{q} \in Q_3$. By (24.3), (24.4) and (24.5) we get for the coefficient of $2\,\beta - 1$

$$\frac{1}{3}\left(q_{(1,0,0)} + q_{(0,1,0)} + q_{(0,0,1)} + 2\left(q_{(1,1,0)} + q_{(1,0,1)} + q_{(0,1,1)}\right) + 3\,q_{(1,1,1)}\right) = \frac{\mu}{3}\,.$$

Thus, we get, using the abbreviations $a := q_{(0,0,0)}$, $b := q_{(1,0,0)} + q_{(0,1,0)} + q_{(0,0,1)}$, $c := q_{(1,1,0)} + q_{(1,0,1)} + q_{(0,1,1)}$ and $d := q_{(1,1,1)}$,

$$\begin{aligned} &Op_3(\mathbf{g}^*, \mathbf{q}) \\ &= (2\,\beta - 1)\,\frac{\mu}{3} + \frac{3}{3}\,q_{(0,0,0)} + \frac{2}{3}\left(q_{(1,0,0)} + q_{(0,1,0)} + q_{(0,0,1)}\right) + \frac{1}{3}\left(q_{(1,1,0)} + q_{(1,0,1)} + q_{(0,1,1)}\right) \\ &= (2\,\beta - 1)\,\frac{\mu}{3} + \frac{1}{3}\left(3\,a + 2\,b + c\right), \qquad (24.10) \end{aligned}$$

where $a + b + c + d = 1$ and $b + 2\,c + 3\,d = \mu$. These boundary conditions yield, by elimination of d, $3\,a + 2\,b + c = 3 - \mu$. Therefore, (24.10) finally simplifies to $Op_3(\mathbf{g}^*, \mathbf{q}) = Op_3^*$ for any $\mathbf{q} \in Q_3$, and the right hand side of (24.9) is fulfilled as equality.

To prove the left hand side of (24.9), we first note that we have for any $\mathbf{g} \in G_3$

$$\begin{aligned} &\Big((1 - g_3)\,(2\,\beta - 1) + g_3\Big) + \Big(1 - g_3 + g_3\,\Big((1 - g_{2,0})\,(2\,\beta - 1) + g_{2,0}\Big)\Big) \\ &\qquad + \Big(1 - g_3 + g_3\,\Big(1 - g_{2,0} + g_{2,0}\,(2\,\beta - 1)\Big)\Big) \\ &= (1 - g_3)\,(2\,\beta + 1) + g_3\,\Big(1 + (1 - g_{2,0})\,(2\,\beta - 1) + g_{2,0} + 1 - g_{2,0} + g_{2,0}\,(2\,\beta - 1)\Big)\Big) \\ &= 2\,\beta + 1 \end{aligned}$$

and

$$\begin{aligned} &\Big((1 - g_3)\,(2\,\beta - 1) + g_3\,\Big((1 - g_{2,1})\,(2\,\beta - 1) + g_{2,1}\Big)\Big) \\ &\qquad + \Big((1 - g_3)\,(2\,\beta - 1) + g_3\,\Big(1 - g_{2,1} + g_{2,1}\,(2\,\beta - 1)\Big)\Big) + \Big(1 - g_3 + g_3\,(2\,\beta - 1)\Big) \\ &= (1 - g_3)\,\Big(2\,(2\,\beta - 1) + 1\Big) \\ &\qquad + g_3\,\Big((1 - g_{2,1})\,(2\,\beta - 1) + g_{2,1} + 1 - g_{2,1} + g_{2,1}\,(2\,\beta - 1) + (2\,\beta - 1)\Big) \\ &= 4\,\beta - 1\,. \end{aligned}$$

Thus, (24.4), (24.6) and (24.7) imply for any $\mathbf{g} \in G_3$

$$Op_3(\mathbf{g}, \mathbf{q}^*) = a_0 + a_1\,(2\,\beta + 1) + a_2\,(4\,\beta - 1) + a_3\,(2\,\beta - 1) = 1 - 2\,(1 - \beta)\,\frac{\mu}{3} = Op_3^*\,,$$

i.e., the left hand side of (24.9) is fulfilled as equality, which completes the proof. □

Let us comment the results of Lemma 24.1: First of all, the optimal strategy (24.6) and (24.7) of the Inspectorate is not unique; we will come back to this important point on p. 437. We can choose, e.g., the probabilities $q_{(j_3,j_2,j_1)}$ such that only the nearest integer values of μ are mixed:

- For $0 < \mu \leq 1$

$$\begin{aligned} q^*_{(0,0,0)} &= 1 - \mu \qquad \text{and} \qquad q^*_{(1,0,0)} = q^*_{(0,1,0)} = q^*_{(0,0,1)} = \frac{\mu}{3} \\ q^*_{(1,1,0)} &= q^*_{(1,0,1)} = q^*_{(0,1,1)} = q^*_{(1,1,1)} = 0\,, \end{aligned} \tag{24.11}$$

- For $1 \leq \mu \leq 2$ by

$$\begin{aligned} & q^*_{(1,0,0)} = q^*_{(0,1,0)} = q^*_{(0,0,1)} = -\frac{\mu}{3} + \frac{2}{3} \qquad \text{and} \qquad q^*_{(1,1,0)} = q^*_{(1,0,1)} = q^*_{(0,1,1)} = \frac{\mu - 1}{3} \\ & q^*_{(0,0,0)} = q^*_{(1,1,1)} = 0\,, \end{aligned} \tag{24.12}$$

- For $2 \leq \mu \leq 3$ by

$$\begin{aligned} q^*_{(1,1,0)} &= q^*_{(1,0,1)} = q^*_{(0,1,1)} = \frac{3 - \mu}{3} \qquad \text{and} \qquad q^*_{(1,1,1)} = \mu - 2 \\ q^*_{(0,0,0)} &= q^*_{(1,0,0)} = q^*_{(0,1,0)} = q^*_{(0,0,1)} = 0\,. \end{aligned} \tag{24.13}$$

Second, for $\mu = 1, 2, 3$, i.e., for an integer (expected) number of inspections, the optimal inspection strategies (24.11) – (24.13) indicate that in these cases the number of inspections is *not* randomized. Instead of using (24.11) – (24.13) for $\mu = 1, 2, 3$, one can, however, also choose a mixed strategy. For $\mu = 2$, e.g., one can choose $a_0 = a_2 = 0$ and thus, by (24.7) one gets $a_1 = 1/6$ and $a_3 = 1/2$. Therefore, another optimal strategy of the Inspectorate is, for $\mu = 2$, given by

$$\begin{aligned} q^*_{(1,0,0)} &= q^*_{(0,1,0)} = q^*_{(0,0,1)} = \frac{1}{6} \qquad \text{and} \qquad q^*_{(1,1,1)} = \frac{1}{2} \\ q^*_{(0,0,0)} &= q^*_{(1,1,0)} = q^*_{(1,0,1)} = q^*_{(0,1,1)} = 0\,. \end{aligned}$$

Note that with Lemma 24.1 we have also found the solution of the corresponding Se-No inspection game with a fixed integer number of inspections. In case of one inspection one just have to put $q_{(0,0,0)} = q_{(1,1,0)} = q_{(1,0,1)} = q_{(0,1,1)} = q_{(1,1,1)} = 0$ in (24.4) to obtain the payoff to the Operator, and then $g^*_3 = 2/3$ and $g^*_{2,0} = 1/2$ as well as $q^*_{(1,0,0)} = q^*_{(0,1,0)} = q^*_{(0,0,1)} = 1/3$ are optimal strategies in the inspection game with (exactly) one inspection.

Third, we compare the results of Lemma 24.1 with those of the generalized Thomas-Nisgav inspection game for $L = 3$, $d = b = 1$ and $k/3 < 1/(2\,(1 - \beta))$. We see that the Operator's optimal strategies (24.5) coincide with the equilibrium strategy (17.31) of the Smuggler

$$\bar{p}^*_{3,k} = \frac{1}{3} = 1 - g^*_3 \qquad \text{and} \qquad \bar{p}^*_{2,k'} = \frac{1}{2} = 1 - g^*_{2,0} = 1 - g^*_{2,1}\,,$$

and that the optimal payoff (24.8) coincides with the equilibrium payoff (17.33) for $k = \mu$. Therefore, if the Operator decides sequentially, then he need not care whether the Inspectorate decides non-sequentially using the expected number μ of inspections, or decides sequentially using the fixed number $k = \mu$ of inspections. These properties of optimal strategies we have already observed in Chapter 6.

Fourth, we compare the results of Lemma 24.1 with those of the Canty-Rothenstein-Avenhaus inspection game for $L = 3$, $a = b = c = d = 1$, $\alpha = 0$ and $k/3 < 1/(2\,(1-\beta))$. Let p_i, $i = 3, 2$, be the probability that the Operator behaves illegally at step i. Then (24.5) implies

$$p_3^* = 1 - g_3^* = \frac{1}{3} \qquad \text{and} \qquad p_2^* = g_3^*\,(1 - g_{2,0}^*) = g_3^*\,(1 - g_{2,1}^*) = \frac{1}{3}\,,$$

i.e., (15.76) for $L = 3$. Let q_j, $j = 3, 2, 1$, denote the probability that the Inspectorate performs an inspection at step j. Then we have

$$\begin{aligned} q_3 &= q_{(1,0,0)} + q_{(1,1,0)} + q_{(1,0,1)} + q_{(1,1,1)} \\ q_2 &= q_{(0,1,0)} + q_{(1,1,0)} + q_{(0,1,1)} + q_{(1,1,1)} \\ q_1 &= q_{(0,0,1)} + q_{(1,0,1)} + q_{(0,1,1)} + q_{(1,1,1)}\,, \end{aligned} \tag{24.14}$$

and (24.11) – (24.13) imply independently of μ that

$$q_3^* = q_2^* = q_1^* = \frac{\mu}{3}\,,$$

which coincides with $q_2^* = q_1^* = q_0^*$ in (15.77) for $k = \mu$ and $L = 3$. Again, the optimal payoff (24.8) coincides with the equilibrium payoff (15.78) for $k = \mu$. Thus, if the Inspectorate decides non-sequentially (either using the expected number μ of inspections or using the fixed number $k = \mu$ of inspections), then it need not care whether the Operator decides sequentially or not. We did not observe these properties of optimal strategies in Chapter 6.

We mentioned on p. 427 that the probabilities $g_{2,0}$ and $g_{2,1}$ need to be distinguished from a modelling view point, because the Operator might choose a different probability depending whether there was an inspection at step 3 or not. The fact $g_{2,0}^* = g_{2,1}^*$ is a result of the game theoretical analysis. However, the game tree in Figure 24.1 points to an interesting *heuristic* argument: Suppose the Operator behaves legally at step 3 (ℓ_3), then the decision situation he faces at step 2 is independent of whether there was an inspection at step 3 or not. For example, if the Inspectorate chooses one of the triples $(0,0,0)$ or $(1,0,0)$, then the (not proper) subgames entered after the decision ℓ_3 are equal. The same is true if the Inspectorate chooses $(0,1,0)$ or $(1,1,0)$, or if it chooses $(0,0,1)$ or $(1,0,1)$, or if it chooses $(0,1,1)$ and $(1,1,1)$. According to this heuristic argument it can be assumed that the probabilities $g_{2,0}$ and $g_{2,1}$ are equal and we abbreviate them by g_2. Thus, the Operator's strategy set is then given by

$$\widetilde{G}_3 := \left\{ \widetilde{\mathbf{g}} := (g_3, g_2) \in [0,1]^2 \right\}. \tag{24.15}$$

Putting $g_{2,0} = g_{2,1} =: g_2$ we obtain for the coefficient of $2\,\beta - 1$ from (24.4), using (24.14),

$$\begin{aligned} &(1 - g_3)\left(q_{(1,0,0)} + q_{(1,1,0)} + q_{(1,0,1)} + q_{(1,1,1)}\right) \\ &+ g_3\,(1 - g_2)\left(q_{(0,1,0)} + q_{(1,1,0)} + q_{(0,1,1)} + q_{(1,1,1)}\right) \end{aligned}$$

$$+ g_3 \, g_2 \Big(q_{(0,0,1)} + q_{(1,0,1)} + q_{(0,1,1)} + q_{(1,1,1)} \Big)$$

$$= (1 - g_3) \, q_3 + g_3 \, (1 - g_2) \, q_2 + g_3 \, g_2 \, q_1 \, . \tag{24.16}$$

The expression in (24.16) is the probability that an inspection is performed at the same step at which the Operator behaves illegally. The payoff to the Operator is in this case $2\,\beta - 1$. The remaining terms in (24.4) simplify, using (24.14), to

$$(1 - g_3) \Big(q_{(0,0,0)} + q_{(0,1,0)} + q_{(0,0,1)} + q_{(0,1,1)} \Big)$$

$$+ g_3 \, (1 - g_2) \Big(q_{(0,0,0)} + q_{(1,0,0)} + q_{(0,0,1)} + q_{(1,0,1)} \Big)$$

$$+ g_3 \, g_2 \Big(q_{(0,0,0)} + q_{(1,0,0)} + q_{(0,1,0)} + q_{(1,1,0)} \Big)$$

which gives by (24.14)

$$(1 - g_3) \, (1 - q_3) + g_3 \, (1 - g_2) \, (1 - q_3) + g_3 \, g_2 \, (1 - q_1) \, , \tag{24.17}$$

i.e., the probability that no inspection is performed at the step at which the Operator behaves illegally. The payoff to the Operator is in this case $+1$. Note that using (24.14) as basis for the Inspectorate's behaviour, its strategy set is now

$$\widetilde{Q}_3 := \Big\{ \widetilde{\mathbf{q}} := (q_3, q_2, q_1) \in [0,1]^3 : q_3 + q_2 + q_1 = \mu \Big\} \, . \tag{24.18}$$

Using (24.4), the Operator's payoff can be expressed, for any $\widetilde{\mathbf{g}} \in \widetilde{G}_3$ and any $\widetilde{\mathbf{q}} \in \widetilde{Q}_3$, by

$$\begin{aligned} \widetilde{Op}_3(\widetilde{\mathbf{g}}, \widetilde{\mathbf{q}}) &:= (2\,\beta - 1) \, ((1 - g_3) \, q_3 + g_3 \, (1 - g_2) \, q_2 + g_3 \, g_2 \, q_1) \\ &\quad + (1 - g_3) \, (1 - q_3) + g_3 \, (1 - g_2) \, (1 - q_3) + g_3 \, g_2 \, (1 - q_1) \, . \end{aligned} \tag{24.19}$$

This expression will be generalized in Section 24.2 to any $L = 4, 5, \ldots$.

24.2 Any number of steps; errors of the second kind

We now consider the case of L steps, and solve first the *simplified* Se-No inspection game with an expected number μ of inspections. "Simplified" refers here to the fact that this game is based on the assumption that the Operator's probabilities at step j, $j = L-1, \ldots, 1$, do not depend on the history of the game, i.e., do not depend on what will have happened at steps $L, \ldots, j+1$. As a consequence – see last section – we can utilize the probability q_j that an inspection is performed at step j; see (24.14) for $L = 3$ steps.

As a generalization of (24.15), the Operator's strategy set in the simplified Se-No inspection game is given by

$$\widetilde{G}_L := \Big\{ \widetilde{\mathbf{g}} := (g_L, g_{L-1}, \ldots, g_2) \in [0,1]^{L-1} \Big\} \, , \tag{24.20}$$

where g_ℓ denotes the probability of not behaving illegally at step ℓ, and that of the Inspectorate, as a generalization of (24.18), is given by

$$\widetilde{Q}_L := \left\{ \widetilde{\mathbf{q}} := (q_L, \ldots, q_1) \in [0,1]^L : \sum_{j=1}^{L} q_j = \mu \right\} \, , \tag{24.21}$$

where q_j denotes the probability that the Inspectorate performs an inspection at step j.

As a generalization of (24.16) and (24.17), the probability that an inspection is performed at the same step at which the Operator behaves illegally is given by

$$\sum_{i=1}^{L} (1-g_i)\, q_i \prod_{\ell=i+1}^{L} g_\ell\,, \tag{24.22}$$

where $\prod_{\ell=L+1}^{L} g_\ell = 1$ and $g_1 = 1$, and the probability that no inspection is performed at the step at which the Operator behaves illegally is given by

$$\sum_{i=1}^{L} (1-g_i)\,(1-q_i) \prod_{\ell=i+1}^{L} g_\ell\,.$$

Therefore, the (expected) payoff to the Operator is, for any $\widetilde{\mathbf{g}} \in \widetilde{G}_L$ and any $\widetilde{\mathbf{q}} \in \widetilde{Q}_L$, given by

$$\widetilde{Op}_L(\widetilde{\mathbf{g}}, \widetilde{\mathbf{q}}) := (2\,\beta - 1) \sum_{i=1}^{L} (1-g_i)\, q_i \prod_{\ell=i+1}^{L} g_\ell + \sum_{i=1}^{L} (1-g_i)\,(1-q_i) \prod_{\ell=i+1}^{L} g_\ell\,. \tag{24.23}$$

Of course, (24.23) simplifies to (24.19) for $L = 3$ steps.

The game theoretical solution of the simplified Se-No inspection game, see Krieger and Avenhaus (2018a), is given in Lemma 24.2. Even though the following result applies to any L, we do not formulate it as a Theorem – as we have done throughout this monograph – but as a Lemma to indicate the provisional nature of our analysis.

Lemma 24.2. *Given the simplified Se-No inspection game with L steps, the expected number $\mu(>0)$ of inspections, and with errors of the second kind. The Operator's set of behavioural strategies is given by (24.20), the Inspectorate's strategy set by (24.21), and the payoff to the Operator by (24.23).*

Then an optimal strategy of the Operator is given by

$$g_\ell^* = \frac{\ell - 1}{\ell}\,, \qquad \ell = L, \ldots, 2\,, \tag{24.24}$$

and an optimal strategy of the Inspectorate by

$$q_\ell^* = \frac{\mu}{L}\,, \qquad \ell = L, \ldots, 1\,. \tag{24.25}$$

The optimal payoff to the Operator is

$$\widetilde{Op}_L^* := \widetilde{Op}_L(\widetilde{\mathbf{g}}^*, \widetilde{\mathbf{q}}^*) = 1 - 2\,(1-\beta)\,\frac{\mu}{L}\,. \tag{24.26}$$

Proof. Obviously, we have $\widetilde{\mathbf{g}}^* \in \widetilde{G}_L$ and $\widetilde{\mathbf{q}}^* \in \widetilde{Q}_L$. We have to prove that

$$\widetilde{Op}_L(\widetilde{\mathbf{g}}, \widetilde{\mathbf{q}}^*) \le \widetilde{Op}_L(\widetilde{\mathbf{g}}^*, \widetilde{\mathbf{q}}^*) \le \widetilde{Op}_L(\widetilde{\mathbf{g}}^*, \widetilde{\mathbf{q}}) \tag{24.27}$$

for any $\widetilde{\mathbf{g}} \in \widetilde{G}_L$ and any $\widetilde{\mathbf{q}} \in \widetilde{Q}_L$. Because (24.23) can be written as

$$\widetilde{Op}_L(\widetilde{\mathbf{g}}, \widetilde{\mathbf{q}}) = \sum_{i=1}^{L} (1-g_i) \Big(1 - 2\,(1-\beta)\, q_i\Big) \prod_{\ell=i+1}^{L} g_\ell\,,$$

we get by (24.25) and $g_1 = 1$

$$\widetilde{Op}_L(\widetilde{\mathbf{g}}, \widetilde{\mathbf{q}}^*) = \Big(1 - 2\,(1-\beta)\,\frac{\mu}{L}\Big) \sum_{i=1}^{L} (1 - g_i) \prod_{\ell=i+1}^{L} g_\ell = 1 - 2\,(1-\beta)\,\frac{\mu}{L} = \widetilde{Op}_L^*$$

for any $\widetilde{\mathbf{g}} \in \widetilde{G}_L$, and by (24.21) and (24.24)

$$\widetilde{Op}_L(\widetilde{\mathbf{g}}^*, \widetilde{\mathbf{q}}) = \sum_{i=1}^{L} \frac{1}{i} \Big(1 - 2\,(1-\beta)\,q_i\Big) \prod_{\ell=i+1}^{L} \frac{\ell - 1}{\ell} = \frac{1}{L} \sum_{i=1}^{L} (1 - 2\,(1-\beta)\,q_i) = \widetilde{Op}_L^*$$

for any $\widetilde{\mathbf{q}} \in \widetilde{Q}_L$, i.e., the saddle point criterion (24.27) is fulfilled as equality. □

Let us comment the results of Lemma 24.2: First, we see that the Operator's optimal strategy (24.24) coincides with that of the generalized Thomas-Nisgav inspection game for $d = b = 1$ and $k/L < 1/(2\,(1-\beta))$ in (17.31). Also, the optimal payoffs (17.33) and (24.26) are the same for $k = \mu$. The relation between the Inspectorate's optimal strategies (17.32) and (24.25), i.e., k'/ℓ resp. μ/L, is, for $\ell = L-1, \ldots, 1$, unclear. Using the Operator's optimal strategy as given by (24.24), we see that $1 - g_2^*, \ldots, 1 - g_L^*$ form a harmonic progression; see also Table 4.1 on p. 72 for an overview of inspection games treated in this monograph with this property.

Second, comparing the results of the Canty-Rothenstein-Avenhaus inspection game for $a = b = c = d = 1$, $\alpha = 0$ and $k/L < 1/(2\,(1-\beta))$, we see that the Operator's optimal strategies can be transformed into each other. Let p_i, $i = L \ldots, 2$, be the probability that the Operator behaves illegally at step i if he does not do so before. Then (24.24) implies for all $i = L, \ldots, 1$

$$p_L = 1 - g_L^* = \frac{1}{L} \qquad \text{and} \qquad p_i^* = g_L^* \ldots g_{i+1}^* \,(1 - g_i^*) = \frac{L-1}{L} \cdots \frac{1}{i+1} = \frac{1}{L}\,,$$

i.e., (15.76). This is intuitive since the Operator who ignores the history in fact behaves non-sequentially, like in the Canty-Rothenstein-Avenhaus inspection game. The Inspectorate's optimal strategies (15.77) and (24.25) obviously coincide for $k = \mu$. Again, the optimal payoffs to the Operator (15.78) and (24.26) are the same.

Third, let us assume that instead of the critical time concept which is the basis of (24.23), the probability to detect the illegal behaviour is considered as the objective function. This probability is, using (24.22), given by

$$\mathbb{P}_L(\text{detect the illegal behaviour}) = (1-\beta) \sum_{i=1}^{L} (1 - g_i)\, q_i \prod_{\ell=i+1}^{L} g_\ell\,. \tag{24.28}$$

It can be shown that (24.24) and (24.25) are also optimal strategies if (24.28) is taken as the objective function. The resulting optimal probability to detect the illegal behaviour is

$$\mathbb{P}_L^*(\text{detect the illegal behaviour}) = (1-\beta)\,\frac{\mu}{L}\,.$$

Thus, it does not matter whether the Inspectorate plans its inspections based on the (expected) payoff (24.23) or based on the probability to detect the illegal behaviour (24.28), because they both lead to the same optimal inspection strategy. This issue is also addressed on p. 391.

Fourth, introducing payoffs to the Operator as in (17.1), it can be shown that he behaves legally if and only if condition (17.34) is fulfilled; see Krieger and Avenhaus (2018a).

Having solved the *simplified* Se-No inspection game with the expected number μ of inspections, we get back to the original game. Again, let g_L be the probability to postpone the illegal behaviour at step L, and let $g_{\ell,(j_L,\dots,j_{\ell+1})}$ be the probability to postpone the illegal behaviour at step ℓ if the inspection history $(j_L,\dots,j_{\ell+1})$ has been observed. The Inspectorate chooses $(j_L,\dots,j_1)$ with probability $q_{(j_L,\dots,j_1)}$. Let the strategy sets G_L and Q_L be defined as appropriate generalizations of (24.2) and (24.3). The payoff to the Operator is, for any $\mathbf{g}\in G_L$ and any $\mathbf{q}\in Q_L$, given by

$$\begin{aligned} Op_L(\mathbf{g},\mathbf{q}) = & \sum_{(j_L,\dots,j_1)\in\{0,1\}^L} q_{(j_L,\dots,j_1)} \Big((1-g_L)\,(1-2\,(1-\beta)\,\mathbb{1}_{j_L}(1)) \\ & + \sum_{n=2}^{L-1} (1-g_{n,(j_L,\dots,j_{n+1})})\, g_L \prod_{\ell=n+1}^{L-1} g_{\ell,(j_L,\dots,j_{\ell+1})}\,(1-2\,(1-\beta)\,\mathbb{1}_{j_n}(1)) \\ & + g_L \prod_{\ell=2}^{L-1} g_{\ell,(j_L,\dots,j_{\ell+1})}\,(1-2\,(1-\beta)\,\mathbb{1}_{j_1}(1))\Big)\,, \end{aligned} \tag{24.29}$$

where, like (11.23), the indicator function is given by

$$\mathbb{1}_i(j) := \begin{cases} 0 & : \; i\neq j \\ 1 & : \; i=j \end{cases} .$$

Then Lemmata 24.1 and 24.2 lead to

Conjecture 24.1. *Given the Se-No inspection game with L steps, the expected number $\mu(>0)$ of inspections, and with errors of the second kind. The Operator's set G_L of behavioural strategies and the Inspectorate's strategy set Q_L are appropriate generalizations of (24.2) and (24.3), and the payoff to the Operator is given by (24.29).*

Then an optimal strategy of the Operator is given by

$$\begin{aligned} & g_L^* = \frac{L-1}{L} \qquad \text{and} \qquad g^*_{\ell,(j_L,\dots,j_{\ell+1})} = \frac{\ell-1}{\ell} \\ & \text{for all} \quad \ell = L-1,\dots,2 \quad \text{and any} \quad (j_L,\dots,j_{\ell+1})\in\{0,1\}^{L-\ell}\,, \end{aligned} \tag{24.30}$$

and an optimal strategy of the Inspectorate as follows: Any of the tuples $(j_L,\dots,j_1)\in\{0,1\}^L$ with k-times "1" is chosen with probability a_k, $k=0,\dots,L$, such that these probabilities satisfy

$$\sum_{k=0}^{L}\binom{L}{k} a_k = 1 \quad \text{and} \qquad \sum_{k=1}^{L} k\binom{L}{k} a_k = \mu\,.$$

The optimal payoff to the Operator is given by (24.26).

The right hand side of the saddle point inequality (24.27) with $Op_L(\mathbf{g},\mathbf{q})$ instead of $\widetilde{Op}_L(\widetilde{\mathbf{g}},\widetilde{\mathbf{q}})$ can be proven as follows: Using (24.30), (24.29) simplifies for any $\mathbf{q}\in Q_L$ to

$$Op_L(\mathbf{g}^*,\mathbf{q}) = 1-2\,(1-\beta) \sum_{(j_L,\dots,j_1)\in\{0,1\}^L} q_{(j_L,\dots,j_1)} \Big(\frac{1}{L}\,\mathbb{1}_{j_L}(1)$$

$$+\sum_{n=2}^{L-1}\frac{1}{n}\,\frac{L-1}{L}\,\frac{L-2}{L-1}\cdots\frac{n}{n+1}\,\mathbb{1}_{j_n}(1)+\frac{L-1}{L}\,\frac{L-2}{L-1}\cdots\frac{1}{2}\,\mathbb{1}_{j_1}(1)\Big)$$

$$=1-2\,(1-\beta)\,\frac{1}{L}\sum_{(j_L,\ldots,j_1)\in\{0,1\}^L} q_{(j_L,\ldots,j_1)}\sum_{n=1}^{L}\mathbb{1}_{j_n}(1)$$

$$=1-2\,(1-\beta)\,\frac{\mu}{L}=Op_L^*\,.$$

So far we were not able to prove the left hand side of the saddle point inequality (24.27) for any L, but only for $L=3$, see the proof of Lemma 24.1, and for $L=4$, not presented here. Even though this proof does not seem to be out of reach, we leave it to our readership, thus encouraging further research in this fascinating area.

Most importantly, we realize, as we did already after Lemma 24.1, that the optimal strategy of the Inspectorate is not unique. A special optimal strategy of the Inspectorate is given by choosing it such that only the nearest integer values of μ are mixed, more precisely: Any of the tuples $(j_L,\ldots,j_1)\in\{0,1\}^L$ with exactly $\lfloor\mu\rfloor$ "1", i.e., $\lfloor\mu\rfloor$ inspections are performed, is chosen with probability[2]

$$\frac{1+\lfloor\mu\rfloor-\mu}{\binom{L}{\lfloor\mu\rfloor}}\,, \qquad (24.31)$$

and any of the tuples $(j_L,\ldots,j_1)\in\{0,1\}^L$ with exactly $\lfloor\mu\rfloor+1$ "1", i.e., $\lfloor\mu\rfloor+1$ inspections are performed, is chosen with probability

$$\frac{\mu-\lfloor\mu\rfloor}{\binom{L}{\lfloor\mu\rfloor+1}}\,. \qquad (24.32)$$

Any of the tuples $(j_L,\ldots,j_1)\in\{0,1\}^L$ with less than $\lfloor\mu\rfloor$ or more than $\lfloor\mu\rfloor+1$ "1" are chosen with probability 0. If μ is an integer, then $\mu-\lfloor\mu\rfloor=0$ and (24.31) and (24.32) yield

$$q^*_{(j_L,\ldots,j_1)}=\begin{cases}\binom{L}{\mu}^{-1} & \text{for } (j_L,\ldots,j_1)\in\{0,1\}^L \text{ with } \sum_{n=1}^{L}\mathbb{1}_{j_n}(1)=\mu\\ 0 & \text{for } (j_L,\ldots,j_1)\in\{0,1\}^L \text{ with } \sum_{n=1}^{L}\mathbb{1}_{j_n}(1)\neq\mu\end{cases}\,. \qquad (24.33)$$

Let us close this chapter with two remarks on the applicability of these results: First, of course, we provide optimal strategies even in the case that for some reason or other the given number of expected inspections is *not* an integer. If it is an integer, then the Inspectorate can either use this fixed number leading to (24.33), or it can use some mixed strategy, e.g., (24.31) and (24.32). In the former case the number of inspections is exactly μ and thus, the Inspectorate's inspection effort is a priori fixed. In the latter case there might be actually more or less than μ inspections, which – in case of more than μ inspections – is a disadvantage for the Inspectorate.

Second, we answer the question "Is there any advantage of using inspection games with an expected number instead of a fixed number of inspections?"; see p. 425. If the optimal payoff to the Operator is used to decide whether an expected or a fixed number of inspections should be preferred, then the Se-No inspection game treated in this chapter shows that both models are equivalent in the sense that they result in the same optimal payoff to the Operator for

[2]The floor function $\lfloor\,\rfloor$ maps x to the greatest integer less than or equal to x.

any integer value of μ. So here, there is neither an advantage nor a disadvantage of using a model with an expected number of inspections, and a model can only be selected by taking into account further criteria such as organizational ones which may favour a solution as given by (24.31) to (24.32), or the initially mentioned *unpredictability*.

If the latter one has the highest priority, then *all* possibilities $k = 0, \ldots, L$ of numbers of inspections should be used, e.g., by

$$q^*_{(j_L,\ldots,j_1)} = \begin{cases} 1 - \frac{\mu}{L}\left(2 - \frac{1}{2^{L-1}}\right) & : \ (j_L, \ldots, j_1) = (0, \ldots, 0) \\ \frac{\mu}{L\,2^{L-1}} & : \ (j_L, \ldots, j_1) \in \{0,1\}^L \setminus \{(0, \ldots, 0)\} \end{cases}$$

if μ is small

$$\frac{\mu}{L}\left(2 - \frac{1}{2^{L-1}}\right) < 1\,,$$

and by

$$q^*_{(j_L,\ldots,j_1)} = \begin{cases} \left(1 - \frac{\mu}{L}\right)\frac{1}{2^{L-1}} & : \ (j_L, \ldots, j_1) \in \{0,1\}^L \setminus \{(1, \ldots, 1)\} \\ 1 - \left(1 - \frac{\mu}{L}\right)\left(2 - \frac{1}{2^{L-1}}\right) & : \ (j_L, \ldots, j_1) = (1, \ldots, 1) \end{cases}$$

if μ is large

$$\frac{\mu}{L}\left(2 - \frac{1}{2^{L-1}}\right) \geq 1\,.$$

This, on the other side, makes planning difficult; perhaps the responsible officials will prefer a solution of the kind given by (24.11) – (24.13) resp. (24.31) and (24.32). Life is not always made easier by freedom of choice.

List of Figures

List of Tables

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