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ABSTRACT

Labor Income Share Dynamics with Variable Elasticity of Substitution*

The accumulation principle suggests that complementarity between capital and labor forces the labor income share to rise in the presence of capital accumulation. The CES model estimates using data from 20 Japanese industries between 1970 and 2012 explain the same outcome but with substitutable factor inputs. To resolve this puzzle, this paper proposes a variable elasticity of substitution (VES-W) framework that embodies a variable elasticity of substitution and a share parameter as a non-linear function of the Weibull distribution of capital-labor ratio. Empirical findings support the choice of a variable elasticity of substitution. While the estimated structural parameters calibrate the actual output level and the movements in factor income shares reasonably well in both the CES and VES-W models, the VES-W model outcomes support the accumulation principle by achieving the same result but with complementary factor inputs.

JEL Classification: E21, E22, E25

Keywords: substitution elasticity, labor income share, production function parameters

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Introduction

The elasticity of factor substitution (σ_{KL}) plays an important role in the analysis of factor income shares. The assumption of a non-unitary σ_{KL} explains the movements in the labor income share over time. On the other hand, production technology governs the characteristics of σ_{KL}[1], and the relationship between the allocative efficiency of factor inputs and the rate of productivity per worker. To this extent, the Constant Elasticity of Substitution (CES) production technology has been a dominant model choice for researchers studying the labor income share. The CES production framework, with constant returns to scale and perfectly competitive factor markets, predicts a stable relationship between the labor income share and capital accumulation where the constant σ_{KL} forces a decrease (increase) in the labor income share with capital accumulation when it is greater than (less than) unity[2]. Empirical evidences, however, do not always support this accumulation principle. For example, both capital-output ratio and the labor income share increased in a majority of the Japanese industrial sectors between 1970 and 2012 but the CES model estimates of σ_{KL} are predominantly greater than unity across sectors.

This paper aims to resolve this puzzle by developing a variable elasticity of substitution (VES) framework as an alternative to the CES model. While the estimated structural parameters calibrate the actual output level and the movements in factor income shares reasonably well in both the CES and VES models, the VES model provides a stronger empirical support to the accumulation principle compared to the CES model. The VES model explains an increase in the labor income share and capital accumulation with an average σ_{KL} less than unity, whereas the CES model achieves the same result with an average σ_{KL} greater than unity.

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[1]Specifically, intra-sectoral substitution of production framework, technological innovations and intersectoral substitution due to commodity substitution are among the determinants of σ_{KL} (Hicks, 1932).

[2] Bentolila and Saint-Paul (2003) derived $L_S = 1 - (c)^{\frac{\sigma-1}{\sigma}}$ where $c = \left[\frac{(K)^{\frac{\sigma-1}{\sigma}}}{(K)^{\frac{\sigma-1}{\sigma}} + (L)^{\frac{\sigma-1}{\sigma}}}\right]^{\frac{\sigma}{\sigma-1}}$, and $\sigma_{KL} = \sigma$. Both Karabarbounis and Neiman (2014) and Piketty (2014) estimate the elasticity of substitution >1.

[Figure 1 is about here]

I begin the discussion by providing evidence on the variability of σ_{KL}[3] in Figure 1. It plots the factor income share ratio against factor share ratio for 20 Japanese industrial sectors over 1970-2012. In most of the sectors, the labor income share shows a negative correlation with the capital-labor ratio, which broadly supports the capital accumulation principle (Karabarbounis and Neiman, 2014; Piketty, 2014). At the same time, the non-linear fits suggest variation in the rate of substitution between capital and labor, albeit considerable uniformity in the rate of substitution across sectors. The technology and share parameters in any production technology are unlikely to remain fixed if we allow σ_{KL} to vary. In a multi-dimensional parameter space, simultaneous changes in the time-variant parameters cause an identification problem as factors other than variable σ_{KL} affect movements in the labor income share. There are also further complications when the value of σ_{KL} crosses over from one side of the unity to other, which forces the direction of the movement in the factor income share to change. Altogether, it suggests that one must exercise caution when interpreting the implications of a variable σ_{KL} on the movements in the labor income share.

The proponents of the "normalized CES technology" (La Grandville, 1989; Klump and La Grandville, 2000; Klump et al, 2007) assume that both technology and share parameters are functions of variable σ_{KL}. The normalization process ties up the share parameter with the capital-labor ratio in a way where the share parameter arises when the capital-labor ratio equals to a benchmark level (unity in the case of a CES framework). However, an arbitrary choice of the benchmark capital-labor ratio does not reduce the parameter space to one dimension (forcing only σ_{KL} to vary). It instead shifts the burden of constancy from the share parameter to the benchmarked capital-labor ratio. This results in the relationship between σ_{KL} and any variable of interest in a normalized CES set-up depending on the benchmarked

[3] Notable theoretical contributions to the literature include Lucas (1990), Laitner (1995) and Turnovsky (2002).

capital-labor ratio, which may become more difficult to interpret the resulting economic implications due to its arbitrary nature (Temple, 2012).

As an alternative strategy, I analyze the relationship between capital accumulation and the labor income share using a variable elasticity of substitution framework. I define the share parameter as a cumulative distribution function of a 3-parameter Weibull distribution of the capital-labor ratio[4]. The choice is not ad-hoc because the Weibull distribution shows the best fit relative to other extreme value distributions[5]. The share parameter as a monotonic transformation of the capital-labor ratio implies that a higher capital-labor ratio leads to a higher share of the capital-output ratio following the explicit parameters of the Weibull distribution of the capital-labor ratio. I extend the variable elasticity framework of Lu and Fletcher (1968) to incorporate time-variant σ_{KL} and time-variant share parameter as non-linear functions of the capital-labor ratio, which I then use to theoretically derive the full set of parameters. I will refer to this as the VES-W model. Using annual data from 20 Japanese industrial sectors over a period of 43 years (1970-2012), I test the evidence of a variable σ_{KL} and estimate a full set of parameters in both CES and VES-W models. Finally, I calibrate the production function and factor income shares using the estimated parameters and compare the descriptive accuracy and predicting power of each model.

The empirical findings suggest that the labor income share increased in 18 out of the 20 industrial sectors (leaving transport and non-government services) between 1970 and 2012. The capital-labor ratio plays a consistent and statistically significant role in explaining the variation the elasticity of substitution between capital and labor. The estimated values of the elasticity of substitution in the VES-W model fluctuates around unity in most of the sectors with an overall average of .91, but do not show any consistent relationship with the capital-

[4] Taking stock of the growing literature on productivity levels driven by an explicit set of technology (Kortun, 1997; Jones, 2005; Growiec, 2008),

[5] I compare the Kormogolov-Smirnov goodness of fit test statistic for six models form the class of extreme generalized value distribution (Frechet with 2 and 3 parameters, General extreme value, Gumbel max and Weibull 2 and 3 parameters) and normal distribution.

labor ratio across sectors. The calibration outcomes based on a full set of estimated parameters do not suggest that VES-W must be a preferred choice of model over CES. Both models fit the actual output and factor income shares reasonably well, but the most striking difference lies in the estimated values of σ_{KL}: an average $\hat{\sigma}_{VES-W}$ = .91 against $\hat{\sigma}_{CES}$ = 1.03. Figure 2 shows a significant discrepancy in the estimated values of σ_{KL} between both models where 14 out of the 20 industrial sectors show a lower average $\hat{\sigma}_{VES-W}$ than $\hat{\sigma}_{CES}$. In the production process, capital and labor show complementarity in most of the industrial sectors based on the VES-W model, whereas the CES model estimates suggest more substitutability between capital and labor. This way the VES-W model accounts for the same labor income share dynamics but with complementary factor inputs at the sectoral level. It remains a subject of further study whether an aggregation of sectoral σ_{KL} produces the same result at the macro level.

[Figure 2 is about here]

This paper contributes to the recent debate on the role of the elasticity of factor substitution behind the secular decline in the labor income share. Using the CES production framework, Piketty (2014) and Karabarbounis and Neiman (2014) estimate the values of elasticity of substitution between capital and labor to be greater than unity. However, the majority of the studies estimate σ_{KL} to be less than one (Leon-Ledesma, McAdam and Willman 2015; Oberfield and Raval 2014; Chirinko and Mallick 2017). There are two recent studies that attempt to resolve this apparent puzzle. Grossman et al. (2017) argue that a decline in the labor income share with σ_{KL} < 1 is feasible if there is a slowdown of labor productivity growth. Paul (2019) shows that it is feasible for labor income share to decline with capital accumulation when $\sigma_{KL} < 1$., by drawing insights from the literature on differential capital–skill substitutability (Krusell et al., 2000) and applying the Morishima elasticity of substitution to identify substitution parameters for different skill-groups. This study extends this literature suggesting that a VES framework can serve as an alternative means to study movements in the labor income share.

The literature that provides empirical validity to the usefulness of VES production technology predates the recent growth in the labor income share research. While most of the studies (Sato and Hoffman 1968; Diwan 1970; Kazi 1980; Meyer and Kadiwala 1974; and Revankar 1971) reject CD and CES model specifications in favor of the VES model, Lovell (1973), Tsang and Yeung (1976) and Zellner and Ryu (1998) provide evidence that in certain sectors, the CES model provides a better fit to the data compared to VES. Since these studies use various estimation strategies methods and different sets of data (both cross-sectional and time-series) at different levels of aggregation (from industries within a country to cross-country countries using aggregate data), it becomes difficult to ascertain a definite answer.

This paper is also related to the literature on the endogenous elasticity of factor substitution. Miyagawa and Papageorgiou (2007) build a static factor-endowment model where σ_{KL} in each period is endogenously determined by the existing endowments of capital and labor and their equilibrium inter-sectoral allocation. Duffy and Papageorgiou (2000) show that σ_{KL}increases as the economy grows. This line of literature does not assume that capital per unit of labor and $\sigma_{KL}$are related based on any functional form. Rather, such a relationship is determined by the market equilibrium conditions of a growing multi-sectoral economy. This paper is directly linked to the endogenous elasticity of factor substitution, but unlike Miyagawa and Papageorgiou (2007), it assumes a non-linear relationship between σ_{KL} and capital per unit of labor.

The rest of the paper is organized as follows. In section 2, I provide a snapshot of the labor income share trends across 20 Japanese industrial sectors. This section also briefly introduces the Regional Japan Industrial Productivity (R-JIP) dataset and its key features. Section 3 is devoted to the discussion of the VES-W production framework. The first part explains the results showing empirical validity to a variable σ_{KL}. The second part derives the parameters of the VES-W model. In section 4, I discuss the empirical results. I begin with the CES estimates, and then consider two forms of VES models: VES-W and a simple VES model where the share parameter remains fixed. Section 5 compares the descriptive accuracy and predicting power of the CES and the VES models, which is followed by a concluding remark.

2. Labor income share trends across Japanese industries, 1970 - 2012

2.1. Data

I use the Regional Japan Industrial Productivity (R-JIP) databases compiled by RIETI (Research Institute of Economy, Trade, and Industry) and Hitotsubashi University, Tokyo.[6] The R-JIP database compiles value-added output in current and constant prices, quality-adjusted labor input, and quality-adjusted capital input for all 23 industrial sectors[7]. In the latest version of the R-JIP data, published in 2017, there is available data for every year from 1970 to 2012. Following Fukao and Perugini (2018), I construct the labor income share by sector (industry) as the ratio of nominal total labor compensation to nominal value added (at current prices). Since nominal total labor compensation includes all types of remuneration, such as employee compensation and mixed income (labor supplied by self-employed and family workers), it automatically adjusts for labor compensation of nonworkers (employees). This makes the labor income share measure less susceptible to measurement errors as highlighted by many researchers (Gollin 2002; Guerriero 2012). Certain industrial sectors (textiles, wholesale and retail trade and private services) contain abnormally high outliers possibly due to these measurement issues. For this reason, I restrict the analysis to 20 main industrials sectors after dropping textiles, wholesale and retail trade and private services.

2.2. Labor Income share trends

[6] https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

[7] These are agriculture, mining, food, textiles, paper, chemicals, petroleum, nonmetallic minerals, primary metals, fabricated metals, machinery, electrical machinery, transport equipment, precision instruments, other manufacturing, construction, utilities (electricity, gas and water supply), wholesale and retail trade, finance and insurance, real estate, transport and communication, private services and government services).

Figure 3 shows labor income share trends for 20 industrial sectors in Japan from 1970 to 2012. The top five sectors in terms of average labor income share (from the highest to the lowest) are construction, non-government services, processed metals, other manufacturing and precision instruments, where the bottom five (from the lowest to the highest) are petroleum, utilities, real estate, food and chemicals. Between 1970 and 2012, the labor income share increased in 18 out of the 20 sectors, leaving transport and non-government services. The labor income share has increased by more than 25 percentage points in real estate and processed metals, while remaining between 10 to 20 percentage points for 50% of the sectors. Industrial sectors including agriculture, food and petroleum register a small increase in the labor income share. As shown in appendix 1, most of the industrial sectors show a steady growth in the capital-labor ratio until the global crisis in 2007-2008. Since the crisis, the capital-labor ratio can at best be described as volatile across all industrial sectors. Overall, for the period from 1970-2012, there is a predominant trend of a rising capital-labor ratio with an increasing labor income share.

[Figure 3 is about here]

3. A Variable elasticity of substitution (VES) framework

3.1. Evidence on the variability of σ_{KL}

The empirical relationship in Figure 1 can be modelled using a three-variable relationship between value added per unit of labor ($\frac{Y}{L}$), the wage rate (W) and the capital–labor ratio ($\frac{K}{L}$). This framework was suggested by Liu and Hildebrand (1965) more than a half-century ago, and has since served as the basis of estimating any production framework. Equation (1) replicates a generalized log-linear version of this relationship, including a constant term (α) and an error term (u). The CES production function (Arrow et al., 1961) assumes that $\beta = 0$, and the value of σ_{KL} depends on the goodness of fit of the empirical relationship between $log\frac{Y}{L}$ and $logW$. It further assumes (i) a relationship between value added per unit of labor and the wage rate independent of the changes in the stock of capital (i.e., $\beta = 0$) and (ii) the elasticity of substitution between factor inputs as a constant (but not unity) along the isoquant. The validity of these assumptions is subject to test especially in the presence of an upward trend in the availability of capital per worker (Acemoglu and Guerrieri 2008).

(1) $log\frac{Y}{L} = \alpha + \varepsilon logW + \beta log\frac{K}{L} + u$

Equation (1) also serves as the basis for a class of production functions with variable elasticity of factor substitutions (VES) that assumes $\beta \neq 0$. This makes σ_{KL} a function of the capital-labor ratio. Karagiannis, Palivos, and Papageorgiou (2005) show that factor income shares can also vary with capital per worker by considering σ_{KL} as a linear function of the capital-labor ratio following Revankar (1971). Taken together, these point to the advantages of a VES framework in addressing movements in the labor income share with varying levels of capital per worker. It remains an empirical question whether a VES is a more realistic model compared to CES to study changes in the factor income shares. The recent growth in the labor income share literature mostly relies on the CES model to derive the relationship between the elasticity of factor substitution and the labor income share.

The log-linear relationship between value added per unit of labor ($\frac{Y}{L}$), the wage rate (W) and the capital-labor ratio ($\frac{K}{L}$) including a constant term (α) and an error term (u) forms Equation 1: $log\frac{Y}{L} = \alpha + \varepsilon logW + \beta log\frac{K}{L} + u$. While the CES production function (Arrow et al., 1961) assumes $\beta = 0$, a class of production functions with variable elasticity of factor substitutions (VES) assumes $\beta \neq 0$. It makes σ_{KL} a function of the capital-labor ratio. In this section, I discuss the estimated results of Equation 1. Each row in Table 1 shows the outcomes for a specific industrial sector. Once regressed on output per worker, the estimated coefficient of the capital-labor ratio is statistically significant at 1% in all sectors except for utilities. Except for two sectors (other manufacturing and utilities), the F-test statistic rejects the null hypothesis that $\beta \neq 0$ in all sectors at a level of 1% significance. The outcomes of the Breusch-Godfrey LM test for autocorrelation suggest that there is no autocorrelation between labor productivity and its first lag at 5% level of significance for all industrial sectors. Overall, the estimated coefficients suggest that capital–labor ratio plays a consistent and statistically significant role in explaining the variation in output per worker over time as well as in the elasticity of substitution between capital and labor.

[Table 1 is about here]

3.2. VES production function

A production function shows the means to which the inputs produce the output. It shows both the technical efficiency and allocative efficiency of the inputs. Economists have typically assumed that the factor inputs are technically efficient and so production functions in economic analyses focus on the allocative efficiency of the factor inputs. Philip Wicksteed (1894) was the first economist to algebraically formulate the relationship between output and n inputs as $Y = f(x_1, x_2, .., x_n)$, though some sources suggest that Johann von Thünen first formulated it in the 1840s (Humphrey 1997). Since the formulation of the CES production function, several attempts[8] have been made to include a variable σ_{KL} in the production function. Mukerji (1963) generalizes the CES production function based on constant ratios of σ_{KL}. Revankar (1967) developed a generalized CES production function with variable returns to scale and elasticity of substitution[9]. Revankar's VES (or the generalized CES production function), does not contain the Leontief production function but shows the Harrod–Domar fixed coefficient model, and both the linear and Cobb-Douglass production

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[8] Bruno (1962); Brown and Cani (1963); Mukerji (1963); Nerlove (1963); Ringstad (1967); Revankar (1967); Lu and Fletcher (1968); Sato and Hoffman (1968); Revankar (1971) and Kadiyala (1972), among others. Please see Mishra (2007) for a comprehensive analysis of the evolution of production functions in economic analysis.

[9] Revankar (1967) considered the production function $Y = AK^{\alpha\left(1-\theta\frac{\sigma-1}{\sigma}\right)}\left[\left(\frac{1-2\sigma}{\sigma}\right)K^{\frac{\sigma-1}{\sigma}} + L\right]^{\alpha\left(\theta\frac{\sigma-1}{\sigma}\right)}$ and derived a linear elasticity parameter of the form $\sigma_{KL} = 1 + \frac{1-2\sigma}{\sigma-\theta(1-\sigma)}\frac{K}{L}$.

function as its special cases. In this model, σ_{KL}has a linear relationship with the capital per unit of labor. However, it does not allow the value of $\sigma_{KL}$to cross over from one side of the unity to the other in the relevant range of the capital–labor ratio. Bruno (1968) formulated constant marginal share (CMS) production function[10], where labor productivity increases with capital per unit of labor, but at a decreasing rate (θ). As a result, σ_{KL} is less likely to be less than unity in a CMS production function. A year later, Lu and Fletcher (1968) developed a VES production function as a generalized function of CES that permits $\sigma_{KL}$to vary with the factor shares. Lu and Fletcher's (1968) VES model successfully overcame the problem of the monotonic relationship between σ_{KL} and the capital–labor ratio, which was an issue that weakened the Revankar (1967) model. They derived a VES model assuming the minimum cost conditions of a perfectly competitive labor and product market. To derive the production function, Lu and Fletcher (1968) used a log-linear form of the relationship between value added per unit of labor, a constant term (β_0), the wage rate (W), the capital-labor ratio ($\frac{K}{L}$) and an error term (ε) as shown in Equation 1.

I work with an extended version of the Lu and Fletcher (1968) model. A production function in a general time series form can be written as $Y = A(t)F(K, L)$, where $A(t)$ measures the technical change in output, K and L are internally homogeneous and are continuously differentiable factors of production, representing capital and labor respectively. $F(K, L)$ is twice differentiable and linearly homogenous. The marginal rate of technical substitution (R) can be expressed as a function of the capital-labor ratio (k), where $f(k) > 0$ and $f'(k) > 0$.

(2) $\quad R = -\frac{dK}{dL} = f(k)$

From equation (2), the elasticity of substitution is calculated using

[10] Bruno (1968) developed a production function with $\frac{Y}{L} = A\left(\frac{K}{L}\right)^{\alpha} - \theta$ and $\sigma_{KL} = 1 - \frac{\theta\alpha}{(1-\alpha)}\frac{L}{Y}$.

(3) $\sigma_{KL} = \frac{dk/k}{dR/R} = \frac{f(k)}{kf\prime(k)}$

Equation 4 presents this model with V as the output produced by capital and labor in a general time-series form. In this model, the labor input is also multiplied by the capital per unit of labor. If $\beta = 0$, it takes the form of a standard CES production function. The VES production function in Equation 4 is directly related to Equation 1 since the parameters ε and β are estimated using Equation 1 (Lu, 1967). A is a constant technical parameter and μ measures the changes in the Hicks-neutral technological progress over time. Lu and Fletcher's (1968) model considered a time-invariant distribution parameter, which is unlikely to be the case if the substitution parameter varies over time (Klump and La Grandville, 2000; Temple, 2009). I extend their model to a time-variant distribution parameter, θ_t, which measures the relative weights of factors in the production of output, V.

(4) $V_t = Ae^{\mu t}\left[\theta_t K_t^{\frac{\varepsilon-1}{\varepsilon}} + (1-\theta_t)\left(\frac{K}{L}\right)_t^{\frac{-\beta}{\varepsilon}} L_t^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$

The production function in Equation 4 has positive marginal products of both input factors, a downward sloping marginal productivity curve over the relevant ranges of inputs, and shows constant returns to scale (homogeneous of degree 1). Equation 5 shows the production function in terms of capital per worker (k) denoted as $g(k)$.

(5) $g(k_t) = \left(\frac{V}{L}\right)_t = Ae^{\mu t}\left[\theta_t k_t^{\frac{\varepsilon-1}{\varepsilon}} + (1-\theta_t) k_t^{\frac{-\beta}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$

I assume that the time-variant share parameter is linked to the distribution of the capital-labor ratio by borrowing from the literature that models production technology as a function of explicit techniques (Kortum, 1997; Jones, 2005; Growiec, 2008), [11]. To find the most suitable distribution for the capital-labor ratio, I compare the Kolmogorov-Smirnov statistics for six models (Frechet with 2 and 3 parameters, General extreme value, Gumbel max and Weibull 2 and 3 parameters) from the class of generalized extreme value distribution with that of the normal distribution. As shown in Appendix 2, in 11 (agriculture, mining, paper, petroleum, chemical, ceramics, machinery, transport equipment, precision instruments, finance and utilities) out of 20 industrial sectors, the Weibull (3 parameter) model shows the best fit, followed by general extreme value distribution showing best fit in 5 industrial sectors (food, basic metals, processed metals, electrical and construction). The Weibull distribution (3 parameters) appears to be the best choice, and I assume that the distribution parameters are drawn from the cumulative density functions of the Weibull distribution of the capital-labor ratio, $F(k|\alpha,\gamma,\pi)$, with parameters α (scale or characteristics parameters), γ (threshold or shift parameter), and π (shape parameter).

(6) $$\theta_t \simeq F(k_t|\alpha,\gamma,\pi) = e^{-\left(\frac{\mathrm{k}_t-\gamma}{\pi}\right)^{\alpha}}$$

Entering θ_t into Equation 6? delivers the VES production function with the distribution parameters based on explicit Weibull draws of k (Equation 7). In this sense, θ_t is defined as a monotonic transformation of k_t, which implies that θ_t increases within the interval between 0 and 1 as k_t increases. In economic terms, an increasing capital per worker leads to an increase in the relative use of capital in the production of output.

(7) $$g(k_t) = Ae^{\mu t}\left[e^{-\left(\frac{\mathrm{k}_t-\gamma}{\pi}\right)^{\alpha}} k_t^{\frac{\varepsilon-1}{\varepsilon}} + \left\{1 - e^{-\left(\frac{\mathrm{k}_t-\gamma}{\pi}\right)^{\alpha}}\right\} k_t^{\frac{-\beta}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

[11] Jones (2005) shows that if the unit factor productivities are drawn from independent Pareto distributions, then with infinite number of draws the production function resembles Cobb-Douglas under certain conditions. In another study, Growiec (2008) links Weibull distribution of unit factor productivity to CES production function.

Differentiating $g(k_t)$ with respect to k_t and combining it with the expression for σ_{KL} in Equation 3, a time-variant elasticity of substitution between capital and labor (σ_{VES}) can be derived as[12]

$$(8)\quad \sigma_{VES} = \frac{g(k_t)}{kg\prime(k_t)} = \frac{\left[\frac{e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}}{1-e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}}\right] k_t^{\frac{\varepsilon+\beta-1}{\varepsilon}} + 1}{\left[\frac{e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}}{1-e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}}\right] k_t^{\frac{\varepsilon+\beta-1}{\varepsilon}} - \frac{\beta}{\varepsilon-1}}$$

In the VES model, σ_{VES} becomes a nonlinear function in k_t and the time-variant distribution parameter, $e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}$. Other studies that examine the productivity of factor inputs also use the Weibull distribution. Growiec (2008), shows that if unit factor productivity follows Weibull draws, then under certain conditions it is possible to derive a CES production function where the substitution parameter is related to the shape parameter of the Weibull distribution. In a recent study, Growiec (2011) shows that if factor-augmenting technologies consist of a large number of complementary components, then the Weibull distribution should approximate the productivity distribution better than any other theoretical model.

4. Estimation of parameters in CES and VES models

4.1. CES model parameters

Consider a CES production function of the following form (Equation 9), where the elasticity of factor substitution (σ_{CES}) is a constant and independent of k.

$$(9)\ Y = Be^{\gamma t}\left[\delta K^{\frac{\sigma_{CES}-1}{\sigma_{CES}}} + (1-\delta)L^{\frac{\sigma_{CES}-1}{\sigma_{CES}}}\right]^{\frac{\sigma_{CES}}{\sigma_{CES}-1}}$$

[12] The detailed algebraic steps are given in appendix 3.

B is a technical parameter, γ measures the changes in the Hicks-neutral technological progress, and δ shows the relative weights of factor use in the production of output Y. As Equation 10 shows, changes in the relative contribution of factor inputs can impact the relative factor shares. From a constrained cost-minimization based on the assumption that cost minimized factor proportions are selected in response to changes in the current factor prices, the marginal rate of technical substitution become equal to factor prices. Combining this result and Equation (2), the relative factor shares become

$$(10) \qquad \frac{R}{k} = \frac{wL}{rK} = \left(\frac{1-\delta}{\delta}\right) k^{\frac{1-\sigma_{CES}}{\sigma_{CES}}}.$$

If $\sigma_{CES} < 1$ (i.e., when capital and labor are complements), an increase in the capital stock relative to labor force increases the labor share of income. Similarly, if $\sigma_{CES} < 1$ (when capital and labor are substitutes), an increase in the capital stock relative to labor force decreases the labor share of income. If Equation 10 follows a log-linear relationship, then it can be expressed as

$$(11) \qquad \ln(R) = \ln\left(\frac{1-\delta}{\delta}\right) + \frac{1}{\sigma_{CES}} ln(k).$$

I follow Lovell (1973) to estimate the full set of parameters of the CES production function in two steps. In stage 1, a partial adjustment model of constrained cost-minimization is estimated based on the assumption that cost minimized factor proportions are selected in response to changes in the current factor prices. Simple algebraic calculations convert Equation (11) into the following expression in general time series form

$$(12) \qquad {k_t}^* = \left[\left(\frac{\delta}{1-\delta}\right)\left(\frac{w}{r}\right)_t\right]^{\sigma_{CES}}.$$

If there is no initial adjustment, the actual adjustment process between k_t and k_{t-1} can be written as

$$(13) \quad \left(\frac{k_t}{k_{t-1}}\right) = \left(\frac{k_t{}^*}{k_{t-1}}\right)^{\varphi},$$

where φ is an adjustment of actual to desired cost-minimized factor proportions. By combining Equations 12 and 13, the following expression for the adjustment process between k_t and k_{t-1} is obtained.

$$(14) \quad k_t = \left[\left(\frac{\delta}{1-\delta}\right)\left(\frac{w}{r}\right)_t\right]^{\varphi\,\sigma_{CES}} (k_{t-1})^{1-\varphi}.$$

Taking log, it becomes

$$(15) \quad \ln k_t = \varphi\,\sigma_{CES} \ln\frac{\delta}{1-\delta} + \varphi\,\sigma_{CES} \ln\left(\frac{w}{r}\right)_t + (1-\varphi)\ln k_{t-1}$$

Assuming the random disturbance terms enter multiplicatively, I use a log-linear stochastic model specification to estimate in Equation 15, as follows

$$(16) \quad \ln k_t = \alpha_0 + \alpha_1 \ln\left(\frac{w}{r}\right)_t + \alpha_2 \ln k_{t-1} + u_t$$

where

$$\alpha_0 = \varphi\,\sigma_{CES} \ln\frac{\delta}{1-\delta}$$

$$\alpha_1 = \varphi\,\sigma_{CES}$$

$$\alpha_2 = 1-\varphi.$$

In the presence of a lagged value of the dependent variable as a regressor, the OLS assumptions suggest the ordinary least square estimates are consistent and asymptotically efficient. The adjustment parameter φ, δ and σ_{CES} can be estimated from equation 16.

In stage 2, the technological parameters, B and γ, are estimated using the estimates of δ and σ_{CES} from step 1. The production function in equation 4 can be written in the log version of output per worker terms and using the estimated values of $\hat{\delta}$ and $\hat{\sigma}_{CES}$ as

$$(17) \qquad \ln\frac{Y}{L} = \ln(Be^{\gamma t}) + \frac{\hat{\sigma}_{CES}}{\hat{\sigma}_{CES}-1}\ln\left[\hat{\delta}K^{\frac{\hat{\sigma}_{CES}-1}{\hat{\sigma}_{CES}}} + (1-\hat{\delta})\right].$$

Rearranging the terms in Equation 17, denoting a shift in labor productivity $(\frac{Y}{L})$ due to technological progress net of capital deepening as Λ_{CES}, and adding a stochastic error term (ν_t) gives Equation 18, which is used to estimate the parameters, B and γ.

$$(18) \qquad (\Lambda_{CES})_t = \beta_0 + \beta_1 t + \nu_t$$

where

$$(\Lambda_{CES})_t = \ln\left(\frac{Y}{L}\right)_t - \frac{\hat{\sigma}_{CES}}{\hat{\sigma}_{CES}-1}\ln\left[\hat{\delta}K^{\frac{\hat{\sigma}_{CES}-1}{\hat{\sigma}_{CES}}} + (1-\hat{\delta})\right],$$

$\beta_0 = \log B$, and

$\beta_1 = \gamma$.

Table 2 presents the results from the first stage of estimation for the CES model. The estimated coefficients of both log factor price ratio and log factor ratio are statistically significant across the board. I calculate $\hat{\sigma}_{CES}$ using the estimated coefficients, which is shown in the last column of Table 2. The estimated values for the elasticity of substitution parameter varies between -.71 to 2.28, with an overall average of 1.03. In 13 out of 20 industries, capital and labor are measured as substitutable components in the production process. The outcomes of the Breusch-Godfrey LM test for autocorrelation suggest no autocorrelation between labor productivity and its first lag at a 5% level of significance for all industrial sectors. The average of the constant share parameter I the CES model is .41. Based on the first stage estimates, I derive the technology parameters in the second stage and then calibrate the output and factor shares. These outcomes are discussed in section 5. As shown in appendix 4, the estimated parameter of the adjustment to capital-labor ratio φ consistently lie between 0 and 1 across the industrial sectors.

[Table 2 is about here]

4.2. VES and VES-W model parameters

VES model parameters are estimated in three stages. In the first stage, I estimate time-invariant parameters, $\hat{\varepsilon}$ and $\hat{\beta}$ directly from equation 1. The second stage estimates the distribution parameters following two ways. I first estimate the time-invariant θ following the estimation procedure proposed by Lu and Fletcher (1968). This requires estimation of the following equation using the estimated values of the parameters $\hat{\varepsilon}$ and $\hat{\beta}$ from the first stage:

(19) $$z1_t = \Psi_0 + \Psi_1 \, z2_t + u_t$$

Where, $z1_t = \left(\frac{Y}{K}\right)^{\frac{\hat{\varepsilon}-1}{\hat{\varepsilon}}}$ and $z2_t = k^{\frac{\hat{\varepsilon}+\hat{\beta}-1}{\hat{\varepsilon}}}$. Then by using the estimated values of Ψ_0 and Ψ_1, one can estimate the distribution parameter as

(20) $$\hat{\theta} = \frac{1}{1+\frac{\Psi_0}{\Psi_1}}$$

Plugging the estimated values of $\hat{\varepsilon}$, $\hat{\beta}$ and $\hat{\theta}$ into Equation 8, the series of the time-variant elasticity of substitution can be calculated from Equation 21.

$$(21) \qquad \widehat{\sigma}_{VES}\big|_{\theta=\widehat{\theta}} = \frac{\left[\frac{\widehat{\theta}}{1-\widehat{\theta}}\right] k_t^{\frac{\hat{\varepsilon}+\widehat{\beta}-1}{\hat{\varepsilon}}} + 1}{\left[\frac{\widehat{\theta}}{1-\widehat{\theta}}\right] k_t^{\frac{\hat{\varepsilon}+\widehat{\beta}-1}{\hat{\varepsilon}}} - \frac{\widehat{\beta}}{\hat{\varepsilon}-1}}$$

The second approach estimates a time-variant distribution parameter using $\theta_t = e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}$, which produces the elasticity parameter using Equation 22.

$$(22) \qquad \widehat{\sigma}_{VES}\Big|_{\theta_t=e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}} = \frac{\left[\frac{e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}}{1-e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}}\right] k_t^{\frac{\hat{\varepsilon}+\widehat{\beta}-1}{\hat{\varepsilon}}} + 1}{\left[\frac{e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}}{1-e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}}\right] k_t^{\frac{\hat{\varepsilon}+\widehat{\beta}-1}{\hat{\varepsilon}}} - \frac{\widehat{\beta}}{\hat{\varepsilon}-1}}.$$

In stage 3, the technological parameters, A and μ, are estimated using the estimates of θ, ε and β and σ_{CES} from stages 1 and 2. This follows the same estimation strategy as described in stage 2 for CES model.

Table 3 reports the results from the first stage of estimation in the VES model (Equation 1). The estimated coefficients $\hat{\beta}$ and $\hat{\varepsilon}$ show statistically significant results for all the industrial sectors. The second last column shows the constant share parameter estimated in the VES model (following Equation 21) and the last column shows the average of the time-variant share parameter estimated in the VES-W model using Equation 22. The averages of the constant and time-variant share parameters across all the industrial sectors are 0.55 and 0.62 respectively. Thus, the average share parameter approximated by the cumulative distribution function of the Weibull distribution of the capital-labor ratio is higher than the constant share parameters in both the CES and the VES model. The correlation of the share parameter between the VES and the VES-W model is 0.14, whereas between the CES and the VES-W model it is 0.12. The outcomes of the Breusch-Godfrey LM test for autocorrelation suggest no autocorrelation between labor productivity and its first lag at 5% level of significance for all industrial sectors.

[Table 3 is about here]

5. Comparison of the CES and VES model outcomes

[Table 4 is about here]

Table 4 compares the estimates of the elasticity of substitution from three models: CES, VES and VES-W. The CES estimates suggest that capital and labor are substitutes in 13 out of 20 industrial sectors, whereas in both VES models this number drops to only seven (comparing with the average σ_{KL} for each sector). As shown in Figure 2, capital and labor are predominantly complementary in the VES-W model (with an average of 0.91) relative to the CES model (with an average of 1.03). At the same time, the estimates of the elasticity of substitution in VES-W model are more volatile compared to the VES model estimates (Appendix 5). Notably, the elasticity of substitution in the VES-W model shows an increasing trend in 10 out of 20 sectors. However, there is no consistent relationship observed between the elasticity of substitution and labor income share over time. In fact, in most of the cases, no relationship exists between these two factors. For example, the labor income share declined the most in processed metal and real estate industries. The elasticity of substitution in processed metal decreases over time whereas in real estate it shows a slight upward trend. Overall, these findings point to the elasticity of substitution having a limited role in explaining the movements in the labor income share in the VES-W framework.

[Figure 4 is about here]

Figure 4 compares the calibrated output levels from the CES, VES and VES-W models with the actual level of output for 20 industrial sectors. The log of output levels are plotted against time variable. The sectoral averages of log output are measured as 15.75, 15.84, and 15.75 in from the CES, the VES and the VES-W models. The Kormogorov-Smirnov goodness of fit test statistics suggest reasonably good fit between the actual values of the output and the calibrated results from the CES and the VES-W model. Finally, Table 5 compares the estimated labor income shares from the three models with the actual labor income share. The Kormogorov-Smirnov goodness of fit test statistics suggest reasonably good fit between the actual values of the labor income share and the calibrated results from the CES and the VES-W model. For both models, the test statistics reject the null hypothesis that the distributions are identical in only 7 out of 20 industrial sectors. On the other hand , the VES model performs poorly relative to the other two models. The calibrated results of VES-W model provide a much better fit to the actual data compared to VES model, which supports the approximation of the share parameters with extreme value distribution of the capital-labor ratio.

[Table 5 is about here]

5. Concluding remarks

This paper makes two contributions to the literature. First, it develops a variable elasticity of substitution (VES-W) framework as an alternative to the CES model to analyze movements in the labor income share. The VES-W model embodies a variable elasticity of substitution (σ_{KL}) and a share parameter, both as non-linear functions of the capital-labor ratio. The share parameter in the VES model is defined as a cumulative density function of 3-parameter Weibull distribution of the capital-labor ratio. The share parameter as a monotonic transformation of the capital-labor ratio implies that a higher capital-labor ratio leads to a higher share of the capital-output ratio. Empirical findings based on annual data from 20

Japanese industrial sectors between 1970 and 2012 support the choice of a variable elasticity of substitution. The elasticity of substitution in the VES-W model fluctuates around unity with an overall average of 0.91. The estimated structural parameters calibrate the actual output level and the movements in factor income are reasonably reflected in both the CES and VES-W models. The calibration outcomes do not suggest that the VES-W model must be a preferred choice over the CES model.

Notably, the VES-W model explains the movements (predominantly, an increase) in the labor income share with an average $\sigma_{KL} < 1$, whereas the same result is achieved in the CES model with an average $\sigma_{KL} > 1$. The outcomes of the VES-W model support the accumulation principle that complementary factor inputs contribute to an increase in the labor income share with capital accumulation, suggesting that the VES-W is a sensible model choice. Based on the results, there is a solid foundation to also argue that a variable elasticity framework remains a potential area to study the role of σ_{KL} in labor income share dynamics. It remains a subject of further study whether an aggregate variable σ_{KL} produces the same result at the macro level since the findings in this paper suggest an increase in the labor income share with complementary factor inputs at the sectoral level. The VES framework developed in this paper can be extended to study the behavior of an aggregate elasticity of substitution, and I leave this goal for future studies to achieve.

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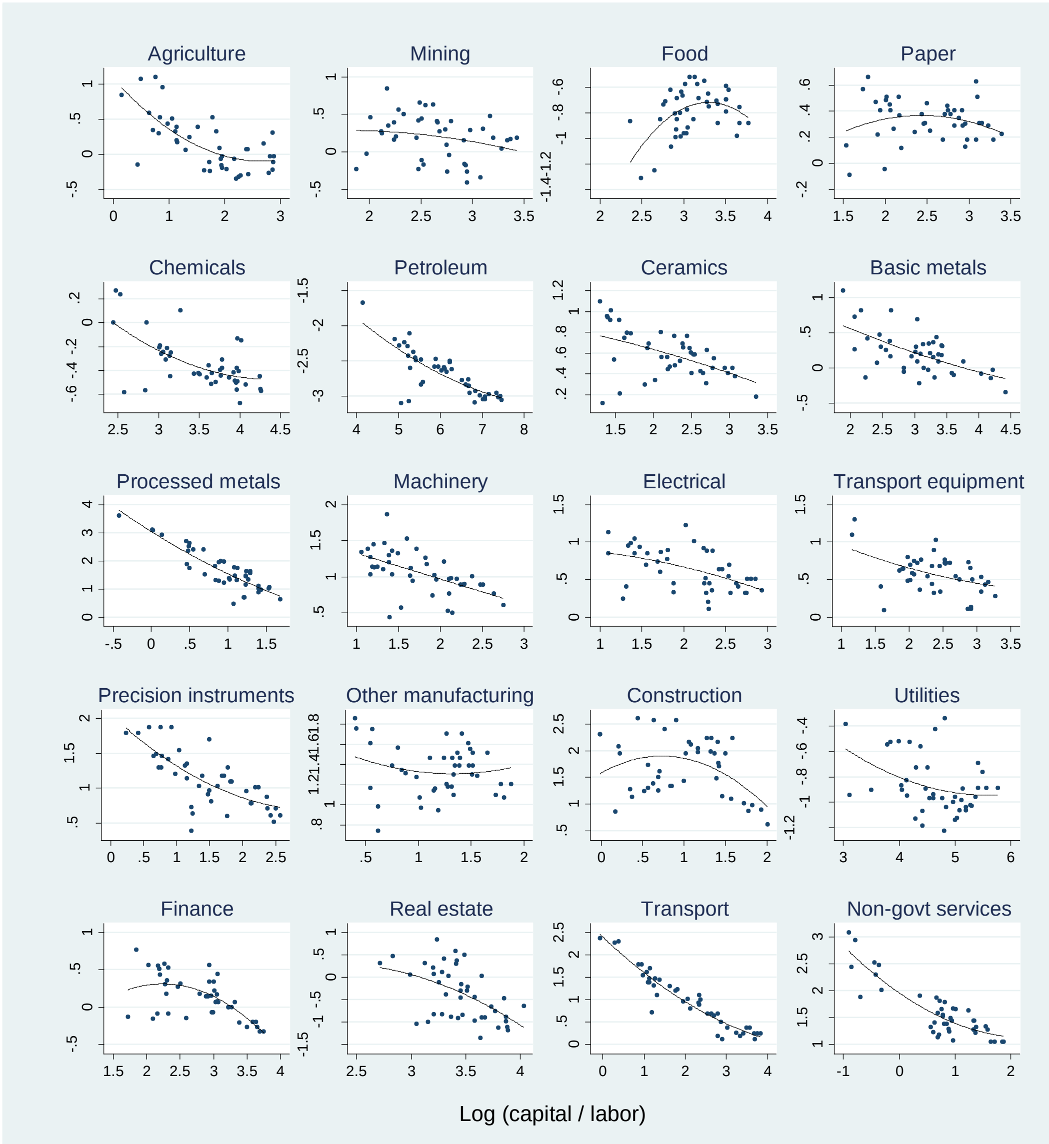
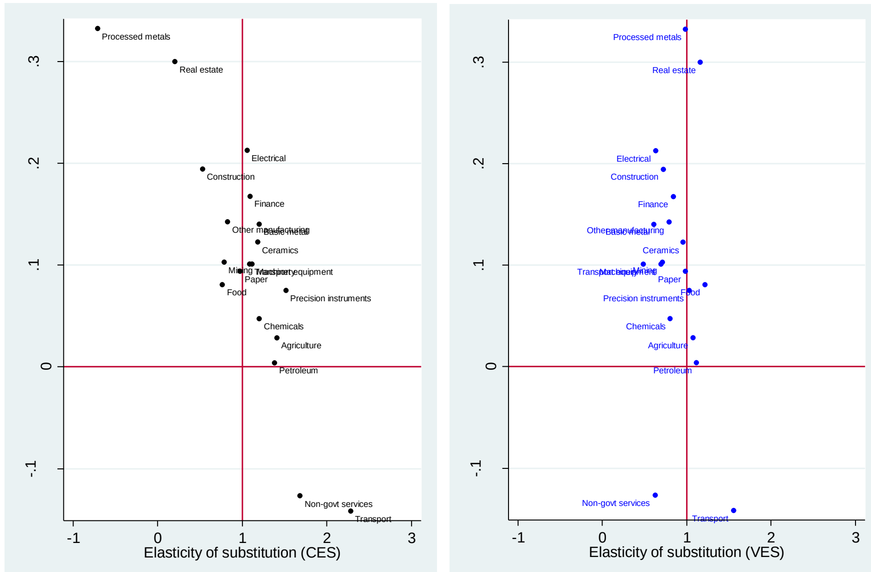
Figure 1. Labor income-capital income ratio and capital-labor ratio across Japanese industries, 1970 - 2012



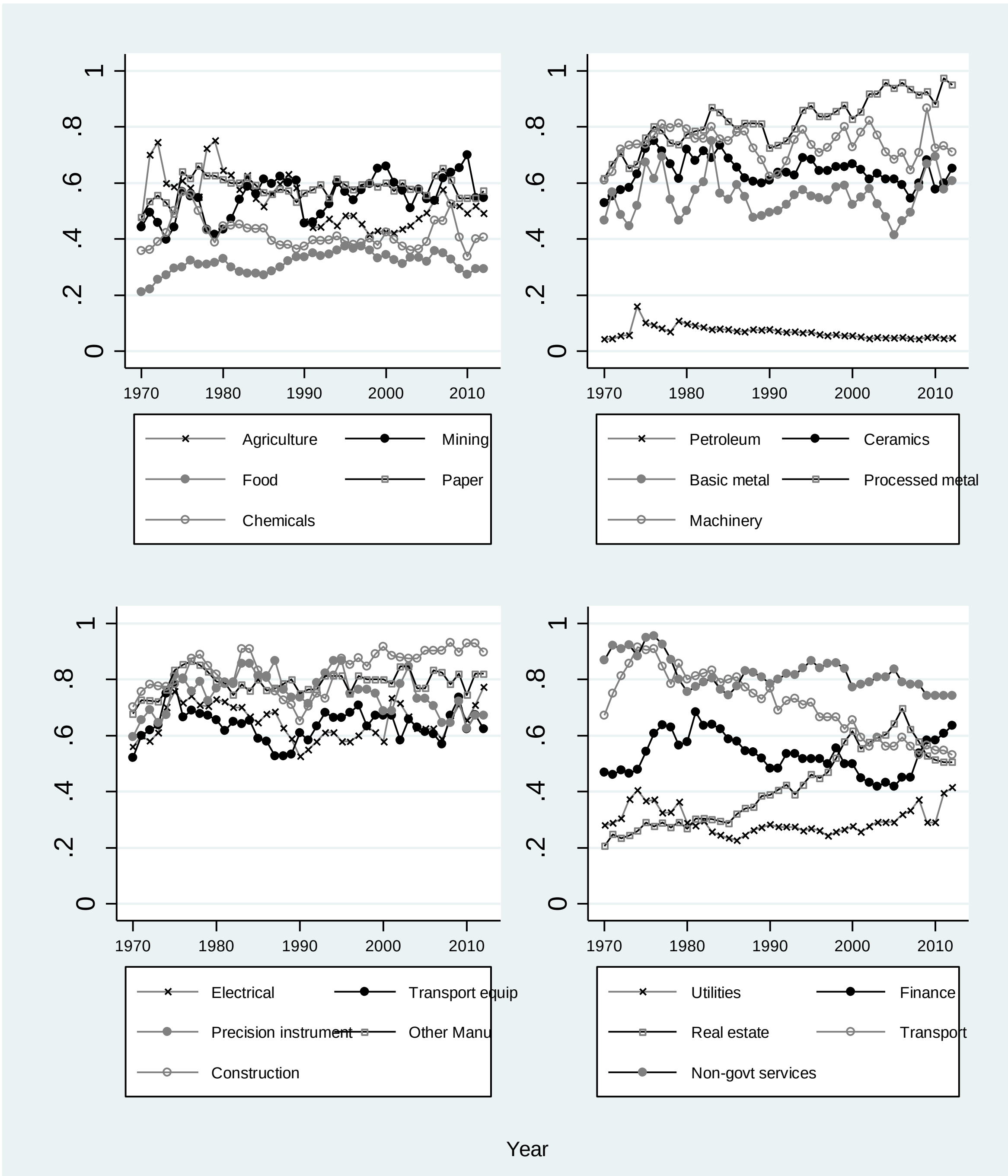
Note: Each dot represents the relationship between the log of factor income ratio and the log of capital labor ratio in a particular year in the period from 1970 to 2012.
Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Figure 2. Changes in the labor income share and estimated elasticity of substitution from CES and VES-W models, 1970-2012



Note: The back and gray circles represent estimated values of the elasticity parameter in the CES and the VES-W model, respectively. For the VES-W model, it shows average of the elasticity of substitution over the period from 1970-2012.
Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Figure 3. Labor income share trends in key industrial sectors, 1970-2012



Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Table 1. Regression outcomes on output per worker, 1970 – 2012

Industry code	Industry name	$\hat{\beta}$ (coefficient of $\log\frac{K}{L}$)	R2	Breusch-Godfrey Statistic	F-test ($\hat{\beta} = 0$)
1	Agriculture	0.210***	0.949	13.395	29.92***
2	Mining	0.209***	0.951	21.933	32.28***
3	Food	0.166***	0.970	23.294	11.35***
5	Paper	0.123***	0.989	5.315	18.9***
6	Chemicals	0.277***	0.982	19.097	59.13***
7	Petroleum	0.487***	0.972	6.472	169.54***
8	Ceramics	0.177***	0.995	6.703	119.39***
9	Basic metal	0.226***	0.982	13.285	100.21***
10	Processed metals	0.127***	0.994	5.471	89.76***
11	Machinery	0.151***	0.989	16.594	56.21***
12	Electrical	0.210***	0.985	25.351	53.72***
13	Transport equipment	0.182***	0.984	21.696	39.53***
14	Precision instruments	0.152***	0.995	6.084	216.07***
15	Other manufacturing	0.128***	0.996	10.195	44.58
16	Construction	0.157***	0.991	21.29	80.67***
17	Utilities	0.086	0.937	25.782	1.89
19	Finance	0.343***	0.991	7.642	188.45***
20	Real estate	0.201**	0.655	32.306	5.32**
21	Transport	0.165***	0.990	6.123	262.53***
22	Non-govt services	0.124***	0.997	20.945	105.21***

Note: *** significance at 1%, ** significance at 5% and * significance at 10%.

This table shows the estimated results of equation (1): $log\frac{Y}{L} = \beta_0 + \varepsilon logW + \beta log\frac{K}{L} + \varepsilon$.

The outcomes of the Breusch-Godfrey LM test for autocorrelation suggest no autocorrelation at 5% level of significance or over.

Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Table 2. Estimated results from the CES model

Industry code	Industry name	$\hat{\alpha}_1$ (coefficient of $\log\frac{w}{r}$)	$\hat{\alpha}_2$ (coefficient of $\log\frac{K}{L}$)	Constant term	R^2	Durbin-Watson statistic	$\hat{\sigma}_{CES}$
1	Agriculture	0.545***	0.613***	-0.325**	0.913	16.95	1.41
2	Mining	0.513***	0.348***	0.279	0.609	13.78	0.79
3	Food	0.523***	0.317***	0.916***	0.863	25.19	0.77
5	Paper	0.694***	0.288***	-0.178	0.928	8.92	0.97
6	Chemicals	0.856***	0.288**	-0.210	0.863	18.69	1.20
7	Petroleum	0.943***	0.318***	0.936***	0.931	13.14	1.38
8	Ceramics	0.821***	0.306***	-0.763***	0.878	14.93	1.18
9	Basic metal	0.834***	0.304***	-0.604*	0.798	8.514	1.20
10	Processed metals	-0.294*	0.587***	1.145**	0.498	12.05	-0.71
11	Machinery	0.643***	0.424***	-0.800**	0.717	9.83	1.12
12	Electrical	0.477***	0.549***	-0.350	0.825	21.84	1.06
13	Transport equipment	0.720***	0.339***	-0.556**	0.804	12.09	1.09
14	Precision instruments	0.640***	0.578***	-1.040***	0.824	14.59	1.52
15	Other manufacturing	0.441***	0.467***	-0.477**	0.772	10.49	0.83
16	Construction	0.194*	0.635***	-0.140	0.622	22.12	0.53
17	Utilities	0.846***	0.184**	0.601**	0.875	26.51	1.04
19	Finance	0.363***	0.668***	-0.120	0.876	29.92	1.09
20	Real estate	0.098	0.520***	1.349**	0.241	9.30	0.20
21	Transport	1.118***	0.510***	-2.412***	0.949	13.99	2.28
22	Non-govt services	0.298**	0.823***	-0.519*	0.908	24.01	1.68

Note: *** significance at 1%, ** significance at 5% and * significance at 10%.

This table shows the estimated results of equation (16): $\ln k_t = \alpha_0 + \alpha_1 \ln\left(\frac{w}{r}\right)_t + \alpha_2 \ln k_{t-1} + u_t$.

The outcomes of the Breusch-Godfrey LM test for autocorrelation suggest no autocorrelation at 5% level of significance or over.

Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Table 3. Estimated results from the VES model

Industry code	Industry name	$\hat{\varepsilon}$	$\hat{\beta}$	R2	$\theta = \hat{\theta}$	$\bar{\theta}_t = Avg\left[e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}\right]$
1	Agriculture	0.809***	0.164***	0.950	0.676	0.526
2	Mining	0.900***	0.262***	0.973	0.749	0.623
3	Food	0.721***	0.149***	0.970	0.926	0.740
5	Paper	0.862***	0.143***	0.989	0.710	0.608
6	Chemicals	0.871***	0.322***	0.984	0.459	0.697
7	Petroleum	0.528***	0.394***	0.974	0.852	0.539
8	Ceramics	0.842***	0.173***	0.995	0.585	0.507
9	Basic metal	0.922***	0.263***	0.988	0.270	0.552
10	Processed metals	0.891***	0.119***	0.994	0.450	0.850
11	Machinery	0.956***	0.185***	0.991	0.328	0.506
12	Electrical	0.920***	0.254***	0.988	0.434	0.590
13	Transport equipment	0.927***	0.234***	0.987	0.327	0.694
14	Precision instruments	0.853***	0.137***	0.995	0.487	0.547
15	Other manufacturing	0.923***	0.172***	0.997	0.557	0.735
16	Construction	0.927***	0.152***	0.994	0.581	0.577
17	Utilities	1.190***	0.119**	0.952	0.835	0.661
19	Finance	0.780***	0.367***	0.992	0.163	0.600
20	Real estate	0.829***	0.082	0.835	0.997	0.805
21	Transport	0.835***	0.077***	0.996	0.104	0.429
22	Non-govt services	0.949***	0.159***	0.998	0.482	0.634

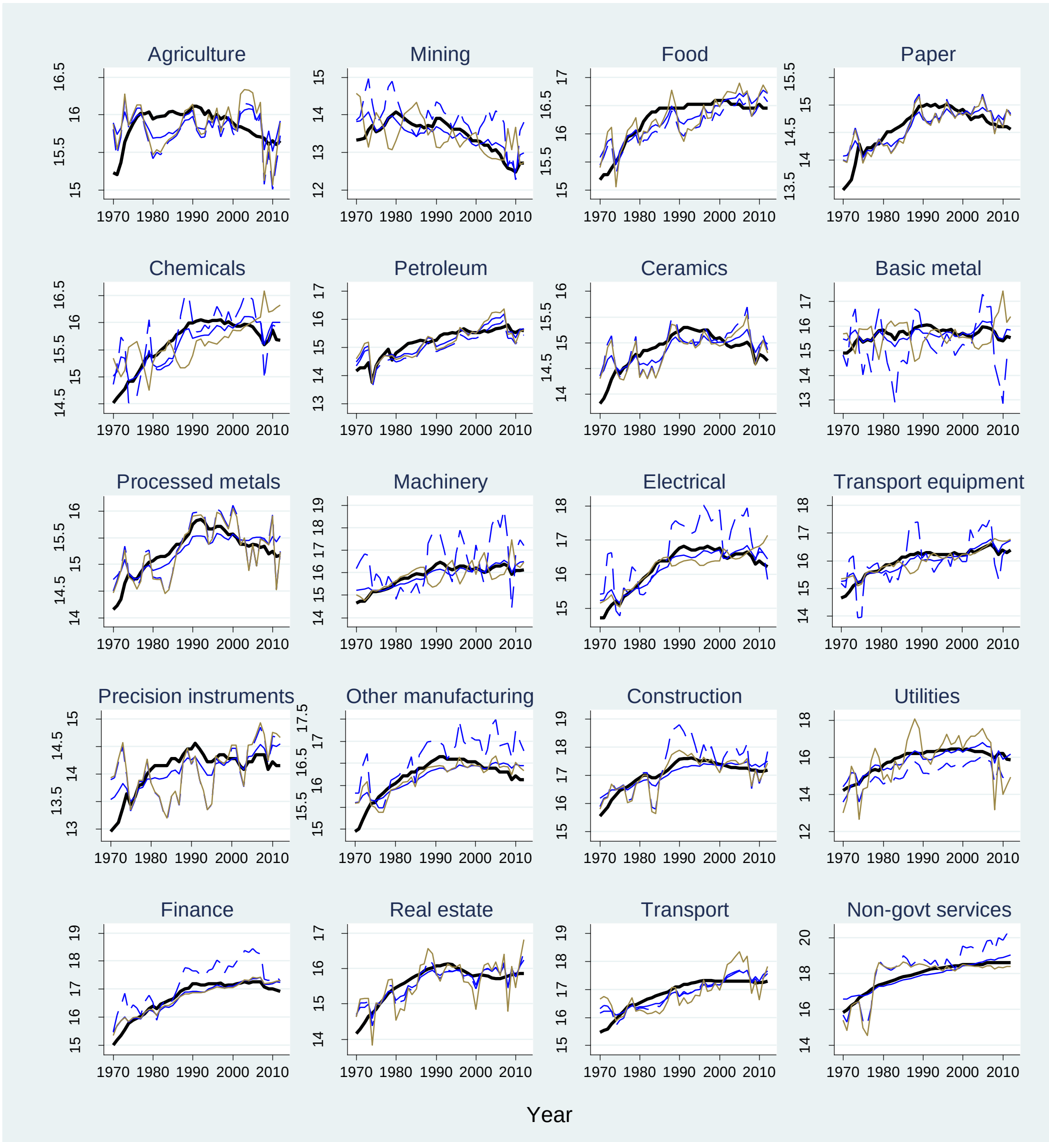
Note: *** significance at 1%, ** significance at 5% and * significance at 10%.
This table shows the estimated results of equation (1): $log\frac{Y}{L} = \alpha + \varepsilon logW + \beta log\frac{K}{L} + u$
Source: Author’s calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database
https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Table 4. Comparison of the estimated elasticity of substitution parameters

Industry code	Industry name	$\widehat{\sigma}_{CES}$	$\widehat{\sigma}_{VES}\big\vert_{\theta=\widehat{\theta}}$	$\widehat{\sigma}_{VES}\Big\vert_{\theta_t=e^{-\left(\frac{k_t-\gamma}{\pi}\right)^{\alpha}}}$
1	Agriculture	1.408	1.050	1.075
2	Mining	0.787	0.782	0.714
3	Food	0.766	1.061	1.215
5	Paper	0.975	0.990	0.986
6	Chemicals	1.202	0.655	0.802
7	Petroleum	1.383	1.052	1.116
8	Ceramics	1.183	0.963	0.956
9	Basic metal	1.198	0.416	0.610
10	Processed metals	-0.712	0.952	0.987
11	Machinery	1.116	0.337	0.485
12	Electrical	1.058	0.496	0.633
13	Transport equipment	1.089	0.441	0.697
14	Precision instruments	1.517	1.036	1.032
15	Other manufacturing	0.827	0.663	0.792
16	Construction	0.532	0.699	0.723
17	Utilities	1.037	1.101	12.838
19	Finance	1.093	0.666	0.840
20	Real estate	0.204	1.000	1.162
21	Transport	2.282	1.953	1.556
22	Non-govt services	1.684	0.488	0.627

Note: The figures for the VES models are averages over the period from 1970-2012.
Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Figure 4 Actual versus calibrated value-added by industrial sectors



Note: The solid black lines measure the actual value-added figures. The Solid-blue lines, dashed-blue lines and brown lines show calibrated value-added figures using CES model, VES model and VES-W model. The Kormogorov-Smirnov test for goodness of fit suggests that the distribution of actual output level is similar yto calibrated output figures from the CES and the VES-W models.

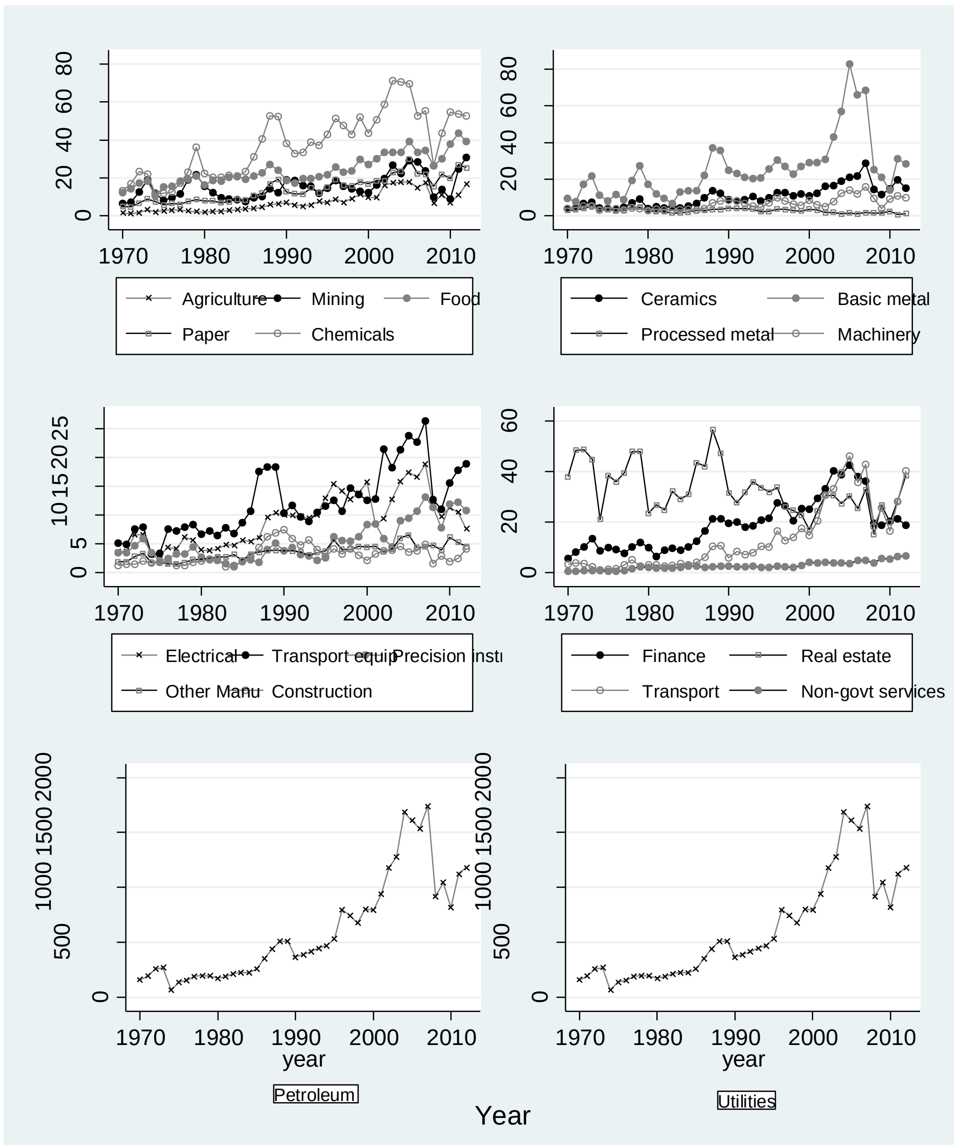
Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Table 5. Comparison of actual and the calibrated labor income share trends

Industry code	Industry name	Average labor income share, 1970-2012			
		Actual	CES	VES	VES-Weibull
1	Agriculture	0.536	0.551	0.61	0.57
2	Mining	0.548	0.566	0.398*	0.598
3	Food	0.314	0.325	0.376*	0.333
5	Paper	0.581	0.599*	0.591*	0.604*
6	Chemicals	0.416	0.43	0.415*	0.446
7	Petroleum	0.067	0.071	0.085	0.074
8	Ceramics	0.641	0.667*	0.653	0.689*
9	Basic metal	0.552	0.573*	1.492*	0.641*
10	Processed metals	0.821	0.843	0.853*	0.879
11	Machinery	0.738	0.76	0.674*	0.8
12	Electrical	0.65	0.666	0.485*	0.673
13	Transport equipment	0.639	0.657*	0.757*	0.66
14	Precision instruments	0.749	0.782*	0.917*	0.897*
15	Other manufacturing	0.79	0.815*	0.555*	0.823*
16	Construction	0.832	0.851*	0.771*	0.926
17	Utilities	0.297	0.304	0.666*	0.596*
19	Finance	0.533	0.55*	0.353*	0.548
20	Real estate	0.414	0.422	0.43	0.469
21	Transport	0.712	0.751	0.762*	0.836*
22	Non-govt services	0.822	0.845*	0.762*	1.188

Note: The figures for VES models are averages over the period from 1970-2012. The Kormogorov-Smirnov test for goodness of fit is performed for each calibrated distribution with the actual one. * indicates the case when two distributions are different at 1% level of statistical significance.
Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Appendix 1. Capital per worker across industrial sectors, 1970-2012



Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Appendix 2. Goodness of fit of seven distributions to capital-labor ratio over time

			Frechet (2 pars)	Frechet (3 Pars)	General Extreme Value	Gumbel Max	Normal	Weibull (2 pars)	Weibull (3 pars)
1	Agriculture	Statistic	0.147	0.110	0.123	0.129	0.159	0.114	0.089
		Ranking	6	2	4	5	7	3	1
2	Mining	Statistic	0.107	0.086	0.073	0.077	0.123	0.100	0.066
		Ranking	6	4	2	3	7	5	1
3	Food	Statistic	0.107	0.097	0.091	0.101	0.152	0.137	0.105
		Ranking	5	2	1	3	7	6	4
5	Paper	Statistic	0.171	0.130	0.126	0.142	0.146	0.119	0.114
		Ranking	7	4	3	5	6	2	1
6	Chemicals	Statistic	0.183	0.125	0.134	0.153	0.146	0.109	0.120
		Ranking	7	3	4	6	5	1	2
7	Petroleum	Statistic	0.129	0.117	0.127	0.134	0.174	0.116	0.103
		Ranking	5	3	4	6	7	2	1
8	Ceramics	Statistic	0.148	0.104	0.095	0.099	0.117	0.089	0.109
		Ranking	7	4	2	3	6	1	5
9	Basic metal	Statistic	0.181	0.110	0.103	0.142	0.205	0.106	0.116
		Ranking	6	3	1	5	7	2	4
10	Processed metals	Statistic	0.195	0.094	0.079	0.122	0.087	0.082	0.082
		Ranking	7	5	1	6	4	2	3
11	Machinery	Statistic	0.118	0.116	0.120	0.127	0.154	0.131	0.095
		Ranking	3	2	4	5	7	6	1
12	Electrical	Statistic	0.210	0.136	0.112	0.135	0.131	0.114	0.158
		Ranking	7	5	1	4	3	2	6
13	Transport equipment	Statistic	0.128	0.100	0.093	0.100	0.128	0.105	0.085
		Ranking	7	4	2	3	6	5	1
14	Precision instruments	Statistic	0.092	0.085	0.092	0.116	0.158	0.109	0.071
		Ranking	4	2	3	6	7	5	1
15	Other manufacturi ng	Statistic	0.202	0.126	0.084	0.132	0.076	0.092	0.114
		Ranking	7	5	2	6	1	3	4
16	Construction	Statistic	0.133	0.105	0.090	0.103	0.123	0.095	0.090
		Ranking	7	5	1	4	6	3	2
17	Utilities	Statistic	0.129	0.078	0.065	0.072	0.121	0.067	0.064
		Ranking	7	5	2	4	6	3	1
19	Finance	Statistic	0.208	0.146	0.139	0.150	0.145	0.129	0.134
		Ranking	7	5	3	6	4	1	2

20	Real estate	Statistic	0.130	0.059	0.064	0.062	0.112	0.097	0.082
		Ranking	7	1	3	2	6	5	4
21	Transport	Statistic	0.135	0.110	0.146	0.145	0.190	0.130	0.114
		Ranking	4	1	6	5	7	3	2
22	Non-govt services	Statistic	0.280	0.119	0.118	0.113	0.169	0.160	0.156
		Ranking	7	3	2	1	6	5	4

Note: The ranking of the theoretical distributions follows the Kormogolov-Smirnov goodness of fit statistic. The first row for each industrial sector shows the test statistics, and the second row shows the ranks.
Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Appendix 3. Derivation of substitution parameter and factor income share in VES model

From Equation 7, differentiating $g(k)$ with respect to k yields the following expression

$$\frac{d[g(k)]}{dk} = Ae^{\gamma t}\left(\frac{\varepsilon}{\varepsilon-1}\right)\left[\theta(k)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\theta)(k)^{\frac{\beta}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}}\left[\left(\frac{\varepsilon-1}{\varepsilon}\right)\theta(k)^{\frac{-1}{\varepsilon}} + \frac{\beta}{\varepsilon}(1-\theta)(k)^{\frac{\beta-\varepsilon}{\varepsilon}}\right].$$

Plugging this expression in equation 3, and after some algebraic calculation produces

$$\sigma_{KL}{}^{VES} = \frac{g(k)}{kg\prime(k)} = \frac{\left(\frac{\theta}{1-\theta}\right)k^{\frac{\varepsilon-1-\beta}{\varepsilon}}+1}{\left(\frac{\theta}{1-\theta}\right)k^{\frac{\varepsilon-1-\beta}{\varepsilon}}+\frac{\beta}{\varepsilon-1}}.$$

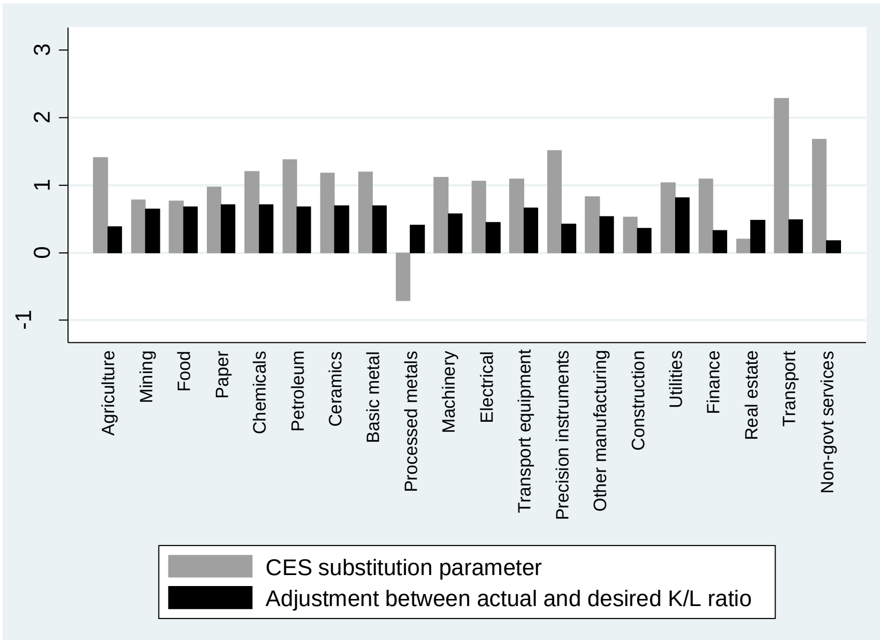
From a constrained cost-minimization based on the assumption that cost minimized factor proportions are selected in response to changes in the current factor prices, the marginal rate of technical substitution become equal to factor prices. The marginal productivities can be calculated from Equation 5. Since, the ratio of factor prices equal the ratio of the marginal productivities $\frac{\frac{\partial V}{\partial L}}{\frac{\partial V}{\partial K}}$, we get the following expression for $\frac{w}{r}$

$$\frac{w}{r} = \frac{Ae^{\gamma t}\left(\frac{\varepsilon}{\varepsilon-1}\right)\left[K^{\frac{\varepsilon-1}{\varepsilon}}+\left(\frac{K}{L}\right)^{\frac{\beta}{\varepsilon}}L^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}}\left(\frac{\varepsilon-1-\beta}{\varepsilon}\right)K^{\left(\frac{\beta}{\varepsilon}\right)}L^{\frac{-1-\beta}{\varepsilon}}}{Ae^{\gamma t}\left(\frac{\varepsilon}{\varepsilon-1}\right)\left[K^{\frac{\varepsilon-1}{\varepsilon}}+\left(\frac{K}{L}\right)^{\frac{\beta}{\varepsilon}}L^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}}\left[\left(\frac{\varepsilon-1}{\varepsilon}\right)K^{\frac{-1}{\varepsilon}}+\frac{\beta}{\varepsilon}K^{\frac{\beta-\varepsilon}{\varepsilon}}L^{\frac{\varepsilon-1-\beta}{\varepsilon}}\right]}.$$

Combining this result with the capital labor ratio, k, the ratio of relative factor shares becomes

$$\frac{R}{k} = \frac{wL}{rK} = \frac{\varepsilon-1-\beta}{\left(\frac{\theta}{1-\theta}\right)(\varepsilon-1)k^{\frac{\varepsilon-1-\beta}{\varepsilon}}+\beta}.$$

Appendix 4. Elasticity of substitution ($\widehat{\sigma}_{CES}$) and adjustment to capital-labor ratio ($\hat{\varphi}$)



Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09

Appendix 5. Variable elasticity of substitution by industrial sectors, 1970-2012

Note: 1 – Agriculture, 2 – Mining, 3 – Food, 5 – Paper, 6 – Chemicals, 7 – Petroleum, 8-Ceramics, 9 - Basic metal, 10 - Processed metals, 11- Machinery, 12 – Electrical, 13 - Transport equipment, 14 - Precision instruments, 15 - Other manufacturing, 16 – Construction, 17 – Utilities, 19 – Finance, 20 - Real estate, 21 – Transport, 22 - Non-govt services.
The blue lines show variable elasticity of substitution from VES-W model, whereas the black lines show the same form VES model.
Source: Author's calculation based on the Regional-level Japanese Industrial Productivity (R-JIP) database https://www.rieti.go.jp/en/database/R-JIP2017/index.html#09