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ABSTRACT

Switching Queues, Cultural Conventions, and Social Welfare

We use queuing-related behavior as an instrument for assessing the social appeal of alternative cultural norms. Specifically, we study the behavior of rational and sophisticated individuals who stand in a given queue waiting to be served, and who, in order to speed up the process, consider switching to another queue. We look at two regimes that govern the possible order in which the individuals stand should they switch to the other queue: a regime in which cultural convention, social norms, and basic notions of fairness require that the order in the initial queue is preserved, and a regime without such cultural inhibitions, in which case the order in the other queue is random, with each configuration or sequence being equally likely. We seek to find out whether in these two regimes the aggregate of the behaviors of self-interested individuals adds up to the social optimum defined as the shortest possible total waiting time. To do this, we draw on a Nash Equilibrium setting. We find that in the case of the preserved order, the equilibrium outcomes are always socially optimal. However, in the case of the random order, unless the number of individuals is small, the equilibrium outcomes are not socially optimal.

JEL Classification: C72, D60, Z13

Keywords: decision processes, queuing, Nash Equilibrium, social customs, social welfare

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1. Introduction

We use queuing-related behavior to assess the social appeal of different cultural norms. In harnessing queuing behavior to this end, we complement stances most often taken in the received queuing literature. Specifically, our study of switching queues sheds light on an intriguing question that is at the heart of social science research: which of two social conventions, each having merit, is superior in the sense of yielding a better social outcome. Thus, the value added of this paper lies in demonstrating that close observation of queuing behavior can serve as an instrument for assessing the social value of fundamental traits such as fairness, equal opportunity, and property rights. It is a novelty of the paper that it *exploits* queuing behavior rather than *explores* queuing behavior per se.

Waiting in line is an experience that very few people enjoy. The advance of technologies such as online services and Amazon Go shop[1] could markedly improve the quality of life of many people by turning waiting in line to be served into a fading memory. For now, though, queuing is still a frequent part of daily life. It is, therefore, of interest and relevance to study the architecture of queuing, and to assess the social efficiency of queuing arrangements.

Queuing arrangements can have significant consequences not only for the individuals queuing, but also for the enterprises that provide the services and sell the goods for which individuals queue.[2] A brief search reveals that this topic has attracted not only the attention of the public and the media,[3] but also that of researchers in a variety of disciplines. Mathematicians and engineers have studied aspects of queues for more than a century now; a seminal paper is that of Erlang (1909) who showed that the Poisson distribution can be applied to study random telephone traffic, which, at the time, was often characterized by long queues. Citing Gross et al. (2008), it appears that the field of inquiry referred to as queuing theory seeks to provide answers to questions such as “How long must a customer wait?” and “How many people will join the line?” and has resorted to rigorous mathematical reasoning. For example, Haight (1958) analyzes

[1] Consider: http://www.theverge.com/2016/12/5/13842592/amazon-go-new-cashier-less-convenience-store

[2] Consider: http://time.com/money/4651994/starbucks-sales-growth/?xid=newsletter-brief

[3] Consider the recent NY Times article: https://mobile.nytimes.com/2016/09/08/business/how-to-pick-the-fastest-line-at-the-supermarket.html

a system of two queues, where individuals arriving choose the shorter queue and then either stay there until they are served, or switch to the other queue if it becomes shorter. The main objective of Haight's analysis is to calculate the probabilities that at time $t = \infty$, the two queues will reach given lengths. Haight's analysis was expanded by Tarabia (2008), who introduced the possibility of moving the first individual at a given point in time from one queue to the other queue when the other queue is empty, and of finite-length queues (restricted to be shorter than some fixed length). In spite of a seeming congruence, our approach and goal in this paper are very different: assuming that the number of individuals as well as the serving time per individual are constant, we employ the tool of Nash Equilibrium to study the behavior of rational and sophisticated individuals who stand in a queue and who consider switching to another queue. Drawing on this underlying infrastructure, we shed light on an issue that is of interest from both an economics perspective and from a social efficiency perspective: under which regime governing switching does the aggregate of the behaviors of self-interested individuals add up to the social optimum?

Recently, the topic of queues has been studied to some extent by behavioral, experimental, and management researchers. Carmon and Kahneman (1996) designed an experimental setting aimed at investigating how characteristics of queues such as the length remaining and the speed of moving influence individuals' real-time and retrospective evaluation of their waiting experience. Koo and Fishbach (2010) analyze whether queuing changes the value individuals attribute to the product or service for which they queue. Janakiraman et al. (2011) study the decision to abandon waiting in a queue. Song et al. (2015) and Shunko et al. (2015) investigate how the architecture of queues (for example, having a single queue compared to having parallel queues) affects the performance of related workers such as cashiers. Kuzu (2015) investigates customer preferences and their perceptions of ticketed queues compared with standard, physical queues.[4] Hassin and Haviv (2003) present an extensive overview of the development of general queue theories. Practical settings in which queuing theory is applied include health care (Kozlowski and Worthington, 2015; Carmen et al., 2018), airplane boarding (Bachmat, 2019), and logistics (Jemaï and Karaesmen, 2005; Wu and McGinnis, 2012).

[4] In a ticket queue, a customer receives a ticket with a number, and thereafter waits until the number is called (and / or appears on a screen). In contrast with a physical queue, there is no need for customers to stand in line.

Socially optimal queueing arrangements were studied by Maniquet (2003) and by Chun (2006) in the case of a single queue, and by Chun and Heo (2008) in the case of two queues. In these three papers, the main interest lies in determining the optimal distribution of N individuals who are heterogeneous with respect to their waiting cost, but homogeneous with respect to their required service time. In the optimal distribution, individuals are assigned a position in the queue(s) being compensated or charged, depending on their position in the queue(s) so that their combined waiting cost is minimized. In several aspects, the approach taken in our paper differs. First, we are not interested in identifying the socially optimal distribution of individuals between two queues but, rather, we inquire under which social norm self-interested individuals will sort themselves between the queues in a socially optimal manner without any exogenous intervention. Second, we do not assume any compensation being provided for individuals who have to wait longer than others. In many real-world situations, such as waiting in line in a supermarket or a post office, compensation of this type is not feasible. Third, we assume a homogenous waiting cost. Because information about waiting costs is not public (Mitra and Mutuswami, 2011), and because self-interested individuals do not endogenize the waiting cost of others, introducing a differential waiting cost would entail considerable complexity without adding insights to the issue of interest. Fourth, we differ from Hassin and Roet-Green (2018) who study a setting in which individuals who arrive at a facility of two servers base their choice of queue on costly inspection. We do not consider an inspection cost, we study the consequences of simultaneous decisions of all the individuals in a queue rather than the choices of individuals who join the queues sequentially, and we analyze the optimal social outcome rather than the equilibrium outcome for a single individual. Although in a sense our approach is similar to that of Economou and Manou (2016), and Wang et al., (2017), where in settings in which individuals join or balk a queue the aggregate of individuals' strategic actions is compared with the socially optimal strategy, our setting of a constant number of individuals and social welfare analysis is different.

The setting that we study is as follows. We consider a constant-size population of N individuals. The individuals stand in front of an about-to-be-opened counter A (we can think of a counter in a supermarket, bank, post office, pharmacy, and so on). The sequence from the first individual in the line to the last individual in the line is $N, N-1, \dots, 1$. We refer to the line in front of counter A as line A. We assume that an adjacent counter B opens. Actually, a light above counter B indicates that soon that counter will open up for business, and that this opening will coincide with the opening of counter A. In other words, both counters will start processing at the very same time.

The individuals can switch lines, but only once. For example, because of a railing (a barrier between the front parts of the lines), switching back is not technically possible. Thus, once made, the choice of where to queue is irrevocable. If line B is equally attractive to an individual as line A (a tie), then the individual stays in line A; when switching confers no gain, switching will not occur. There are no new entries into the population queuing, and we rule out the possibility that the individuals will give up being served altogether (that is, we do not allow walking away). This condition is equivalent to assuming that the reservation utility away from the facility is sufficiently low to ensure that staying in any of the lines is preferable to not waiting in line. Thus, we confine our interest to jockeying between queues. We rule out any payments (monetary transfers) between the individuals. The individuals are rational, namely they prefer to be served earlier than later, they are sophisticated (farsighted), and they are risk neutral. The individuals are homogeneous in their preferences, they do not differ in their valuation of being served, and are also homogeneous in their waiting costs (for example, they all have the same number of items in their baskets to be processed by a cashier). Consequently, once reaching a counter, any individual will be served at the same pace, namely within a fixed time span. In a given queue, the individuals can be served only one at a time. To simplify, we normalize the service time of an individual as one minute. For example, the first individual in an operational line will be out of the premises after one minute, the second individual in a line will be out after two minutes, the third individual will be out after three minutes, and so on. The individuals cannot coordinate with each other their arrival time at the service facility (a Waze type program is not available to that end). The individuals seek to minimize their waiting time, defined as the time it takes until processing is completed, namely the time taken until they leave the facility after being served.

A study of the dynamics of the division between queues in a population of a given size - the individuals are already in the facility - rather than tracking the dynamics of new arrivals supplements the related literature. Fixing the size of the population allows us to focus on the main issue of interest. In the typical "case of arrivals," individuals face choices such as whether and when to arrive (for example, based on the length of the existing queues), would-be arrivals need to bear in mind the likely previous arrival of others, and so on. These and related issues can and should be set aside when the purpose of the study is to ascertain the repercussions of one single choice - that of location/relocation. Our use of simplifying assumptions such as a fixed population does not come at the cost of eliminating the appeal of our handling of the issue at stake: for exogenous reasons (say, it is late in the evening) no new customers arrive at the premises (say, a supermarket). A given number of customers who have finished filling their shopping trolleys (each with a similar number of items) await processing at a cashier, seeking to leave for home earlier rather than later. (That the processing time is the same for all the customers who stand in line to be served is typical in cases such as renewing a driver's license, obtaining a passport, mailing a parcel.) This type of setup is pure in the sense that it allows us to abstract from extraneous considerations, yet is rich enough to support developing a clean analytical protocol aimed at ranking modes of behavior in terms of their repercussions for aggregate wellbeing. This stance of ours happens to align with a received approach of studying social welfare in the context of queuing; see, for example, Chun and Heo (2008).

We consider two regimes or disciplines that govern the possible order of the individuals in line B. In the first regime, cultural convention and social norm have it that the order in line A is preserved in line B. That is, on switching to line B, priority is the same as in line A. In the second regime, the cultural inhibition is that once people leave an existing order, any order is possible; such an alternative convention can arise from a perception like that of equal opportunity regardless of initial conditions; the order in line B is then random (equivalent to a lottery), with any configuration or sequence being equally likely, and being considered by the individuals as such. The choice of the two regimes is inspired by informal observations made in the course of one week in a supermarket in Austria (where it was noticed that when people switched the order was preserved), and in a supermarket in the US (where it was seen that when people switched the order was random). We seek to find out how the individuals will sort themselves between the two queues, and whether the aggregate of the individuals' behaviors yields the optimal social welfare outcome, namely minimization of the total waiting time. Social welfare is utilitarian, where drawing on the assumption that the individuals do not differ in their valuation of being served, all the individuals are weighted equally.

When individuals are indifferent to the consequences of their actions for the wellbeing of others, should we expect the aggregate of their actions to constitute the socially preferred outcome? We find that the answer to this question depends on the type of social norm and cultural convention that define the regime under which the individuals act. The results that we obtain can be summarized as follows.

1. If on switching queues the initial order of the individuals is preserved, then the equilibrium distribution of the individuals between the two queues is socially optimal: a convention of honoring an existing order gives rise to a desirable social outcome.

2. If the initial order is not preserved, namely when once the individuals switch to the second queue their position there is random, then, generally speaking, an equilibrium distribution between the queues is not socially optimal. In equilibrium, about one third of the population is located in line A, and the remainder is located in line B. The absence of a social convention of honoring the prevailing, pre-switching order thus penalizes a population harshly. Only in the cases of population size of 2, 3, 4, 5, or 7 is a socially optimal equilibrium possible.

In the remainder of this paper we proceed as follows. In Section 2 we study the case in which on switching queues the initial order of the individuals is preserved. In Section 3 we analyze the case of random order. In Section 4 we present refinements and comment on robustness. In Section 5 we discuss our results and conclude.

2. On switching queues, the order is preserved

As described in the Introduction, a realistic scenario to consider will be one in which a given number of customers have filled their shopping trolleys (each with a similar number of items), await processing at a cashier, and want to leave for home earlier rather than later.

The assumption that the individuals are sophisticated implies that every individual accurately anticipates the behavior of the other individuals and tailors his behavior accordingly. As a solution concept, we thus resort to Nash Equilibrium (NE).

Our goal is to find out whether the Nash Equilibria obtained constitute a socially efficient outcome. We define social optimum as a distribution of the individuals that minimizes the combined (aggregate) time it takes until their processing is completed (namely until they leave the premises having been served). For such an optimum to occur, the individuals have to be distributed between the queues as evenly as possible: namely if, for a given NE, N is the number of individuals, and K is the number of individuals who stay in line A, then this NE is socially optimal if and only if $\frac{N-1}{2} \le K \le \frac{N+1}{2}$. When on switching queues the same order is preserved, then the resulting distribution of the individuals is socially optimal for any N.

Claim 1. Let $N > 1$. When after switching queues the order is preserved, then there exists only one NE. In this NE, individual $N-k$ stays in line A if k is even, and he switches to line B if k is odd ($k \in \{0,1,\ldots,N-1\}$).

Proof. First, we note that the waiting time of individual i, $i \in \{1,2,\ldots,N\}$, depends only on his choice of strategy and on the choices of individuals $\{i+1,i+2,\ldots,N\}$, but it does not depend on the choices of individuals $\{1,2,\ldots,i-1\}$ because they can never be ahead of him in the final ordering in any queue according to the assumption that the initial order is preserved. Therefore, we can find the NE sequentially, that is, by identifying the optimal strategy for individual N, then by identifying the optimal strategy for individual $N-1$ (given that he can infer the optimal strategy of individual N), then by identifying the optimal strategy for individual $N-2$ (given that he can infer the optimal strategy of individuals N and $N-1$), and likewise for each individual down to individual 1, as long as each individual can infer the strategy choices of all the individuals ahead of him in the initial queue. Bearing this in mind, we can proceed with the proof of the claim using induction on k. Individual N is the first in line A. If he were to switch queues, he would also be the first in line B. Because he has nothing to gain by switching, he stays put, and the other individuals know that he will do so. Therefore, the basis of the induction holds true for $k=0$: namely 0 is even, individual $N-0=N$ stays in line A, and the other individuals know that.

For the inductive step, we assume that for $k \in \{0,1,\ldots,k_0\}$, individual $N-k$ stays in line A if k is even and switches to line B if k is odd, and that the other individuals know that. We then need to prove that the same is true for $k = k_0 + 1$.

First, we assume that $k_0 + 1$ is odd. Then, from the induction protocol, individual $N - (k_0 + 1)$ knows that from the group of $k_0 + 1$ individuals originally preceding him in line A, $\frac{k_0}{2} + 1$ stay in line A, and $\frac{k_0}{2}$ switch to line B. Therefore, if individual $N - (k_0 + 1)$ stayed in line A, then his waiting time there would be $\frac{k_0}{2} + 2$ minutes, and if he were to switch to line B, his waiting time there would be $\frac{k_0}{2} + 1$ minutes. Therefore, individual $N - (k_0 + 1)$ moves to line B, and the other individuals know that.

Next, we assume that $k_0 + 1$ is even. Then, from the induction protocol, individual $N - (k_0 + 1)$ knows that from the group of $k_0 + 1$ individuals originally ahead of him in line A, $\frac{k_0 + 1}{2}$ individuals stay in line A, and $\frac{k_0 + 1}{2}$ individuals switch to line B. Therefore, individual $N - (k_0 + 1)$ knows that his waiting time will be $\frac{k_0 + 3}{2}$ minutes, no matter whether he stays put or switches queues. Because of the assumption that "if line B is equally attractive to an individual as line A, then the individual stays in line A," individual $N - (k_0 + 1)$ stays put, and every other individual knows that.

Because both the induction basis and the induction step have been shown to hold, the induction as a whole holds, and the distribution described in Claim 1 is the only possible NE. Q.E.D.

From Claim 1 we infer that when on switching queues the order is preserved, the individuals end up distributing themselves between the two queues in a NE so that the aggregate waiting time is minimized (namely the distribution of the individuals between the queues is equal when *N* is even, or is equal but one when *N* is odd) and, therefore, social welfare is maximized.

3. After a switch, the order is random

We now analyze a regime where the social convention has it that on switching queues, the order is random, that is, with any configuration or sequence being equally likely, and considered by the individuals as such. For example, if three individuals from line A decide to switch to line B, we will have $3! = 6$ possible sequences, each occurring with probability $\frac{1}{6}$.

In terms of guiding behavior, when it comes to switching queues, a social convention is the equivalent of a heritage acquired from experiencing similar situations many times before, so that when the opportunity to switch queues presents itself, the rules of engagement need not be inferred or learned from sequential observations of the responses of others. Thus, rational and sophisticated individuals proceed simultaneously and replicate solutions proven to be favored in similar circumstances, namely, NE applies. As before, our interest is in finding out whether a socially optimal equilibrium, defined as minimization of the total waiting time, can emerge from the aggregate of the individuals' actions. In this section we find that there are Nash equilibria that are not socially optimal. Specifically, when N is sufficiently large (namely when $N > 7$), the Nash equilibria are never socially optimal.

As a brief illustration of how a social convention gives rise to a switching outcome when after a switch the order can be any, we consider the case of $N = 4$. In this case, there are three random-order sub-conventions. We name them "Upper half stay - bottom half switch;" "Stay - switch intermittently;" and "Edges stay - in-between switch." Common to the three sub-conventions is that each constitutes a NE: the resulting distribution is stable in the sense that no individual has an incentive to change his location decision when no-one else changes theirs.

The distribution of the four individuals between the two queues where each distribution constitutes a NE are delineated in Figure 1: [5]

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[5] The reasoning why the "Upper half stay - bottom half switch" distribution constitutes a NE follows (the reasoning why each of the other two distributions constitutes a NE is analogous). (i) Because individual 4 who is in line A is already in the best possible position, he prefers to stay there. (ii) In the case of this distribution, the waiting time in line A of individual 3 is two minutes. (Again, the waiting time is defined as the time it takes until processing is completed, namely until individual 3 leaves the premises having been served.) If he were to switch to line B, he would be one of three individuals in that line, and his expected waiting time there would be $\frac{1+2+3}{3} = 2$ minutes.

Upper half stay - bottom half switch		Stay - switch intermittently		Edges stay - in-between switch	
Line A	*Line B*	*Line A*	*Line B*	*Line A*	*Line B*
4	$\{2,1\}$	4	$\{3,1\}$	4	$\{3,2\}$
3		2		1	

Figure 1. Distributions constituting Nash Equilibria for $N = 4$ when on switching queues the order in line B is random.

Note: In line B we use the notation {n, m} to denote that individuals *n* and *m* switched to line B, where their order is random.

Clearly, we cannot predict which of the three social sub-conventions underlying the three distributions will prevail. However, we know that as Nash equilibria, the three distributions are the only ones to which a social convention of a random order on a switch can give rise.[6]

Claim 2. Let $N > 1$. When after switching queues the order is random, then each NE satisfies the following condition: $\left\lfloor \frac{N+2}{3} \right\rfloor$ individuals stay in line A, and $N - \left\lfloor \frac{N+2}{3} \right\rfloor$ individuals switch to line B.[7]

Because of the assumption that "if line B is equally attractive to an individual as line A, the individual stays in line A," individual 3 stays in line A. (iii) In the case of this distribution, each of the two individuals 2 and 1 is in a similar situation: his expected waiting time in line B is one and a half minutes, and if he were to stay in line A, his waiting time there would be three minutes. Therefore, both individuals 2 and 1 prefer to abide by the "Upper half stay - bottom half switch" social sub-convention, and they switch to line B. Thus, under the social convention "Upper half stay - bottom half switch," none of the four individuals has any incentive to deviate by changing his decision; the distribution constitutes a NE.

[6] The assumption that a social sub-convention is in place is essential. Had the individuals not known which of the NE will result from the prevailing social norm, they would have needed to *guess* the behavior of other individuals, and if they failed to do so, they would end up with an outcome that does not constitute a NE. For example, without any prior knowledge of the social convention, individual 3 may aim at the "Upper half stay - bottom half switch" distribution, individual 2 - at the "Stay-shift intermittently" distribution, and individual 1 - at the "Edges stay - in-between shift" distribution, thereby yielding the result that all the individuals end up staying in line A.

We also note that the set of Nash equilibria here is markedly different from the set in the parallel case when the order in line B preserves the order in line A. In Section 2 for $N = 4$ there is only one possible NE (individuals 4 and 2 stay in line A, individuals 3 and 1 move to line B).

[7] $\lfloor x \rfloor$ denotes the largest integer that is not greater than x.

Proof. We analyze the following distribution of the individuals between the two queues so as to find for which $K > 0$ the distribution in which exactly K individuals stay in line A constitutes a NE. We can define such a distribution by dividing the individuals between two sets: individuals $N^A, (N-1)^A, \ldots, (N-K+1)^A$ who stay in line A and retain their ordering, and individuals $(N-K)^B, (N-K-1)^B, \ldots, 1^B$ who move to line B where they are ordered randomly. $(N^A, (N-1)^A, \ldots, (N-K+1)^A, (N-K)^B, (N-K-1)^B, \ldots, 1^B)$ is a permutation of the set $\{N, N-1, \ldots, 1\}$ such that for any two numbers k and l, $k > l$ implies that $k^i > l^i$ for $i \in \{A, B\}$. For example, for $N = 4$ and $K = 2$, we could have $4^A = 4,\ 3^A = 2,\ 2^B = 3,\ 1^B = 1$ (the "Stay - switch intermittently" distribution in Figure 1). To check whether the distribution displayed above constitutes a NE, we ask whether any individual from line A would rather be in line B, and whether any individual from line B would rather be in line A.

We start with the A-to-B switch. We look at individual $\left(N-K+1\right)^A$ who is the most likely individual in line A to prefer being in line B because his waiting time in line A is the longest. His waiting time in line A is K minutes, whereas his expected waiting time in line B would be $\frac{N-K+2}{2}$ minutes. This individual will prefer to stay in line A if and only if $K \le \frac{N-K+2}{2}$. This inequality can be transformed into $K \le \frac{N+2}{3}$.

We next look at a hypothetical B-to-A switch. The expected waiting time in line B is $\frac{N-K+1}{2}$. The waiting time of any individual k^B from line B if he were to queue in line A instead would be no longer than $K+1$ minutes because there would be no more than K individuals before him in line A. If the analyzed setting constitutes NE, then no individual who is in line B would rather be in line A. Thus, $K+1 > \frac{N-K+1}{2}$, which can be rewritten as $K > \frac{N-1}{3}$.

In sum, if the distribution

$$\begin{array}{cc}
\textit{Line A} & \textit{Line B} \\
N^A & \left\{(N-K)^B, (N-K-1)^B, \ldots, 1^B\right\} \\
(N-1)^A & \\
\vdots & \\
(N-K+1)^A &
\end{array}$$

constitutes a NE, then $\frac{N-1}{3} < K \le \frac{N+2}{3}$ holds. K is an integer so, therefore, $K = \left\lfloor \frac{N+2}{3} \right\rfloor$. Q.E.D.

For a large *N*, it is straightforward to see that $K \approx \frac{N}{3}$, implying that about $\frac{1}{3}$ of the individuals will stand in line A, and about $\frac{2}{3}$ of the individuals will stand in line B. If $(N-K)^B > (N-K+1)^A$, then the condition $K = \left\lfloor \frac{N+2}{3} \right\rfloor$ may not be sufficient for the analyzed setting to constitute a NE. Additionally, the condition $N+1-(N-K)^B > \frac{N-K+1}{2}$ will need to hold as well, because otherwise individual $(N-K)^B$ would prefer to stay in line A. This last condition can be rewritten as $(N-K)^B < \frac{N+K+1}{2}$, which states that in NE, if the individual is assured of being sufficiently close to the front of line A, then the individual stays in line A.

The results obtained tell us that a socially optimal outcome will be possible only for very small populations. Indeed, for $N=2$ and $N=3$ we have $K = \left\lfloor \frac{N+2}{3} \right\rfloor = 1$, and for $N=4$ and $N=5$ we have $K = \left\lfloor \frac{N+2}{3} \right\rfloor = 2$, and thus every NE for $N \le 5$ is socially optimal. For $N=6$ we have $K = \left\lfloor \frac{N+2}{3} \right\rfloor = 2$, but in this case, $K=2$ yields a NE that is not socially optimal. For $N=7$ we have $K = \frac{N+2}{3} = 3$, which yields a socially optimal NE. In general, for NE to be socially optimal, $K \geq \frac{N-1}{2}$ must hold. Therefore, for $N > 7$, NE is never socially optimal because then $\frac{N-1}{2} > \frac{N+2}{3} \geq \left\lfloor \frac{N+2}{3} \right\rfloor$.

4. Refinements and robustness

The results described in this paper suggest that from the perspective of social welfare, a regime in which on switching queues the initial order is preserved is better than a regime in which on switching queues the order is random.

We are able to shed some light on the question as to which regime will be preferred by a rational individual who does not know where he will be placed in the original queue. If the initial order is preserved, then for an even population size *N*, the expected waiting time in each of the two queues is the same and is equal to $\frac{N}{4}+\frac{1}{2}$; for an odd population size *N*, it is $\frac{N}{4}+\frac{1}{4}$ in the shorter queue, and $\frac{N}{4}+\frac{3}{4}$ in the longer queue. If after a switch the order is random, then, as shown in Section 3, in NE the queues will have different lengths, with line B being longer. On the basis of the result in Section 3, that about $\frac{1}{3}$ of the individuals will stay in line A, we can conclude that the expected waiting time in line A will be $\frac{N}{6}+\frac{1}{2}$, which is shorter than $\frac{N}{4}+\frac{1}{2}$, and that the expected waiting time in line B will be $\frac{N}{3}+\frac{1}{2}$, which is longer than $\frac{N}{4}+\frac{1}{2}$. Nevertheless, *if* an individual assumes that he will be in each place in the original queue with equal probability, then, in case of a random order in line B, he expects to end up in line B with a probability of about $\frac{2}{3}$. Therefore, his expected waiting time to be served will be approximately $\frac{5N}{18}+\frac{1}{2}$, which is longer than $\frac{N}{4}+\frac{1}{2}$. Therefore, the individual will prefer the regime in which the initial order is preserved.

The assumption that switching queues is costless can be relaxed. Suppose that the value of obtaining a gain of one position (namely one slot) is one, and suppose that on switching queues the individuals incur a cost $c > 0$. Then, in the case in which the order is preserved and $N - \lfloor c \rfloor$ is even, $\frac{N + \lfloor c \rfloor}{2}$ individuals will stay in line A, and $\frac{N - \lfloor c \rfloor}{2}$ individuals will stay in line B. In the case in which the order is preserved and $N - \lfloor c \rfloor$ is odd, $\frac{N + \lfloor c \rfloor + 1}{2}$ individuals will stay in line A, and $\frac{N - \lfloor c \rfloor - 1}{2}$ individuals will stay in line B. Thus, the results reported in Section 2 will hold if $N - \lfloor c \rfloor$ is odd and $c < 1$, or if $N - \lfloor c \rfloor$ is even and $c < 2$. If N is large and c is relatively small, both queues will be of approximately the same length. In the case in which the order is random, $\left\lfloor \frac{N + 2 + 2c}{3} \right\rfloor$ individuals will stay in line A, and $N - \left\lfloor \frac{N + 2 + 2c}{3} \right\rfloor$ individuals will stay in line B. Once again, if N is large and c is relatively small, then these results mimic the ones reported in Section 3 where approximately one third of the population stays in line A. If c is relatively large, then the results can significantly differ from the ones reported in Section 3. Interestingly, a specific large value of the cost of switching $c = \frac{N - 4}{4}$ yields a socially optimal equilibrium in which individuals are distributed evenly between the queues if N is even, or evenly but for one individual if N is odd. An analysis of the case in which the individuals incur a positive cost of switching queues is provided in the Appendix.

Suppose that whereas the individuals are rational - they prefer to be served earlier than later - they are not sophisticated (not farsighted), that switching queues is costless, and that switching back and forth is technically possible. We refer to the stages in the progression of the switching steps as "periods." Then, the result specified in Claim 1 will hold. Specifically, by period $N / 2$ if N is even, or by period $(N - 1) / 2$ if N is odd, none of the individuals will have an incentive to switch queues again. Then, the N individuals will be divided between the two lines evenly if N is even, or evenly but for one individual if N is odd. To see this, we note that in period 1, individual N who occupies the first spot in line A does not have an incentive to switch, whereas all the other individuals will move to line B in order to gain a better position. Next, each of the individuals $N-3, N-4, \ldots, 1$ observes that he can obtain a better position (second) if he were to move back to line A. Thus, individuals $N-3, N-4, \ldots, 1$ move to line A, and in period 2 the distribution of the individuals will be N in line A, $N-1$ and $N-2$ in line B, and the remainder of the individuals in line A. (Individual $N-2$ will not move back to line A because of the assumption of no switching when there is a tie.) Once again, some individuals from line A, specifically $N-5, N-6, \ldots, 1$, will have an incentive to move to line B. We can see that by period $N/2$ if N is even, or by period $(N-1)/2$ if N is odd, all comings and goings will come to halt. What remains to be characterized is the queue in which of individuals 1 and 2 will stand. It turns out that the whereabouts of these two individuals depends on whether N is even or odd, and on whether when N is even, whether N or $N-2$ is a multiple of 4, and when N is odd, whether $N-1$ or $N-3$ is a multiple of 4. Specifically, we have the following characterization. When N is even, then 1 and 2 are in different lines: if N is a multiple of 4, then 1 is in line A, and 2 is in line B; if $N-2$ is a multiple of 4, then 2 is in line A, and 1 is in line B. When N is odd, then 1 and 2 are in the same line: if $N-1$ is a multiple of 4, then they are in line A; if $N-3$ is a multiple of 4, then they are in line B.

The results obtained in the two preceding sections reveal a difference in terms of the social welfare outcome between the two social norms studied: the one in which after a switch the initial order is preserved, the other in which after a switch the order is random with every possible order being equally likely. It is of some interest to ask how the results obtained would be affected when the social norm in place is a "mix" of these two norms. Specifically, we could consider a regime in which after a switch order is generally random, but the sequences are such that on switching queues, individuals who were closer to the counter in line A will be more likely to be closer to the counter in line B. For example, such a constellation could arise when the two lines are parallel to each other with no barrier between them so that upon a switch individuals who occupy a position in the front of line A have a shorter distance to cover to reach the front of line B.

Suppose that M individuals, namely $b_1, b_2, \ldots, b_M$, such that $b_i < b_j$ iff $i < j$ moved to line B. There are various ways of formalizing the probability of an ordering of these individuals in line B under a "mix" social norm. As an example, we consider the following procedure: first, we assign a position to individual b_M. The probability that this individual will end up occupying the k^{th} position in line B is given by

$$P_k^M = \frac{(M+1-k)^2}{\sum_{i=1}^{M}(M+1-i)^2}. \tag{1}$$

The probabilities of occupying different positions by individual b_M in the sequence of M individuals sum up to 1. Having assumed that individuals who were closer to the counter in line A will be more likely to be closer to the counter in line B, the highest probability is accorded to the outcome that individual b_M will be the first in line B. After the position of individual b_M is assigned according to the probabilities assessed, we consider next individual b_{M-1}, and we use the same assignment rule to accord him a position in the shorter sequence, namely in the sequence $b_1, b_2, \ldots, b_{M-1}$, where the probability that he will end up occupying the k^{th} position among individuals $b_1, b_2, \ldots, b_{M-1}$ is equal to $P_k^{M-1} = \frac{(M-k)^2}{\sum_{i=1}^{M-1}(M-i)^2}$. After assigning a position to individual b_{M-1}, we repeat this procedure for individuals $b_{M-2}, b_{M-3}, \ldots, b_2, b_1$. As a result, for each permutation $\varphi: \{1, 2, \ldots, M\} \to \{1, 2, \ldots, M\}$, when we define for each $i \in \{1, 2, \ldots, M\}$

$$\Psi(i) \equiv \#\{j > i : \varphi(j) < \varphi(i)\},$$

(with $\#$ denoting the number of elements of a given set), the probability that the ordering of the individuals in line B from the first to the last is $(b_{\varphi^{-1}(1)}, b_{\varphi^{-1}(2)}, \ldots, b_{\varphi^{-1}(M)})$, namely that for each $i \in \{1, 2, \ldots, M\}$ individual b_i takes the $\varphi(i)$-th place in the line, is given by

$$P_\varphi = \prod_{i=1}^{M} P^i_{\varphi(i)-\Psi(i)}.$$

Obviously, the probabilities of all possible permutations of the set $\{1, 2, \ldots, M\}$ add up to 1.

As a numerical illustration, we consider the following distribution of $N = 8$ individuals between the two queues:

Line A	*Line B*
8	$\{7,5,4,2\}$
6	
3	
1	

Here, $M=4$, $b_1=2$, $b_2=4$, $b_3=5$, and $b_4=7$. To demonstrate how the probability of a given ordering of the individuals in line B is calculated, we consider the specific sequence $(4,7,5,2)$ such that individual 4 is the first in line B, individual 7 is the second in line B, and so on. We begin with individual 7 who according to the social norm "mix" has the highest probability amongst {7, 5, 4, 2} of being the first in line B. We assume that the probability that he will end up occupying the k^{th} position in line B is given by (1) for $M=4$, namely $P_k^4=\frac{(5-k)^2}{\sum_{i=1}^{M}(5-i)^2}$. In the specific sequence $(4,7,5,2)$, individual 7 is second in line B, an outcome occurring with probability $P_2^4=0.3$. After the probability for individual 7 is assessed, we analyze a smaller set of individuals without him, namely the set {5, 4, 2}, and we consider the individual who, according to the social norm "mix," has the highest probability of being the first in line B if individual 7 has not taken that position, or has the highest probability of being the second in line B if individual 7 has taken the first position. In this particular case, the considered individual is 5, so we calculate the probability that in the set {5, 4, 2} he will be the second. This probability is $P_2^3=0.286$. We thereafter repeat the procedure; we analyze the set without individual 5, namely the smaller set {4, 2}, and we calculate the probability that individual 4 will be the first in this set. This probability is $P_1^2=0.8$. Lastly, individual 2 is obviously the first in the set {2} and this, of course, happens with probability $P_1^1=1$. It follows then that the probability of the sequence $(4,7,5,2)$ in line B is $P_2^4\cdot P_2^3\cdot P_1^2\cdot P_1^1=0.0686$.

In a situation in which after a switch the order is random, with every possible order being equally likely, we found that for $N>7$ socially optimal Nash equilibria are not possible. For example, for $N=8$, three individuals will stay in line A, and five individuals will switch to line B. In contrast, under a "mix" social norm, the socially optimal distribution

Line A	*Line B*
8	$\{7,5,4,2\}$
6	
3	
1	

constitutes a unique NE.[8]

An analysis of the general case of $N > 7$ individuals is complex and will depend heavily on the functional form of the expression appearing in (1). For example, if in (1) we were to eliminate the raising to the power of 2, then there will be no socially optimal NE for the case of eight individuals. This consideration suggests that, quite intuitively, when in the regime "mix" the probability of the sequence with the preserved order is high, then the results will be closer to the ones obtained for the regime of preserved order.

5. Discussion

We have made several assumptions which, while applying in some real-life situations, may not hold in others. These assumptions include that of a constant number of individuals in the queue, and a constant serving time per individual. Here we comment on what could follow when these assumptions are relaxed. We refer, first, to the assumption that there is a constant number of individuals in the queue. A situation allowing individuals to join and leave is considered in the majority of the received literature on queuing theory, as already noted in the Introduction. Our case is different, and thus complements received treatments: other writings on queuing theory focus, in the main, on the process of forming and disbanding the queues, while we study the behavior of individuals who are already in a queue and who do not leave the facility until the game that we model is concluded. Therefore, relaxing the assumption of a constant number of individuals (N in our paper) will not deliver added value. To reiterate: what we consider is a single game, based on a single decision, and without any repetitions and, therefore, the course of time is not a factor in our model. To illustrate this vividly, consider a situation in which m

[8] Because proving that such a configuration constitutes a unique NE is tedious, it is omitted here. The proof is available from the authors on request.

individuals are already waiting to be served, and k individuals are about to join the queue. These last can join either before or after line B is formed and everyone who is queuing then decides if he is going to stay in line A or move to line B. If the k individuals join before the decisions of the m individuals are made, then they will be positioned at the end of line A, and we solve the game for $N = m + k$ individuals, as was already done earlier in the paper. If the k individuals join after the decision is made, we solve the game for $N = m$ before k join so that k have no impact on the solution. Therefore, allowing the number of individuals to change will not be all that meaningful. Put somewhat differently, we have studied a specific setting in which individuals are already standing in front of an about-to-be-opened counter A, and an adjacent counter B, which will start processing at the same time as counter A, will also be opened soon. Assuming that switching from line A to line B happens immediately after the individuals observe the opening of counter B, then the arrival of new individuals will presumably not affect the choice of the individuals who are *already* in the queue, assuming that it is unlikely that new people will arrive at the same time as individuals who are already in the queue find out about the opening of counter B. Nor is it likely that new arrivals will join the queue in front of counter A anywhere but at the end of the queue.

On the matter of changing the serving time per individual, we have the following thoughts. If we assume that the serving time per individual is not fixed, but rather follows a distribution that the individuals know, then assuming risk-neutrality, the individuals will be concerned only with the mean of this distribution, which should not substantively affect our reported results. And if we allow the serving time to vary but the expected time of service is the same, nothing will change. (Guo et al., 2011 study a queue setting in which individuals who have only partial information on service time adopt the maximum entropy principle in order to obtain more information.) If each individual expects to wait for a different time, then anything can happen. We can imagine an extreme case in which when $N = 100$, and the serving time of each individual but of N is 1 and the serving time of individual N is 500, then the solution will be that N stays in line A and everyone else moves to line B. Considering cases such as this will not provide us with any valuable insight into human behavior. However, even if it is rational to expect that the service time for different individuals will not be the same, it may not be realistic in some situations to assume that individuals in the queue are able to assess in advance the serving time of every other individual and incorporate that in their calculations of their own expected waiting time. We have commented on this point early on: “Because information about waiting costs is not public (Mitra and Mutuswami, 2011), and because self-interested individuals do not internalize the waiting cost of others, introducing a differential waiting cost would entail considerable complexity without adding insights to the issue.”

The problem of switching queues can be seen as a nice test of the rationale for the prevalence and sustainability of a particular social convention. We have seen that under the “guide” of the preservation of a queue order, but not (except when the population is particularly small) under the guide of a random order, the behavior of selfish individuals who are not concerned about the effects of their conduct on others adds up to the socially optimal outcome. On the face of it, each of the two social conventions referred to in this paper has merit. Preserving an order aligns with the notion of “preservation of property rights” and with an interpretation of the concept of fairness, while a random order aligns with the notion of “equal opportunity” and with yet another interpretation of the concept of fairness. It is difficult to identify the socially preferable convention on the basis of abstract reasoning. The observed behavior of the individuals switching queues under each of the alternative conventions can then be construed as a social laboratory experiment that helps identify the socially preferable convention.

As a closing reflection, we note that it is possible to consider different regimes governing the order of individuals on switching queues and (nearly analogously to the reasoning presented in Section 2) even to prove a stronger version of Claim 1: if there exists *any* fixed, deterministic regime that governs the possible order of the individuals in line B known and accepted by all the individuals, namely if a permutation φ of the set $\{1,2,\ldots,N\}$ exists such that individual i precedes individual j in line B if and only if $\varphi(i) > \varphi(j)$, then there exists only one possible NE, and that NE is socially optimal. That being said, we elected not to pursue this track because no other deterministic regime seemed to us to be as natural and accepted as the one that we studied in Section 2 (namely when $\varphi(k) = k$ for $k \in \{1,2,\ldots,N\}$).

Appendix

In this Appendix we inquire whether the results reported in Sections 2 and 3 hold when switching queues subjects the individuals to a cost, $c > 0$. We begin with the case in which on switching the order is preserved, as described in Section 2.

Claim A1. Let $N > 1$. When after switching queues the order is preserved, and on switching queues individuals incur cost $c > 0$, then there exists only one NE. In this NE, individual $N - \lfloor c \rfloor - k$ stays in line A if k is even, and he switches to line B if k is odd ($k \in \{0,1,\ldots,N-\lfloor c \rfloor -1\}$). Additionally, all the individuals from N to $N - \lfloor c \rfloor + 1$ stay in line A.

Proof. The proof is analogous to the proof of Claim 1. The difference is that now, on account of the cost, individuals from N to $N - \lfloor c \rfloor$ stay in line A because switching queues does not confer any gain. Then, an analogous induction reasoning can be made, with individual $N - \lfloor c \rfloor$ as a base of induction. Q.E.D.

The distribution of the individuals between the queues is then as follows. When $N - \lfloor c \rfloor$ is even, $\frac{N-\lfloor c \rfloor}{2} + \lfloor c \rfloor = \frac{N+\lfloor c \rfloor}{2}$ individuals will stay in line A, and $\frac{N-\lfloor c \rfloor}{2}$ individuals will stay in line B. When $N - \lfloor c \rfloor$ is odd, then $\frac{N-\lfloor c \rfloor+1}{2} + \lfloor c \rfloor = \frac{N+\lfloor c \rfloor+1}{2}$ individuals will stay in line A, and $\frac{N-\lfloor c \rfloor-1}{2}$ individuals will stay in line B. If $c \ge 2$, then K (as in Sections 2 and 3, this is the number of individuals who stay in line A) satisfies the inequality: $K \ge \frac{N+\lfloor c \rfloor}{2} > \frac{N+1}{2}$. Additionally, when N is even and $c \ge 1$, then $K \ge \frac{N+\lfloor c \rfloor}{2} \ge \frac{N+1}{2}$ and because $\frac{N+1}{2}$ is not an integer, $K > \frac{N+1}{2}$. Therefore, the obtained NE will be socially optimal only if N is even and $c < 1$, or if N is odd and $c < 2$. If N is large and c is relatively small, then the two queues will approximately be of the same length.

We next turn to the case of random ordering of Section 3.

Claim A2. Let $N > 1$. When after switching queues the order is random, and when on switching queues individuals incur cost $c > 0$, then each NE satisfies the following condition: $\left\lfloor \frac{N+2+2c}{3} \right\rfloor$ individuals stay in line A, and $N - \left\lfloor \frac{N+2+2c}{3} \right\rfloor$ individuals switch to line B.

Proof. Analogously to the proof of Claim 2, we analyze the following distribution of the individuals between the two queues, so as to find for which $K > 0$ their distribution constitutes a NE:

$$\begin{array}{cc} \textit{Line A} & \textit{Line B} \\ N^A & \left\{(N-K)^B, (N-K-1)^B, \ldots, 1^B\right\} \\ (N-1)^A & \\ \vdots & \\ (N-K+1)^A & \end{array}$$

We will refer to a waiting cost instead of a waiting time because now we are assuming that the individuals value one minute as one. Alternatively, the cost c could be counted in time units, namely minutes.

As in Section 3, we start with the A-to-B switch. We consider individual $(N-K+1)^A$ who is the most likely individual in line A to prefer being in line B because his waiting time in line A is the longest. His waiting cost in line A is K, whereas his expected waiting cost in line B would be $\frac{N-K+2}{2}+c$. This individual will prefer to stay in line A if and only if $K \le \frac{N-K+2}{2}+c$. This inequality can be transformed into $K \le \frac{N+2+2c}{3}$.

We next look at a hypothetical B-to-A switch. The expected waiting cost in line B is $\frac{N-K+1}{2}+c$. The waiting cost of any individual k^B from line B if instead he were to queue in line A would be not higher than $K+1$ because there would be no more than K individuals preceding him in line A. If the analyzed setting constitutes NE, then no individual who is in line

B would rather be in line A. Thus, $K+1>\frac{N-K+1}{2}+c$, which can be rewritten as $K>\frac{N-1+2c}{3}$.

In sum, if the distribution

$$\begin{array}{cc}
\textit{Line A} & \textit{Line B} \\
N^A & \left\{(N-K)^B, (N-K-1)^B, \ldots, 1^B\right\} \\
(N-1)^A & \\
\vdots & \\
(N-K+1)^A &
\end{array}$$

constitutes a NE, then $\frac{N-1+2c}{3}<K\leq\frac{N+2+2c}{3}$ holds. Because K is an integer, we obtain that $K=\left\lfloor\frac{N+2+2c}{3}\right\rfloor$. Q.E.D.

If c is fixed, and N is large, then, similarly as in Section 3, $K\approx\frac{N}{3}$. On the other hand, if the cost of switching queues is relatively large, then that can significantly alter the results reported in Section 3. Interestingly, when the large value of the cost of switching is $c=\frac{N-4}{4}$, then $K=\left\lfloor\frac{N}{2}\right\rfloor$ and, therefore, NE will be socially optimal, with the individuals distributed evenly between the queues if *N* is even, or evenly but for one individual if *N* is odd.

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