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Patrick Emerson
Oregon State University and IZA

Shawn D. Knabb
Western Washington University

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IZA – Institute of Labor Economics

Schaumburg-Lippe-Straße 5–9
53113 Bonn, Germany

Phone: +49-228-3894-0
Email: publications@iza.org

www.iza.org

ABSTRACT

Demographic Uncertainty and Generational Consumption Risk with Endogenous Human Capital

This paper uses a model with overlapping generations to demonstrate that human capital accumulation can potentially attenuate factor price movements in response to birth rate shocks. Specifically, we show that if education spending per child is inversely related to the size of the generation, then there will be less movement in factor prices in response to the relative size of each generation. The degree of this attenuation effect will depend on the effectiveness of education spending in producing human capital. We also demonstrate that this attenuation effect tends to concentrate generational consumption risk around the generation subject to the birth rate shock. In a limiting case, we show that an i.i.d birth rate shock translates into an i.i.d. generational consumption shock. In other words, each generation bears all of the risk associated with their own demographic uncertainty. As a final exercise, we demonstrate that if the tax rate funding education spending varies with the size of the generation rather than education spending per child, then human capital does not influence the dynamic behavior of the economy in response to a birth rate shock.

JEL Classification: J12, E21, I26, I31, J11

Keywords: human capital, consumption risk, factor price movements, fertility shocks

Corresponding author:
Patrick M. Emerson
School of Public Policy
Oregon State University
Corvallis, OR 97331
USA

E-mail: patrick.emerson@oregonstate.edu

Demographic Uncertainty and Generational Consumption Risk with Endogenous Human Capital

I. Introduction

Demographic shocks carry intergenerational risk. Those born into larger generations face lower prices for their relatively abundant labor and higher prices for relatively scarce capital with which to transfer wealth into retirement. Those born into smaller generations will find both a high price for their relatively scarce labor when working and lower prices for relatively abundant capital, making saving for retirement cheaper (Abel 2001; Bohn 2002, 2001, and 1997; and Easterlin 1987).[1] Thus, being born into a larger generation can result in lower consumption throughout life while being born into a smaller generation can lead to higher consumption. This well-known result has implications for systems of intergenerational risk sharing such as the Social Security program in the United States. The focus of this paper is to formally model a human capital response to generational size and study its impact on factor prices and consumption.

We begin by demonstrating that the addition of an education sector and human capital accumulation can influence factor price movements generated by a birth rate shock. When we *assume* there is an inverse relationship between education spending and generational size, the result is that a relatively smaller generation will be more productive in the labor force.[2] Since education spending attenuates changes in the effective labor force, this, in turn, attenuates changes to the return on capital investments and the effective wage rate. It is also shown that this attenuating effect applies to all current and future generations. This result is consistent with the findings of Ludwig et al. (2012), who study the effects of human capital in a model with overlapping generations.[3] We then formally demonstrate that the degree of attenuation depends

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[1] The debate about how demographic changes affect the return to capital and the price of capital can be found in Abel (2003, 2001) and Poterba (2001). The discussion in these papers is a little more nuanced than the simple argument presented in the introduction and body of this paper.

[2] If there is positive birth rate shock the opposite interpretation applies to the model. Clearly, the attenuation effects remain for factor price movements.

[3] Ludwig et al. (2012) use a large scale model with overlapping generations to study the welfare effects of two social security system types (constant replacement rate, constant tax rate) in presence of human capital accumulation. Our paper is different in the following ways. First, we look at the effects of education spending on human capital accumulation, whereas, Ludwig et al. focus on the time investment decision. Both affects are important to the human capital accumulation process. Second, we look at institutional design in terms of education spending rather than the type of social security system. Finally, we use a simple framework to highlight how the effectiveness of education spending determines the dynamic properties of the economy and the risk exposure of different generations to birth rate shocks. There are also a large number of papers studying the relationship between fertility, factor price

on the effectiveness of education spending in producing human capital. In our limiting case, where a 1% increase in education spending per child results in a 1% increase in individual productivity, we show that factor prices are independent of the birth rate shock for current and future generations. In effect, education spending and human capital completely "smooth out" factor price movements in response to a birth rate shock for all generations.

Given that education spending and human capital accumulation smooth out factor price movements following a birth rate shock, we then go on to demonstrate that this attenuation effect extends to the consumption profiles for all but one generation. Specifically, we show that the less movement there is in factor prices, the less movement we observe in the consumption profiles for all generations except the generation subject to the birth rate shock. This is perhaps the most important result in the paper. The addition of an education sector to the economy tends to concentrate the consumption risk around the generation subject to shock. In our limiting case, where a 1% increase in education spending per child results in a 1% increase in individual productivity, the consumption effect is completely concentrated on the generation subject to the shock. In other words, even with endogenous human capital accumulation and physical capital accumulation, an *i.i.d.* birth rate shock translates into an *i.i.d.* generational consumption shock.

Although we do not formally address the issue of optimal policy design, our results suggest that education and human capital accumulation has important implications for the design of intergenerational risk sharing programs. For example, the literature demonstrates that a defined benefit pay-as-you-go social security program, fully indexed to earnings, will reduce generational consumption risk by redistributing resources from the 'lucky' small generations to the 'unlucky' large generations. This argument, however, is driven by factor price movements following a birth rate shock (Bohn 2006, 2002, and 1997). We demonstrate that the introduction of education and human capital accumulation reduces factor price movements, which calls into question this line of reasoning. However, as we show, the less movement there is in factor prices implies education spending is more productive, meaning a smaller generation will have more human capital and a greater amount of total earnings and lifetime consumption while the reverse

movements and human capital that we do not cite because of space limitations. Ludwig and Vogel (2010) provide an excellent set of references.

is true for larger generations. Thus, redistribution from the 'lucky' small generations to the 'unlucky' large generations is still appropriate from an ex-ante risk sharing perspective. [4]

It is also worth noting here that our model only studies the effect of a birth rate shock on factor price movements and the consumption profiles of current and future generations. For example, if a relatively large generation (a baby-boom) is followed by a relatively small generation (a baby-bust) we show that there is a smaller decrease in the return on capital when the baby-boom generation retires. Thus, our model suggests that we might not observe a 'market meltdown' when the baby boomers retire. However, this factor price smoothing tends to concentrate consumption risk (movements) on each generation. The less movement we observe in the return on capital implies that a baby-boom generation will see a greater decline in lifetime consumption and a baby-bust generation will see a greater increase in lifetime consumption. These results may not directly extend to other types of aggregate risk in the economy, such as, longevity risk (Heijdra and Reijnders 2015; Bohn 2006 and 2002), productivity risk (Bohn 2009; Ball and Mankiw, 2007; Krueger and Kubler 2006; Gordon and Varian 1988; Enders and Lapan 1982)[5], or asset price (valuation) risk (Bohn 2009, 2006, and 2002; Abel 2003). However, the model does suggest that a productive education sector with endogenous spending per child can potentially influence the amount of risk each generation bears under different policy rules governing investment in the young and transfers to the elderly for these other forms of aggregate risk.

As a final exercise we look at an alternative policy where education spending does not respond to the size of the generation. Here, we assume that education spending per child remains constant across generations and the tax rate (or share of spending on children) systematically varies across generations in response to a birth rate shock. In contrast to the results above, human capital no longer affects the dynamic behavior of the economy off the balanced growth path. That is, education spending will no longer attenuate factor price movements. This implies that a larger generation will now be subject to the standard adverse factor price movements. A smaller generation will no longer benefit from an increase in human capital, but will benefit from factor

[4] As in Demange and Laroque (1999) the economy is Pareto efficient if the concept of interim efficiency (optimality) is used to evaluate welfare. Peled (1982) discusses the concept of interim efficiency as well. For an excellent discussion on the topic of ex-ante or interim efficiency, see Rangel and Zeckhauser (2001).

[5] There is a large literature on the risk sharing properties relating social security programs to productivity risk, so we only provide a few here for reference.

price movements. In effect, a model with fixed spending on each child is qualitatively equivalent to a model that omits the education sector.

This last result highlights an important property of models with human capital subject to birth rate shocks. If human capital is going to smooth out factor prices in response to a birth rate shock, then we must assume there is an inverse relationship between education spending and the size of the generation. However, as argued above, this will concentrate consumption movements on the generation subject to the shock. If, on the other hand, we assume that spending per child remains constant, then there is no factor price smoothing and there is less consumption movement for the generation subject to the shock. Thus, factor price smoothing and the concentration of the consumption risk on the generation subject to the shock are fundamentally tied together.[6] In models where households choose the optimal level of human capital for their children, as in Glomm and Kaganovich (2003) and Glomm and Ravikumar (1992), the answer lies somewhere in between. The amount of factor price smoothing will depend on how much households respond to the change in the fertility rate, but the consumption concentration effect will likely remain.

The paper proceeds as follows: Section II describes the general equilibrium framework. Section III derives the log-linear system of equations for the economy. Section IV derives a two-variable system of equations governing the dynamic properties of the economy when education spending per child is endogenous. This section also discusses the implications of the model under this fiscal policy rule. Section V demonstrates that the two variable system collapses to a single variable equation without human capital when the education tax rate is endogenous. Finally, section VI concludes. A majority of the algebra and one of the proofs are relegated to the appendix provided at the end of the paper.

II. The Model

For our formal analysis we utilize a simple stochastic three period model with overlapping generations that builds on the work of Bohn (2009, 2002, and 2001) and Abel (2003

[6]Determining which policy a society will choose is beyond the scope of the paper. The political economy literature related to social security design and education spending is vast. We cite a few here for reference. For the social security literature see Rangel and Zeckhauser (2001) and Boldrin and Rustichini (2000). For the education literature, see Glomm and Ravikumar (1992). For a political economy model including both sides of the policy ledger, see Pecchenino and Utendorf (1999).

and 2001).[7] In the first period of life each child receives an education and lives with his or her parents, where the number of children born each period is subject to a demographic disturbance (birth rate shock). In the second period the working age adult consumes, saves and pays taxes. In the third period of life the elderly do not work and consume all of their savings and any transfers from the government (social security). This setting allows us to derive a relatively simple system of equations that govern the dynamic properties of the economy and clearly demonstrates the role of education spending in our model.[8]

II.A. Households

Since the child lives in the parents' household the economic decision process begins during adulthood, or the working years. We assume consumption during the working years is a composite of household consumption between adults and children. The household maximizes the following utility function.

$$U(t) = \frac{1}{1-\frac{1}{\eta}}\left(c_W(t)^{1-\frac{1}{\eta}} + \beta\mu c_R(t+1)^{1-\frac{1}{\eta}}\right) \tag{1}$$

The parameter $\eta \in (0,1]$ measures the elasticity of intertemporal substitution, when $\eta = 1$ the period specific utility function takes on the natural log form. The parameter $\beta > 0$ is the standard discount rate. The parameter $\mu \in (0,1]$ is the survival probability, which implies savings and the social security payment are distributed to those who survive into old age. In addition, we assume perfect annuity markets.[9] The variables $c_W(t)$ and $c_R(t+1)$ measure consumption during the working years and consumption during the retirement years, respectively.

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[7] Our setup is also similar to those found in Boldrin and Montes (2009, 2005) and Pecchenino and Utendorf (1999), who employ a three period OG model with an education sector and a public pension system that highlights the linkage across generations.

[8] We do recognize that there is some loss of generality when compared to the multi-period models found in the literature based on the original of Auerbach and Kotlikoff (1987). However, we believe our work compliments this large scale framework. We also assume a closed economy in our model. Krueger and Ludwig (2007) develop a multi-period open economy model without education. We do not attempt this here because any cross-country open economy comparison would need to take into account the different fiscal policy rules with respect to education spending and social security design. For a discussion about international risk sharing in the context of social security alone, see Shiller (1999).

[9] However, the model could also assume that μ is a length of life parameter. In this case, savings and the social security payment are spread out over the remaining lifetime of the retiree. The results that follow do not depend on the interpretation. This additional parameter helps with the calibration of the model.

The household allocates disposable income between consumption and savings $s(t)$ during their working years.

(2) $$c_W(t) + s(t) = w(t)e(t-1)\big(1 - \tau_E(t) - \tau_S(t)\big).$$

Disposable income equals the product of the wage rate $w(t)$ and human capital $e(t-1)$ after paying for the education of the next generation and transfers to the current elderly. The household allocates a fraction of total earnings $\tau_E(t)$ to the education of the next generation of workers, the education tax rate, and a fraction of total earnings $\tau_S(t)$ to finance the consumption of current retirees, the social security tax rate.

Consumption during the retirement years for those who survive into old age equals $R(t+1)s(t)$ plus the social security payment $SS(t+1)$, where $R(t+1)$ is the gross return to capital.

(3) $$c_R(t+1) = \mu^{-1}\big(s(t)R(t+1) + SS(t+1)\big)$$

Under the assumption of fair annuities, the parameter μ distributes savings to the old age survivors. For simplicity, we also assume the government distributes tax revenue to old age survivors. The Euler equation below, given a fair annuities market, along with the period specific budget constraints, determine the optimal decision rules for the household.

(4) $$c_W(t)^{-1/\eta} = \beta R(t+1)c_R(t+1)^{-1/\eta}$$

II.B. Factor Payments

On the production side of the economy there are a large number of firms that operate in competitive output and factor markets. The model also assumes there are no capital adjustment costs and the depreciation rate for capital across generations equals one. The aggregate production technology is Cobb-Douglas, which also equals gross output.

(5) $$Y(t) = K(t)^{\alpha}[N_W(t)e(t-1)]^{1-\alpha}$$

The effective labor force equals the number of workers $N_W(t)$ multiplied by each worker's human capital. The aggregate capital stock is $K(t)$ and the parameter $\alpha \in (0,1)$ is the capital share of output.

The aggregate production technology results in the following factor payments,

$$w(t) = (1 - \alpha)k(t)^{\alpha}e(t-1)^{-\alpha} \tag{6}$$

$$R(t) = \alpha k(t)^{\alpha-1}e(t-1)^{1-\alpha}, \tag{7}$$

where $w(t)$ is the wage rate paid to each unit of human capital and $R(t)$ is the gross return on capital. Also note that equations (6) and (7) are expressed in terms of capital per worker, $k(t) = K(t)/N_W(t)$.

II.C. Demographics

The demographic properties of the economy are a function of trend population growth and a demographic disturbance. The trend component assumes that each worker has $n \geq 1$ children each period.

$$L_Y(t) = nL_W(t) \tag{8}$$

In the equation above, $L_W(t)$ measures the number of workers in period t and $L_Y(t)$ measures the number of children (young) in period t for a given rate of population growth. We also assume, for simplicity, that each child survives into the working period and only a fraction of workers μ survive into the retirement period. This implies the following demographic properties for each generation (along trend).

$$L_W(t+1) = L_Y(t) \tag{9}$$

$$L_R(t+2) = \mu L_W(t+1) \tag{10}$$

The variable $L_R(t+2)$ measures the number of surviving retirees.

To model the demographic disturbance, we assume that each generation is subject to an i.i.d. birth rate shock $\varepsilon(t) \sim N(0, \vartheta^2)$. This implies that each young generation in period t equals:

(11) $$N_Y(t) = L_Y(t)exp\{\varepsilon(t)\}.$$

That is, the number of young born in period t equals the number of young born along trend multiplied by the exponential function of the birth rate shock (which has a lognormal distribution).[10]

Assuming the same survival rates for the realized population, $N_Y(t) = N_W(t+1)$ and $N_R(t+2) = \mu N_W(t+1)$, we can derive the two key population ratios for our economy. The first ratio is the child-worker ratio $N_Y(t)/N_W(t)$, which measures the number of children relative to the number of workers in period t. This ratio can be found by substituting (8) into (11), $N_Y(t) = nL_W(t)exp\{\varepsilon(t)\}$. Next, backdate equation (11):

$$N_Y(t-1) = L_Y(t-1)exp\{\varepsilon(t-1)\} \Leftrightarrow N_W(t) = L_W(t)exp\{\varepsilon(t-1)\}.$$

Finally, take the ratio of the two equations above (equation (11) and the lag of equation (11)),

(12) $$\frac{N_Y(t)}{N_W(t)} = \frac{nL_W(t)exp\{\varepsilon(t)\}}{L_W(t)exp\{\varepsilon(t-1)\}} = nexp\{\varepsilon(t) - \varepsilon(t-1)\}.$$

Finally, backdating equation (12) and using the survival probability, gives us the second key population ratio, the worker-retiree ratio (the inverse of the old age dependency ratio in our model).

(13) $$\frac{N_W(t)}{N_R(t)} = \frac{N_Y(t-1)}{\mu N_W(t-1)} = \left(\frac{n}{\mu}\right) exp\{\varepsilon(t-1) - \varepsilon(t-2)\}$$

II.D. Education

Education spending per child determines the amount of human capital each child supplies to the labor market next period when they are of working age. There are many ways to specify

[10] This approach to modeling the demographic properties of the economy, using a trend component and a stochastic component, allows for mean reversion without assuming a negative correlation between birth rate shocks. For example, consider another approach where new births are: $N_Y(t)/N_W(t) = \omega_t$ and $\omega_t \geq 0$ is an i.i.d. shock to each generation. In this setup, if a large generation were to be followed by a smaller generation then this would require a smaller shock next period. That is, the correlation between $(\omega_{t+1}, \omega_t) < 0$.This point is made in Abel (2001).

this mapping from education spending to human capital, for the purpose of this paper we keep the production function simple.

(14) $$e(t) = Ah(t)^{\gamma} H(t)^{1-\gamma}$$

(15) $$H(t) = gH(t-1)$$

Equation (14) expresses human capital as a geometric weighted average of education spending per child $h(t)$ and the existing knowledge base $H(t)$. The knowledge base follows the exogenous process described by equation (15), where $g \geq 1$ represents the rate of knowledge growth. The parameter A is a technology index and the parameter $\gamma \in [0,1]$ is the elasticity of human capital with respect to education spending per child.

Since the education expenditure elasticity is the primary parameter of interest in the paper, it is appropriate to discuss its interpretation in a little more detail. Although the functional form in equation (14) is rather simple, this representation of the education technology can conceptually be thought of as an aggregator function that maps education spending into human capital. That is, the function above heuristically represents different education technologies by systematically varying the value of $\gamma \in [0,1]$. The closer γ is to zero the less effective is education spending at producing human capital and the closer γ is to one the more effective is education spending at producing human capital.[11]

To finance education each working household allocates the fraction $\tau_E(t)$ of income to the education process. This spending component can be thought of as a composite of public spending and private household spending on education, however, we refer this parameter as an education tax rate. The education budget constraint equates total revenue with total expenditures.

(16) $$N_W(t)w(t)e(t-1)\tau_E(t) = N_Y(t)h(t)$$

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[11] For example, let effective units of labor be determined by the more complex composite function $e(t) = f\left(\boldsymbol{x}(sp(t))\right)$. Here, the education technology $f(\cdot)$ combines a vector of inputs $\boldsymbol{x}(sp(t))$, where each input is a function of spending on that particular input. Now define $h(t) = \sum_i sp(i)$ as total spending on education. Next, take the total differential of the education technology with respect to spending, $de(t) = \sum_i f_i \frac{dx(i)}{dsp(i)} dsp(i)$, and define the education expenditure elasticity for the function $f(\cdot)$ as, $\gamma_f = \sum_i f_i \frac{dx(i)}{dsp(i)} dsp(i) \left(\frac{h(t)}{dh(t)e(t)}\right)$. This results in the following relationship for the general functional form: $\frac{dlne(t)}{dlnh(t)} = \gamma_f$.

To determine education spending per child, use the inverse of the young-worker ratio in equation (12) to express spending in terms of the birth rate shocks.

(17) $$h(t) = \left(\frac{1}{n}\right) w(t)e(t-1)\tau_E(t)exp\{\varepsilon(t-1) - \varepsilon(t)\}$$

This equation and the education technology determines how human capital will evolve over time.

II.E. Social Security

Transfers from the current working generation to current retirees occur through a Pay-As-You-Go (PAYGO) social security system. The model does not include a social security trust fund, thus the social security budget constraint is:

(18) $$N_R(t)SS(t) = N_W(t)w(t)e(t-1)\tau_S(t).$$

The left-hand side is total transfers to the elderly, which equals the product of number of current retirees, $N_R(t)$, and individual social security payments, $SS(t)$. The right hand-side is the total revenue available to fund the social security payments.

The social security payment to each retiree depends on the earnings of the current working generation, $w(t)e(t-1)$, that is, there is complete wage indexation.

(19) $$SS(t) = [w(t)e(t-1)]\theta(t)$$

The proportionality factor $\theta(t) \in [0,1]$ represents the average replacement rate, which is the fraction of earnings social security replaces during retirement.

To determine the tax rate necessary to fund social security, substitute (19) into (18), and then solve for the tax rate using the inverse of the worker-retiree ratio in equation (13).

(20) $$\tau_S(t) = \left(\frac{\mu}{n}\right) exp\{\varepsilon(t-2) - \varepsilon(t-1)\}\theta(t)$$

The tax rate depends on the birth rate shocks in the previous two periods.

II.F. Market Clearing

The market clearing condition is standard for most models with overlapping generations. Next period's capital stock, the capital that will be available for the current young, equals aggregate savings of the current working generation: $K(t+1) = N_W(t)s(t)$. Rewriting the aggregate equation in per worker form,

$$k(t+1) = \left(\frac{1}{n}\right) exp\{\varepsilon(t-1) - \varepsilon(t)\}s(t). \tag{21}$$

The equation above uses the relationship $\frac{K(t+1)}{N_W(t+1)} = \frac{N_W(t)}{N_W(t+1)}s(t) = \frac{N_W(t)}{N_Y(t)}s(t)$, along with equation (12).

III. The Log-Linear System of Equations

The system of equations above is nonlinear and does not allow for a closed form solution, so we will therefore follow the approach of Bohn (2009) and Campbell (1994) and log-linearize each equation around the steady state. This log-linear procedure allows for the derivation of simple impulse response functions that highlight the relative importance of the education expenditure elasticity in determining the “approximate” behavior of the economy following a demographic disturbance.

Since the detrending of the system and the steady state solution are standard in the literature we relegate this discussion to Appendix A, and turn our attention to the derivation of the log-linear system. To demonstrate the process, consider the human capital production function. First, take the natural logarithm of both sides of the production function (14) in stationary form and the steady state condition for this equation.

$$lne(t) = lnA + \gamma lnh(t)$$
$$lne = lnA + \gamma lnh$$

The variables without the time dimension represent the steady state in the stationary economy. Next, subtract the steady state from the time dependent equation.

(22) $$lne(t) - lne = \gamma(lnh(t) - lnh) \Leftrightarrow e^d(t) = \gamma h^d(t)$$

The new variables $e^d(t)$ and $h^d(t)$ represent deviations from the balanced growth path. Also note that the constant lnA drops out of the equation (which will apply to all constants in the system).

We now apply this procedure to the rest of the equations. Education spending per child takes on the following log-deviation form.

(23) $$h^d(t) = w^d(t) + e^d(t-1) + \tau_E^d(t) + \varepsilon(t-1) - \varepsilon(t)$$

The social security payment, the social security tax rate, factor payments, and the market clearing condition take on the following forms.

(24) $$SS^d(t) = w^d(t) + e^d(t-1) + \theta^d(t)$$

(25) $$\tau_S^d(t) = \theta^d(t) + (t-2) - \varepsilon(t-1)$$

(26) $$R^d(t) = (1-\alpha)e^d(t-1) - (1-\alpha)k^d(t)$$

(27) $$w^d(t) = \alpha k^d(t) - \alpha e^d(t-1)$$

(28) $$k^d(t+1) = s^d(t) + \varepsilon(t-1) - \varepsilon(t)$$

All of the equations, so far, have not required an approximation because they are already in log-linear form. However, this is not the case for the household period specific budget constraints, so we need to approximate these equations around the steady state. The log-linear form of each equation is as follows.

(29) $$\lambda_1 c_W^d(t) + (1-\lambda_1)s^d(t) = w^d(t) + e^d(t-1) - \lambda_2\tau_E^d(t) - \lambda_3\tau_S^d(t)$$

(30) $$c_R^d(t+1) = \lambda_4 R^d(t+1) + \lambda_4 s^d(t) + (1-\lambda_4)SS^d(t+1).$$

The parameter $\lambda_1 = [c_W/(c_W + s)]$ measures the fraction of income the household consumes after paying their social security taxes and investing in education in the steady state. The parameter $\lambda_2 = [\tau_E/(1-\tau_S-\tau_E)]$ represents the loss (gain) of income due to an increase (decrease) in current education spending. The parameter $\lambda_3 = [\tau_S/(1-\tau_S-\tau_E)]$ represents the loss (gain) of income due to an increase (decrease) in the current social security tax. The parameter $\lambda_4 = [Rs/(Rs+SS)]$ measures the share of retirement income from private savings. Finally, the Euler equation is also already in log-linear form so no approximation is required.

(31) $$c_W^d(t) = c_R^d(t+1) - \eta R^d(t+1)$$

For reference, the unrestricted system of equations is provided in **Table 1**.

IV. Competitive Equilibrium: Fiscal Policy Rule 1

To determine how education spending and human capital accumulation influence the dynamic properties of the economy, we must first define the fiscal policy rule describing how the government responds to a birth rate shock. This provides us with two additional restrictions that allow us to solve the system. The first fiscal policy rule assumes that the fraction of income a household allocates to education, or invests in the next generation, is constant. This implies that a relatively smaller generation will receive more education spending per child (obviously symmetry applies to a positive disturbance). Since there is more spending per child, this will increase human capital for this generation.

Fiscal Policy Rule 1: Education spending per child $h(t)$ is endogenous and responds to the state of the economy and the education expenditure share $\tau_E(t) = \tau_E$ is held constant. The replacement rate $\theta(t)$ is endogenous and responds to the state of the economy and the social security tax rate $\tau_S(t) = \tau_S$ is held constant.

As we demonstrate shortly, this fiscal policy rule attenuates movement in the effective labor force.

IV.A. System of Equations for Fiscal Policy Rule 1

Under this policy rule, we solve the system of equations in **Table 1** given $\tau_E^d(t) = 0$ for $\forall t$ and $\tau_S^d(t) = 0$ for $\forall t$. Specifically, we map this larger system of equations into a two-equation system expressed as a function of contemporaneous and past birth rate shocks, and lags of the two state variables. The solution is rather straightforward, but requires a good bit of algebra, so we relegate this derivation to Appendix B. The dynamic properties of the economy follow a VARMA process in terms of savings and human capital.

(32) $$s^d(t) = \Omega_{s,s1} s^d(t-1) + \Omega_{s,e1} e^d(t-1) + \Omega_{s,\varepsilon 0}\varepsilon(t) + \Omega_{s,\varepsilon 1}\varepsilon(t-1) + \Omega_{s,\varepsilon 2}\varepsilon(t-2)$$

(33) $$e^d(t) = \Omega_{e,s1} s^d(t-1) + \Omega_{e,e1} e^d(t-1) + \Omega_{e,\varepsilon 0}\varepsilon(t) + \Omega_{e,\varepsilon 1}\varepsilon(t-1) + \Omega_{e,\varepsilon 2}\varepsilon(t-2)$$

Here the coefficients Ω represent the partial elasticities of the state variables with respect to the appropriate influencing variables. The first $subscript = s, e$ defines the equation of interest (s for the savings equation and e for the human capital equation) and the second $subscript$ defines which variable the partial elasticity applies to in each equation. For example, $\Omega_{s,\varepsilon 0}$ measures the partial elasticity of savings with respect to a contemporaneous demographic shock and $\Omega_{e,e1}$ is the partial elasticity of human capital with respect to last period's human capital. **Table 2** and **Table 3** provide the parametric form for each partial elasticity.

IV.B. Implications of the Model

We begin with some illustrative calculations to demonstrate how different values of the education expenditure elasticity affect the key variables of interest in the economy, such as the return to capital, earnings, and the consumption profiles for each generation. Since this is a potentially contentious parameter, we simulate the economy using different values ranging from zero to one, $\gamma = \{0, 0.15, 0.5, 0.85, 1\}$, to represent the range of possible values in the literature. For Example, Ludwig et al. (2010) calibrate the elasticity as 0.6, based on the findings in Browning et al. (1999). This value is close to the estimate of 0.7 found in Jackson et.al. (2016), who find a 10% increase in per pupil spending each year for the entire twelve years of education results in a 7% increase in the total earnings of an individual. These values give us a sense of where this elasticity lies and a sense of the effectiveness of education spending in our model, which is somewhere between 0.5 and 0.85 in our simulation. We also show that when $\gamma = 0$ the model collapses to the standard model with overlapping generations (no education or human capital) and when $\gamma = 1$ the model generates endogenous growth. An interesting relationship between the growth rate of the economy and the birth rate arises from this model in the latter case. As we show in this section with our numerical examples, and prove in the next section, the birth rate shock cancels out as it works its way through the dynamic system. Thus, the demographic disturbance has no long-run growth effect on the economy in the endogenous growth setting.[12]

The calibration of the model uses standard parameter values found in the literature for the remaining parameters, so we don't spend any time on them here. The values can be found in **Table 4**. Also, for demonstrative purposes we simulate the economy using a negative 10% birth rate shock at time zero, $\varepsilon(0) = -10$. This implies that the current young generation (children) is 10% smaller than the current working generation (parents). Since the model is log-linear, a larger or smaller disturbance will only have scale effects, and if there is a positive birth rate shock the signs will change. Finally, we assume that all future shocks are zero and the economy is in the steady state prior to the shock. This allows us to trace out the impact effect and the impulse response function for each variable of interest.

Let's start with the results for the contemporaneous generations alive at the time of the demographic disturbance, or birth rate shock. The relevant contemporaneous generations are the current working and the current young generation (the current young generation is subject to the disturbance). We report the results for these generations in the second and third columns of Table 5 and Table 6 under the headings, Generation (-1) which represents the current working generation and Generation (0) which represents the current young.

Implication 1: The education expenditure elasticity attenuates the movement of factor prices for the contemporaneous generations for a given demographic disturbance. The closer the education expenditure elasticity is to one, the greater the degree of attenuation.

The reasoning is as follows. In the first box of Table 5, we see that human capital does not change for the current working generation (-1) in response to the birth rate shock for all values of γ. This is because this generation received their education in the previous period. The same result applies to the effective wage and earnings for the current working generation, presented in the second box of Table 5. However, when we turn our attention to the savings response of the current working generation and the return on savings (next period), we do see a

[12] This last result is sensitive to our policy rule. If the social security tax rate adjusts to the birth rate shock or complete real wage indexation is not in place this result does not hold in the endogenous growth case. The result also fails to hold for other demographic disturbances, such as longevity shocks.

response to the birth rate shock (third and fourth box in Table 5). For each value of $\gamma < 1$ we see that the current working generation increases their savings in response to the decrease in the return on capital (given our parameterization, the income effect dominates the substitution effect). We can also see that this response is attenuated by the education expenditure elasticity. That is, the more effective education spending is at producing human capital, the smaller the savings response to the birth rate shock.[13]

This last result is important because it potentially sheds some light on the movement of the stock market as the 'baby-boomers' start to retire. In our model we can interpret the current working generation as the baby-boom generation (a relatively larger generation) and the current young generation in our model as the baby-bust generation (a relatively smaller generation). Although one needs to be careful in directly relating the return on capital to the return on stocks, see Bohn (2001) and Abel (2001), our model suggests that the more productive education spending is at producing human capital, the less movement we will observe in the future return to capital if there is (was) an inverse relationship between the size of a generation and education spending per child.

We also observe this factor price smoothing effect for the current young generation (0) in Table 5 (these results are in the third column of each box). In the first box of table 5, we observe that human capital for this generation increases proportionally with the education expenditure elasticity. This is because the 'smaller' generation receives more spending per child. This increase in human capital attenuates the movement in the effective labor force, there are less workers, but each worker is more productive. Given that there is less movement in the effective labor force as the education expenditure elasticity increases, there is less movement in the return to capital and savings for current working generation next period. This implies there is less movement in the effective wage rate next period (listed as Wage in Box 2-Table 5). Note here that Earning and the wage rate paid to each unit of human capital are different measures. Earnings will actually increase as human capital increases for the current young during their working years, although the wage effect is muted. Also note that the return on capital for the current young (two periods ahead) is also attenuated by the effectiveness of education spending

[13] This result is consistent with the findings of Ludwig et al. (2012). They show that endogenous human capital attenuates the movement in the return on capital relative to the case where human capital is exogenous. Our results show how this attenuation critically depends on the effectiveness of education spending. We will also demonstrate shortly that this result also depends on a society's institutional design.

in producing human capital. This result (shown in Box 4- Table 5) is due to the fact that the current young's earnings will be higher during their working years, which will increase the amount of education funding available for the next period's young, Generation (1).

<u>Implication 2</u>: Although the education expenditure elasticity attenuates the movement of factor prices for the <u>contemporaneous</u> generations, it tends to concentrate the consumption effect on the generation subject to the demographic disturbance. That is, the more productive education spending is in the economy, the greater the consumption variation for the generation subject to the birth rate shock.

We report these results in Table 6. Here we see that the greater the education expenditure elasticity, the less movement in consumption we observe for the current working generation following the birth rate shock. The logic is simple. Since this generation's earnings are not affected by the demographic disturbance the only effect that matters is the response of savings to the movement in the rate of return on capital and the social security payment. Since the movement in the return on capital is inversely related to the education expenditure elasticity, there is less movement in savings and consumption over this generation's lifetime. Thus, education spending <u>does</u> attenuate movement in the consumption profile for the current working generation (the parents). However, this is not the case for the current young, or the generation directly subject to the birth rate shock. Here we observe that as the education expenditure elasticity increases, the consumption movement for this generation actually increases in magnitude. The logic here is, that as the education expenditure elasticity increases, there is less movement in factor prices (specifically the effective wage), but this relatively small generation is more productive so total earnings increase (see Earnings response in Table 5). This results in an increase in consumption over their lifetime. It is important to note here that for a positive birth rate shock all of the results would have the opposite sign, thus there would be larger consumption losses for the generation subject to the demographic disturbance.

Now let us turn our attention to the consumption profiles of the next two generations, or future generations, to get a sense of the long-run implications of the model. We report the results in columns (4) and (5) of Table 6 under the headings Generation (1) and Generation (2), respectively. First, we observe that as the education expenditure elasticity increases and approaches one, there is less movement in consumption over each future generation's lifetime. This is because the closer the education expenditure elasticity is to one, the less movement there is in the effective labor force and factor payments for all future generations (see Table 5). This argument is most clearly demonstrated when $\gamma = 1$.

Implication 3: When the education expenditure elasticity equals one ($\gamma = 1$), there is no movement of factor prices for all generations, and the consumption effect falls on the generation subject to the birth rate shock. In other words, the *i.i.d.* demographic disturbance translates to an *i.i.d* consumption disturbance.

The results under this assumption are presented in the bottom row of each Table (and Box). Here we see that the increase in human capital completely offsets the birth rate shock through education spending. That is, there is no change in the effective labor force when $\gamma = 1$. This, in turn, results in constant factor payments for all generations. The only generation that observes any movement in earnings and consumption is the generation subject to the birth rate shock. This generation will receive higher earnings because productivity is now higher (more human capital per worker), even though there is no movement in the effective wage rate. Future generations do not see any movement in factor payments or earnings because the following generation (immediately after the generation subject to the birth rate shock) returns to the balanced growth path. The increase in the relatively smaller generation's earnings exactly offsets the increase in the number of children in the subsequent period in terms of education spending per child. Finally, since the only generation that realizes a change in earnings is the generation subject to the disturbance, this will be the only generation that realizes a change in consumption.[14]

Combining all of the results above, we see that the education expenditure elasticity is a critical parameter in determining the effects of the demographic disturbance on factor price movements and the consumption profiles of current and future generations. As the elasticity

[14] There is a nonlinear relationship for some of the variables and the education expenditure elasticity for future generations. This is the result of the interaction between the relative importance of education spending and the relative size of the generations. Specifically, there are two effects for future generations following the demographic disturbance. The first is that the generations following the (negative) demographic shock will receive less spending on their respective educations, thus lowering their earnings because education spending matters. This effect will be stronger the larger the elasticity. However, the larger the elasticity, the greater the increase in earnings for the generation subject to the demographic disturbance will be, which increases spending per child.

increases in value, there is less movement in factor prices, but the consumption effect become more concentrated on the generation subject to the birth rate shock. This last result potentially has important risk sharing implications for any economy where education spending responds to the relative size of the generation. Although we do not attempt to derive any optimal risk sharing policy rules or assess the efficiency of the current policy rule, we can offer some insights into the potential design of the system when an education sector is included in the model.

If we evaluate policy from an ex-ante perspective and initially assume the education expenditure elasticity equals one, then any demographic shock will only affect the generation subject to the shock (Implication 3). This implies that each generation bears all of the risk associated with their own future demographic disturbance (or lottery). As argued in Gordon and Varian (1988), this generational idiosyncratic risk should be shared with all future generations. Thus, any optimal ex-ante policy design should pool the demographic risk across all future generations and drive the risk exposure of each generation to zero (all future generations should share the demographic risk with all previous generations, or, in other words, risk should be diversified over all future generations). This is consistent with the consumption metric proposed by Bohn (2006) to measure risk-sharing. Each generation should bear a share of the risk, proportional to their normal consumption opportunities, which implies all generations should see an infinitesimally small change in lifetime consumption. One possibility is to use optimal debt management policies to move consumption gains (or losses) across time. In the current period, debt could potentially be reduced through higher taxes on the relatively smaller generation (Bohn 2002). Another possibility is to further refine the social security program to share risk through a more robust trust fund management scheme.

This line of reasoning becomes more complicated when the education expenditure elasticity is below one. In this case the capital stock dynamics and the human capital dynamics propagate the *i.i.d.* demographic disturbance across all current and future generations, a result that is also found in Bohn (2009, 2002, and 2001), Gordon and Varian (1988), Ball and Mankiw(2007). However, we can see that the closer the education expenditure elasticity is to one, the greater the potential for risk sharing across generations. To evaluate (and derive a method for) the different policy rules, from an ex-ante risk sharing perspective, we would also need to look at the other side of the generational transfer program, social security, to fully evaluate any optimal policy design.

IV.C. Understanding the Simulation Results

To better understand the simulation results above, we now discuss a few key analytical results with respect to the education expenditure elasticity. Again we assume that the economy starts in the steady state and impose the following parameter restrictions:

(A1) Assumption 1: The economy starts in the non-linear system steady state.
(A2) Assumption 2: Assume that $\gamma \in [0,1]$ and $\eta \in (0,1)$.

Our first result demonstrates the key relationship between human capital and the demographic disturbance. For this proposition, and all that follow, we assume that $\varepsilon(0) \neq 0$ and $\varepsilon(i) = 0 \; for \; \forall i \neq 0$.

(P1) Proposition 1: *a.) There is an inverse relationship between human capital for the current young generation and the birth rate shock. b.) The greater the education expenditure elasticity, that is, the closer γ is to one, the more responsive human capital will be to the birth rate shock for the current young generation.*

Proof:
Follows directly from equation (33) and the definition of the response coefficient in Table 3. For reference, $\frac{\partial e^d(0)}{\partial \varepsilon(0)} = \Omega_{e,\varepsilon 0} = -\gamma < 0$ ■

As in our numerical example, a relatively smaller generation will receive more education spending per child and be relatively more productive. This effect is stronger the more effective education spending is in terms of producing human capital. The implication here is that there will be less movement in the effective labor force following a demographic disturbance the closer γ is to one.

Now let's look at the savings response of the current working generation following the birth rate shock.

(P2) Proposition 2: *a.) There is an inverse relationship between the current working generation's savings and the current period's birth rate shock. b.) The greater the education expenditure elasticity, that is, the closer γ is to one, the less responsive the current working generation's savings is to the current period's birth rate shock.*

Proof:

Follows directly from equation (32) and the definition of the response coefficient in Table 2. For reference, $\frac{\partial s^d(0)}{\partial \varepsilon(0)} = \Omega_{s,\varepsilon 0} = -(1-\gamma)\left(\frac{\lambda_5}{1-\lambda_5}\right) < 0$, where $\lambda_5 = \lambda_1(1-\alpha)(1-\eta) \in (0,1)$ given $\lambda_1 = \frac{c_w}{c_w+s}$, $\alpha \in (0,1)$ and $\eta \in (0,1)$ via (A2). Here we can see that as $\gamma \to 1$, we have $\frac{\partial s^d(0)}{\partial \varepsilon(0)} \to 0$. Thus, the more effective education spending is at producing human capital the less movement we see in savings.■

Why does this result hold? To answer this question we need to look at the return on capital.

(P3) Proposition 3: *a.) There is a positive relationship between the current working generation's return on capital next period (current savings) and the current period's birth rate shock. b.) The greater the education expenditure elasticity, that is, the closer γ is to one, the less responsive the current working generation's return on capital is to the current period's birth rate shock.*

Proof:

From the factor payment relationship in table 1, we have the following:

$$\frac{\partial R^d(1)}{\partial \varepsilon(0)} = -(1-\alpha)\frac{\partial k^d(1)}{\partial \varepsilon(0)} + (1-\alpha)\frac{\partial e^d(0)}{\partial \varepsilon(0)}.$$

From the capital accumulation equation (also in table 1), (P2) and (P1) we know:

$$\frac{\partial k^d(1)}{\partial \varepsilon(0)} = \frac{\partial s^d(0)}{\partial \varepsilon(0)} - 1;\ \frac{\partial s^d(0)}{\partial \varepsilon(0)} = -(1-\gamma)\left(\frac{\lambda_5}{1-\lambda_5}\right);\ \frac{\partial e^d(0)}{\partial \varepsilon(0)} = -\gamma.$$

Substituting these conditions into the return equation:

$$\frac{\partial R^d(1)}{\partial \varepsilon(0)} = -(1-\alpha)\left(\frac{\partial s^d(0)}{\partial \varepsilon(0)} - 1 - (-\gamma)\right) = -(1-\alpha)\left(-(1-\gamma)\left(\frac{\lambda_5}{1-\lambda_5}\right) - (1-\gamma)\right).$$

Finally, factor out $-(1-\gamma)$:

$$\frac{\partial R^d(1)}{\partial \varepsilon(0)} = (1-\alpha)(1-\gamma)\left(\frac{1}{1-\lambda_5}\right) > 0.$$

Here we can see that as $\gamma \to 1$, we have $\frac{\partial R^d(1)}{\partial \varepsilon(0)} \to 0$. Thus, the more effective education spending is at producing human capital the less movement we see in the return to capital. ■

By combining our results from the propositions above, we get a basic picture of how a birth rate shock will affect the initial economy and the contemporaneous generations. For example, in our simulation we assumed a negative demographic disturbance, which resulted in an increase in education spending per child (as in (P1)). This increase in education spending offsets the decrease in physical labor in the economy. There are now fewer workers relative to trend, but each worker is more productive. This, in turn, implies that we will see less movement in the return on capital (as in (P3)). Since there is less movement in the effective labor force the closer γ is to one (the more effective education spending is at producing human capital), the less movement we see in the capital to effective labor ratio. Given that there is less movement in the return on capital the closer γ is to one, there is less need to offset this lower return by saving more (as in (P2)). Recall that the elasticity of intertemporal substitution is less than one, so the income effect is greater than the substitution effect.

We can also see what happens when we set the intertemporal elasticity equal to one, $\eta = 1$ (relax assumption A2). In this case, there is no savings response to the demographic disturbance. This is because the income and substitution effects cancel out in the optimal decision rule for the household, or mathematically in (P2), $\lambda_5 = \lambda_1(1-\alpha)(1-\eta) = 0$. However, this result does not overturn the factor price movement implications of the model. From (P3), we can see the return on capital still responds to movements in the effective labor force. Also, the more effective education spending is at producing human capital the less movement we observe in the return on capital following a demographic disturbance. Thus, the main conclusions from the simulation do not qualitatively depend on the value of the elasticity of intertemporal substitution.

The last issue we want to address is what happens when the education expenditure elasticity equals one? Before answering this question, we need to understand how the education expenditure elasticity affects the effective wage rate for the generation subject to the demographic shock.

(P4) Proposition 4: *a.) There is an inverse relationship between the effective wage rate next period (current young's payment per unit of human capital next period) and the current period's birth rate shock. b.) The greater the education expenditure elasticity, that is, the closer* γ *is to one, the less responsive next period's effective wage rate is to the current period's birth rate shock.*

Proof:

From the factor payment relationship in table 1, we have the following:

$$\frac{\partial w^d(1)}{\partial \varepsilon(0)} = \alpha \frac{\partial k^d(1)}{\partial \varepsilon(0)} - \alpha \frac{\partial e^d(0)}{\partial \varepsilon(0)}.$$

From the capital accumulation equation (also in table 1), (P2), and (P1) we know:

$$\frac{\partial k^d(1)}{\partial \varepsilon(0)} = \frac{\partial s^d(0)}{\partial \varepsilon(0)} - 1; \ \frac{\partial s^d(0)}{\partial \varepsilon(0)} = -(1-\gamma)\left(\frac{\lambda_5}{1-\lambda_5}\right); \ \frac{\partial e^d(0)}{\partial \varepsilon(0)} = -\gamma.$$

Substituting these conditions into the wage equation:

$$\frac{\partial w^d(1)}{\partial \varepsilon(0)} = \alpha\left(\frac{\partial s^d(0)}{\partial \varepsilon(0)} - 1 - (-\gamma)\right) = \alpha\left(-(1-\gamma)\left(\frac{\lambda_5}{1-\lambda_5}\right) - (1-\gamma)\right).$$

Finally, factor out $-(1-\gamma)$:

$$\frac{\partial w^d(1)}{\partial \varepsilon(0)} = -\alpha(1-\gamma)\left(\frac{1}{1-\lambda_5}\right) < 0.$$

Here we can see that as $\gamma \to 1$, we have $\frac{\partial w^d(1)}{\partial \varepsilon(0)} \to 0$. Thus, the more effective education spending is at producing human capital the less movement we see in the effective wage rate. ■

With all of these results in hand, we now have the following proposition.

(P5) Proposition 5: *a.) If the human capital elasticity equals one* $(\gamma = 1)$*, then there is no movement in factor prices* $\left(w^d(\cdot),\ R^d(\cdot)\right) = (0{,}0)$ *for <u>all</u> generation. b.) If the human capital elasticity equals one* $(\gamma = 1)$*, then the change in the consumption profile for the current young generation is* $\left(c_W^d(1),\ c_R^d(2)\right) = \left(\varepsilon(0), \varepsilon(0)\right)$ *and the change in the consumption profile for all other generations is* $\left(c_W^d(\cdot),\ c_R^d(\cdot)\right) = (0{,}0)$*.*

Proof: In Appendix C.

This result proves Implication 3. That is, if the education expenditure elasticity equals one, then factor prices are independent of the demographic disturbance for all generations. This proposition also demonstrates that consumption for all generations is independent of the demographic disturbance except for the generation subject to the birth rate shock, which observes a one-to-one relationship between the shock and their consumption profile. In effect, given our assumption that the birth rate shock is an $i.i.d$ normal random variable, this implies that each generations consumption profile (in log-deviation form) is an $i.i.d$ normal random variable.[15]

V. Competitive Equilibrium: Policy Rule 2

In the previous section we demonstrated that when education spending per child is inversely related to the size of the generation there is an attenuation effect on factor price movements across the generations. We also demonstrated that the more effective education spending is at producing human capital the greater this attenuation effect will be and that the consumption effect becomes concentrated on the generation subject to the demographic disturbance. All of these results hinge critically on the assumption that there is an inverse relationship between education spending per child and the size of the generation (Policy Rule 1).

If we change this assumption to one where education spending per child is held constant across generations then the results change in a significant way. Specifically, we modify the Fiscal Policy Rule as follows:

Fiscal Policy Rule 2: The education expenditure share $\tau_E(t)$ is endogenous and responds to the state of the economy and education spending per child $h(t) = h$ is held constant. The replacement rate $\theta(t)$ is endogenous and responds to the state of the economy and the social security tax rate $\tau_S(t) = \tau_S$ is held constant.

[15] The results in Proposition 5 might be best thought of in terms of a limiting argument, $\gamma \rightarrow 1$, since the steady-state coefficient values depend on stationarity. However, if $\gamma = 1$, this would only change the coefficient values in the log-linear system found in Appendix C, assuming we expressed the variables in terms of stationary ratios (e.g. $k(t)/h(t)$) along the balanced growth path. A similar argument can be found in Campbell (1994), where he discusses issues surrounding non-stationarity in real business-cycle models.

Under this policy rule education spending per child does not change, which implies that human capital is the same for each generation (along trend). This is in contrast to the previous policy rule where human capital could deviate from the balanced growth path. Also note here that in both policy rules the replacement rate adjusts to meet current obligations for a given social security tax rate, which allows us to focus on education side of the economy.[16]

Given this new policy rule, the system collapses to one where there is only one state variable, savings. This result is easy to verify. In fact, this is one of the major advantages of the simplified model. From the education sector we have the following equation (Table 1).

(22) $$e^d(t) = \gamma h^d(t)$$

Given that education spending per child is held constant by assumption (Fiscal Policy Rule 2) for all generations, $h^d(t) = 0$ for $\forall t$, this implies that human capital is constant, $e^d(t) = 0$ for $\forall t$. Since human capital is constant we can drop the variable $e^d(t)$ from all of the equations in the system for every time period. We show in appendix D that the solution to the system (in Table 1) under Fiscal Policy Rule 2 takes on the following form.

(34) $$s^d(t) = \Omega_{s,s1} s^d(t-1) + \Omega_{s,\varepsilon 0}\varepsilon(t) + \Omega_{s,\varepsilon 1}\varepsilon(t-1) + \Omega_{s,\varepsilon 2}\varepsilon(t-2)$$

The dynamic behavior of the economy is now governed by a simple ARMA process rather than a two variable VAR model that includes human capital. In other words, although education spending enters the model and matters for the level variables (and other potential shocks not modeled), the response to a birth rate shock is absent. The partial elasticity coefficients are presented in Table 7. Also note that the education expenditure elasticity is absent from all of the coefficients. The argument above gives us the following result.

Implication 4: If education spending does not respond to the demographic disturbance then education spending will not attenuate factor price movements or concentrate the consumption effects on the generation subject to the demographic disturbance.

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[16] Obviously there are other combinations of partial adjustment policies that could be applied to the economy over the appropriate policy space. We do not address this issue in the current paper.

This result does not mean that the education sector of the economy is completely absent from the model, or, that the model is equivalent to a model with overlapping generations that excludes the education sector all together. This can be seen in equation (23). Although the human capital variable no longer enters any of the equations directly, the birth rate shock now affects the education tax rate.

$$\tau_E^d(t) = -w^d(t) - \varepsilon(t-1) + \varepsilon(t) \tag{35}$$

If there is a negative demographic shock in period t, this implies that the education tax rate for the current working generation will fall. This change in the tax rate will feedback into the system through savings, and therefore, affect movements in all of the other endogenous variables.

For completeness, we demonstrate how a birth rate shock affects the economy under fiscal policy rule 2. We once again assume a negative demographic shock $\varepsilon(0) = -10$ and present our results in Table 8. First, we observe that the earnings for the generation subject to the shock will increase during their working years because labor is relatively scarce and the relatively larger generation (parents) save more to offset the decline in their return on savings. The consumption profile for the generation subject to the shock is still higher and the consumption profile for all other generations is lower. Since the parameter γ does not matter, there is only one required simulation run for all values of education expenditure elasticity. These results are consistent with those found in Bohn (2006).

When looking at linkages across generations through fiscal policy rules that include education spending and social security, we not only are making an assumption about the education parameters in the model economy, such as the education expenditure elasticity, but also the policy rule itself. In effect, if we assume Fiscal Policy Rule 2 describes how society responds to a birth rate shock, we get the same *qualitative* results, and roughly the same quantitative results, as the standard OLG-Life Cycle models discussed in the first paragraph of the introduction. This last result introduces interesting political economy questions, as the ones addressed Boldrin and Rustichini (2000) who look at the social security side of political economy, Glomm and Ravikumar (1992) on the education side of political economy, and Iturbe-Ormaetxe and Valera (2012) who incorporate both sides of political economy ledger (education and pension spending). Which type of policy rule will a society choose before a given birth rate shock? This question is beyond the scope of the current paper. However, the answer will likely depend on the risk exposure properties of the policy, as well as growth, and the intergenerational distribution of resources discussed in the political economy literature.[17]

VI. Conclusion

In this paper we demonstrate that education spending can attenuate factor price movements in a model with overlapping generations *if* there is an inverse relationship between education spending per child and the relative size of the generation receiving an education. We also demonstrate that the degree of attenuation depends on the effectiveness of education spending at producing human capital. In the extreme case, where a one percent change in education spending per child results in a one percent change in human capital, factor prices are independent of the size any generation. We, also demonstrate that as the effectiveness of education spending increases in our model there is a consumption concentration effect on the generation subject to the demographic disturbance (birth rate shock). That is, the generation subject to the birth rate shock bears more of the consumption risk. Again, in our extreme case, the generation subject the demographic disturbance bears all of the consumption risk, and every other generations' lifetime consumption pattern remains unaffected. These results demonstrate that education spending and human capital can potentially 'smooth-out' factor price movements for all generations, but does not necessarily 'smooth-out' consumption movements for all generations.

We then go on to show that factor price movements and generational consumption patterns are not only sensitive to the effectiveness of education spending in producing human capital, but the policy rule itself. Specifically, we show that if society chooses a policy rule where education spending per child is held constant rather than a policy rule where there is an inverse relationship between the size of a generation and education spending per child, the dynamic properties of the economy are independent of education spending. In other words, if education spending per child does not respond to the relative size of the generation then education spending does not attenuate factor price movements or the consumption patterns for some generations. This last result demonstrates the importance of the policy rule assumptions in

[17] There is a relatively large literature on political economy issues surrounding education financing and social security policy design. The reader is referred to the references found in the papers listed above for additional sources on these topics.

a stylized economy attempting to evaluate the potential effects of human capital and education spending in a model subject to a demographic disturbance.

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Table 1

Log-Linear System of Equations: The representative variable is: $x^d(t) = lnx^s(t) - lnx^s$

Education Sector

$$e^d(t) = \gamma h^d(t)$$
$$h^d(t) = w^d(t) + e^d(t-1) + \tau_E^d(t) - \varepsilon(t) + \varepsilon(t-1)$$

Household

$$c_W^d(t) = c_R^d(t+1) - \eta R^d(t+1)$$
$$\lambda_1 c_W^d(t) + (1-\lambda_1)s^d(t) = w^d(t) + e^d(t-1) - \lambda_2 \tau_E^d(t) - \lambda_3 \tau_S^d(t)$$
$$c_R^d(t+1) = \lambda_4 R^d(t+1) + \lambda_4 s^d(t) + (1-\lambda_4)SS^d(t+1)$$

Factor Payments

$$w^d(t) = \alpha k^d(t) - \alpha e^d(t-1)$$
$$R^d(t) = -(1-\alpha)k^d(t) + (1-\alpha)e^d(t-1)$$

Social Security

$$SS^d(t) = w^d(t) + e^d(t-1) + \theta^d(t)$$
$$\tau_S^d(t) = \theta^d(t) - \varepsilon(t-1) + \varepsilon(t-2)$$

Market Clearing Condition

$$k^d(t+1) = s^d(t) - \varepsilon(t) + \varepsilon(t-1)$$

Table 2

Partial elasticity coefficients for the savings equation:

$$s^d(t) = \Omega_{s,s1} s^d(t-1) + \Omega_{s,e1} e(t-1) + \Omega_{s,\varepsilon 0} \varepsilon(t) + \Omega_{s,\varepsilon 1} \varepsilon(t-1) + \Omega_{s,\varepsilon 2} \varepsilon(t-2)$$

Coefficient	Value
$\Omega_{s,s1}$	$\frac{\alpha(1-\gamma\lambda_5)}{1-\lambda_5}$
$\Omega_{s,e1}$	$\frac{(1-\alpha)(1-\gamma\lambda_5)}{1-\lambda_5}$
$\Omega_{s,\varepsilon 0}$	$(-)\frac{\lambda_5(1-\gamma)}{1-\lambda_5}$
$\Omega_{s,\varepsilon 1}$	$(-)\left(1-\frac{(1-\alpha)(1-\gamma\lambda_5)}{1-\lambda_5}\right)$
$\Omega_{s,\varepsilon 2}$	$\frac{\alpha(1-\gamma\lambda_5)}{1-\lambda_5}$
λ_5	$\lambda_1(1-\alpha)(1-\eta)$
λ_4	$\frac{Rs}{Rs+SS}$
λ_3	$\frac{\tau_S}{1-\tau_E-\tau_S}$
λ_2	$\frac{\tau_E}{1-\tau_E-\tau_S}$
λ_1	$\frac{c_W}{c_W+s}$

Table 3
Partial elasticity coefficients for the human capital accumulation equation:
$$e^d(t) = \Omega_{e,s1} s^d(t-1) + \Omega_{s,e1} e(t-1) + \Omega_{e,\varepsilon 0}\varepsilon(t) + \Omega_{e,\varepsilon 1}\varepsilon(t-1) + \Omega_{e,\varepsilon 2}\varepsilon(t-2)$$

Coefficients	Values
$\Omega_{e,s1}$	$\gamma\alpha$
$\Omega_{e,e1}$	$\gamma(1-\alpha)$
$\Omega_{e,\varepsilon 0}$	$(-)\gamma$
$\Omega_{e,\varepsilon 1}$	$\gamma(1-\alpha)$
$\Omega_{e,\varepsilon 2}$	$\gamma\alpha$

Table 4
Parameter Values

Parameter Definitions	Value or Values
Capital share: α	0.3
Intertemporal elasticity: η	0.2
Trend population growth: n	1.02 annual
Knowledge growth: g	1.02 annual
Education Elasticity: γ	0; 0.15; 0.5; 0.85; 1
Gross rate of return: R	1.04 annual
The education spending rate: τ_E	0.10
The social security tax rate: τ_S	0.124
Survival probability (retirement period): μ	0.65

Discussion of Parameters: The parameters for all of the variables other than the education spending elasticity are standard. The social security tax rate is chosen to match the OASDI rate. The education tax rate matches relative expenditures on the young relative to retirees, given the OASDI rate (Edelstein et.al. 2016). Assuming a 30 year working period, the choice of the survival rate approximates a 20 year retirement period (on average). It is worth noting that the qualitative results do not depend on these parameter values. The values above result in a replacement rate of approximately 35%.

Table 5

This series of tables assumes a negative 10% demographic shock at time zero ($\varepsilon(0) = -10$). A positive demographic shock would change the sign on the values below.

Education (Human Capital)

	Generation(-1) (initial working)	Generation(0) (initial young)	Generation(1)	Generation(2)
$\gamma = 0$	0	0	0	0
$\gamma = 0.15$	0	1.5	-0.6568	-0.3692
$\gamma = 0.5$	0	5	-1.2877	-0.9561
$\gamma = 0.85$	0	8.5	-0.6568	-0.6060
$\gamma = 1$	0	10	0	0

Earnings Response

	Generation(-1) (initial working)		Generation(0) (initial young)		Generation(1)		Generation(2)	
	Wage	Total	Wage	Total	Wage	Total	Wage	Total
$\gamma = 0$	0	0	4.8491	4.8491	-2.4971	-2.4971	-1.2112	-1.2112
$\gamma = 0.15$	0	0	4.1218	5.6218	-1.8046	-2.4613	-1.0145	-1.3837
$\gamma = 0.5$	0	0	2.4246	7.4246	-0.6244	-1.9122	-0.4636	-1.4197
$\gamma = 0.85$	0	0	0.7274	9.2274	-0.0562	-0.7129	-0.0519	-0.6579
$\gamma = 1$	0	0	0	10	0	0	0	0

Savings Response

	Generation(-1) (initial working)	Generation(0) (initial young)	Generation(1)	Generation(2)
$\gamma = 0$	6.1637	1.6743	-4.0373	-1.9577
$\gamma = 0.15$	5.2392	3.3279	-3.7509	-2.1086
$\gamma = 0.5$	3.0819	6.6308	-2.5015	-1.8572
$\gamma = 0.85$	0.9246	9.1559	-0.7789	-0.7187
$\gamma = 1$	0	10	0	0

Return on Savings (Capital)

	Generation(-1) (initial working)	Generation(0) (initial young)	Generation(1)	Generation(2)
$\gamma = 0$	-11.3146	5.8280	2.8261	1.3704
$\gamma = 0.15$	-9.6174	4.2107	2.3672	1.3308
$\gamma = 0.5$	-5.6573	1.4570	1.0818	0.8032
$\gamma = 0.85$	-1.6972	0.1311	0.1210	0.1117
$\gamma = 1$	0	0	0	0

Table 6

This series of tables assumes a negative 10% demographic shock at time zero ($\varepsilon(0) = -10$). A positive demographic shock would change the sign on the values below.

Consumption: Working Years

	Generation(-1) (initial working)	Generation(0) (initial young)	Generation(1)	Generation(2)
$\gamma = 0$	-2.8880	6.3367	-1.7764	-0.8614
$\gamma = 0.15$	-2.4548	6.6965	-1.8571	-1.0440
$\gamma = 0.5$	-1.4440	7.7964	-1.6360	-1.2147
$\gamma = 0.85$	-0.4332	9.2608	-0.6820	-0.6294
$\gamma = 1$	0	10	0	0

Consumption: Retirement Years

	Generation(-1) (initial working)	Generation(0) (initial young)	Generation(1)	Generation(2)
$\gamma = 0$	-5.1509	7.5023	-1.2112	-0.5873
$\gamma = 0.15$	-4.3783	7.5387	-1.3837	-0.7779
$\gamma = 0.5$	-2.5754	8.0878	-1.4196	-1.0541
$\gamma = 0.85$	-0.7726	9.2871	-0.6579	-0.6070
$\gamma = 1$	0	10	0	0

Table 7

Partial elasticity coefficients for the savings equation under Social Contract 2:

$$s^d(t) = \Omega_{s,s1}s^d(t-1) + \Omega_{s,\varepsilon 0}\varepsilon(t) + \Omega_{s,\varepsilon 1}\varepsilon(t-1) + \Omega_{s,\varepsilon 2}\varepsilon(t-2)$$

Coefficient	Value
$\Omega_{s,s1}$	$\dfrac{\alpha(1+\lambda_2)}{1-\lambda_5}$
$\Omega_{s,\varepsilon 0}$	$(-)\left(\dfrac{\lambda_2 + \lambda_1(1-\alpha)(1-\eta)}{1-\lambda_5}\right)$
$\Omega_{s,\varepsilon 1}$	$\dfrac{\lambda_2 - (1+\lambda_2)\alpha + \lambda_1(1-\alpha)(1-\eta)}{1-\lambda_5}$
$\Omega_{s,\varepsilon 2}$	$\dfrac{\alpha(1+\lambda_2)}{1-\lambda_5}$

Table 8

This stable assumes a negative 10% demographic shock at time zero ($\varepsilon(0) = -10$) under Social Contract 2. A positive demographic shock would change the sign on the values below.

Earnings Response

	Generation(-1) (initial working)	Generation(0) (initial young)	Generation(1)	Generation(2)
Earnings	0	5.4740	-2.4775	-1.3562
Savings	8.2467	1.7416	-4.5207	-2.4746
Return on Savings	-12.7727	5.7809	3.1645	1.7322
Consumption: Working Years	-1.9715	6.3663	-1.9891	-1.0889
Consumption: Retirement Years	-1.0664	8.7742	-2.2300	-0.9218

Appendix A

To express the original system of equations in stationary form we detrend each variable by the knowledge available to each generation in their youth. Starting with the education technology, divide both sides of equation (14) by the amount of knowledge available at time t, which results in the following trend stationary equation for effective units of labor,

(A1) $$e^s(t) = Ah^s(t)^\gamma.$$

The superscript s denotes variables in their stationary form. For example, the variable $e^s(t) = e(t)/H(t)$ expresses human capital in terms of the amount of knowledge available to this particular generation.

To determine the stationary level of education spending per child divide both sides of equation (17) by $H(t)$ and use equation (15), $H(t+1)/H(t) = g$.

(A2) $$h^s(t) = \left(\frac{1}{ng}\right) w(t)e^s(t-1)\tau_E(t)exp\{\varepsilon(t-1) - \varepsilon(t)\}$$

The tax rate $\tau_E(t)$ is already in stationary form, so we do not use a superscript for this variable. This result also applies to the effective wage rate as we demonstrate shortly.

The next step is to detrend the working generation's decision problem. The children of period t become adults in period $t+1$, so the detrending variable for adults next period is also $H(t)$. This results in the following stationary Euler equation and period specific budget constraints.

(A3) $$c_W^s(t+1)^{-1/\eta} = \beta R(t+2)c_R^s(t+2)^{-1/\eta}$$

(A4) $$c_W^s(t+1) + s^s(t+1) = w(t+1)e^s(t)\big(1 - \tau_E(t+1) - \tau_S(t+1)\big)$$

(A5) $$c_R^s(t+2) = \mu^{-1}\big(R(t+2)s^s(t+1) + SS^s(t+2)\big)$$

For reference, the trend stationary variables for these equations are: $c_W^s(t+1) = c_W(t+1)/H(t)$, $c_R^s(t+2) = c_R(t+2)/H(t)$, $s^s(t+1) = s(t+1)/H(t)$, and $SS^s(t+2) = SS(t+2)/H(t)$.

Given the constant returns to scale production function and competitive markets assumption, factor payments are stationary. To see this last result, rewrite equations (6) and (7) as follows:

(A6) $$w(t+1) = (1-\alpha)k^s(t+1)^\alpha e^s(t)^{-\alpha}$$

(A7) $$R(t+1) = \alpha k^s(t+1)^{\alpha-1}e^s(t)^{1-\alpha}$$

Factor payments are now a function of the trend stationary human capital and trend stationary capital per worker, $k^s(t+1) = k(t+1)/H(t)$. Since human capital and physical capital per worker both grow at the same rate g along the balanced growth path, factor payments are stationary (the ratio $k(t+1)/e(t)$ is constant along the balanced growth path).

Applying the same process to the social security payment, the social security tax rate, and the market clearing condition we have the following stationary equations.

(A8) $$SS^s(t) = \mathrm{g}[w(t)e^s(t-1)]\theta(t)$$

(A9) $$\tau_S(t) = \left(\frac{\mu}{n}\right)\theta(t)exp\{\varepsilon(t-1) - \varepsilon(t-1)\}$$

(A10) $$k^s(t+1) = \left(\frac{1}{ng}\right)exp\{\varepsilon(t-1) - \varepsilon(t)\}s^s(t).$$

We can now use the stationary system of equations above to determine the steady state for the economy. First, assume a constant policy set $\{\tau_E, \tau_S\}$, where variables without t denote steady state values. Also assume that all shocks are zero, $\varepsilon(t) = 0$ for $\forall t$. Finally, normalize effective units of labor to one in the steady state. This last assumption allows the model to nest the standard two period economy with overlapping generations when education spending is completely ineffective. Given these assumptions we can solve for the steady state capital stock for a given value of the return to capital R using equation (A7). The steady state capital stock can then be used to find the effective wage in equation (A6) and household savings in equation (A10). The steady state effective wage rate, along with the policy set $\{\tau_E, \tau_S\}$, gives us the steady state replacement rate in equation (A9) and education spending per child in equation (A2). These values and the policy set can be used to find household consumption during the working period, equation (A4). Finally, the steady state social security payment can be found using equation (A8) and retirement consumption can be found using equation (A5). The free parameters are A in the education production function and β in the Euler equation. With this steady state we can log-linearize the system above to express the variables in log-deviation form, which we provide in Table 1 and discussed in the body of the paper.

Appendix B

This appendix solves the system of equations in Table 1 for policy rule 1. In this case, we set the policy variables $\tau_E^d(t) = 0$ and $\tau_S^d(t) = 0$ for all t. This implies education spending per child and the replacement rate adjust to any changes in the demographic composition of society.

Start with the Euler equation in deviation form (the equation numbering in this appendix is the same as in the body of the paper):

(31) $$c_W^d(t) = c_R^d(t+1) - \eta R^d(t+1).$$

Now, use equation (29) to express the left-hand side of the Euler equation as follows:

$$c_W^d(t) = \frac{1}{\lambda_1} w^d(t) + \frac{1}{\lambda_1} e^d(t-1) - \frac{1-\lambda_1}{\lambda_1} s^d(t).$$

Note that this equation imposes the policy restrictions under the first policy rule. Now eliminate the wage rate, (27) $w^d(t) = \alpha k^d(t) - \alpha e^d(t-1)$, and collect like terms:

$$c_W^d(t) = \frac{\alpha}{\lambda_1} k^d(t) + \frac{1-\alpha}{\lambda_1} e^d(t-1) - \frac{1-\lambda_1}{\lambda_1} s^d(t).$$

Finally eliminate capital, (28) lagged one period $k^d(t) = s^d(t-1) + \varepsilon(t-2) - \varepsilon(t-1)$, and collect like terms:

$$c_W^d(t) = -\frac{1-\lambda_1}{\lambda_1} s^d(t) + \frac{\alpha}{\lambda_1} s^d(t-1) + \frac{1-\alpha}{\lambda_1} e^d(t-1) - \frac{\alpha}{\lambda_1} \varepsilon(t-1) + \frac{\alpha}{\lambda_1} \varepsilon(t-2).$$

Now, use equation (30) to express the right-hand side of the Euler equation as follows:

$$c_R^d(t+1) - \eta R^d(t+1) = \lambda_4 R^d(t+1) + \lambda_4 s^d(t) + (1-\lambda_4) SS^d(t+1) - \eta R^d(t+1).$$

Eliminate the social security payment using equations (24) and (25), updated one period, $SS^d(t+1) = w^d(t+1) + e^d(t) + \varepsilon(t) - \varepsilon(t-1)$ and collect like terms.

$$\ldots = (\lambda_4 - \eta) R^d(t+1) + \lambda_4 s^d(t) + (1-\lambda_4)\left(w^d(t+1) + e^d(t) + \varepsilon(t) - \varepsilon(t-1)\right)$$

Eliminate factor payments using equations (27) and (28), updated one period, and collect like terms.

$$\begin{aligned} \ldots = \left(\alpha(1-\eta) - (\lambda_4 - \eta)\right) k^d(t+1) + (1-\alpha)(1-\eta) e^d(t) + \lambda_4 s^d(t) + (1-\lambda_4)\varepsilon(t) \\ - (1-\lambda_4)\varepsilon(t-1) \end{aligned}$$

Finally, eliminate capital and collect like terms.

$$\begin{gathered} c_R^d(t+1) - \eta R^d(t+1) = (\alpha(1-\eta) + \eta) s^d(t) \\ +(1-\alpha)(1-\eta) e^d(t) + (1-\alpha)(1-\eta)\varepsilon(t) - (1-\alpha)(1-\eta)\varepsilon(t) \end{gathered}$$

Use the Euler equation, equating both sides using the equations above, and solve for current period savings, $s^d(t)$:

(B1) $$\begin{aligned} s^d(t) = -\left(\frac{\lambda_5}{1-\lambda_5}\right) e^d(t) + \left(\frac{\alpha}{1-\lambda_5}\right) s^d(t-1) + \\ \left(\frac{1-\alpha}{1-\lambda_5}\right) e^d(t-1) + \left(\frac{\lambda_5}{1-\lambda_5}\right)\varepsilon(t) - \left(\frac{\lambda_5 - \alpha}{1-\lambda_5}\right)\varepsilon(t-1) + \left(\frac{\alpha}{1-\lambda_5}\right)\varepsilon(t-2), \end{aligned}$$

where $\lambda_5 = \lambda_1(1-\alpha)(1-\eta)$.

We now solve the system for our second state variable, human capital. First substitute equation (23) into equation (22).

$$e^d(t) = \gamma\left(w^d(t) + e^d(t-1) + \varepsilon(t-1) - \varepsilon(t)\right)$$

Next, eliminate the effective wage rate and collect like terms.

(B2) $$e^d(t) = \gamma\alpha s^d(t-1) + \gamma(1-\alpha) e^d(t-1) - \gamma\varepsilon(t) + \gamma(1-\alpha)\varepsilon(t-1) + \gamma\alpha\varepsilon(t-1)$$

Equations (B1) and (B2) result in a block recursive system of equations. To express this system in terms of a VAR model, substitute (B2) into (B1). This gives us equations (32) and (33) in the body of the paper. The partial elasticities (reduced form coefficients) of the system are provided in Table 2 and Table 3.

Appendix C

This appendix provides the proof for proposition 4. First, set $\gamma = 1$ for each of the coefficients found in Table 2 and Table 3. This results in the following system of equations governing the two key state variables, savings and human capital ($s^d(t)$ and $e^d(t)$).

(C1) $\quad s^d(t) = \alpha s^d(t-1) + (1-\alpha)e^d(t-1) - \alpha\varepsilon(t-1) + \alpha\varepsilon(t-2)$

(C2) $\quad e^d(t) = \alpha s^d(t-1) + (1-\alpha)e^d(t-1) - \varepsilon(t) + (1-\alpha)\varepsilon(t-1) + \alpha\varepsilon(t-2)$

Next, conjecture a demographic shock $\varepsilon(0) \neq 0$ and assume that the system starts in the steady state at the time of the shock (time period zero).

Given this set of assumptions, the sequence for each of the state variables can be found using equations (C1) and (C2).

Period 0

$$s^d(0) = \alpha s^d(-1) + (1-\alpha)e^d(-1) - \alpha\varepsilon(-1) + \alpha\varepsilon(-2) = 0$$

$$e^d(0) = \alpha s^d(-1) + (1-\alpha)e^d(-1) - \varepsilon(0) + (1-\alpha)\varepsilon(-1) + \alpha\varepsilon(-2) = -\varepsilon(0)$$

Period 1

$$s^d(1) = \alpha s^d(0) + (1-\alpha)e^d(0) - \alpha\varepsilon(0) + \alpha\varepsilon(-1) = -(1-\alpha)\varepsilon(0) - \alpha\varepsilon(0) = -\varepsilon(0)$$

$$e^d(1) = \alpha s^d(0) + (1-\alpha)e^d(0) - \varepsilon(1) + (1-\alpha)\varepsilon(0) + \alpha\varepsilon(-1) = -(1-\alpha)\varepsilon(0) + (1-\alpha)\varepsilon(0) = 0$$

Period 2

$$s^d(2) = \alpha s^d(1) + (1-\alpha)e^d(1) - \alpha\varepsilon(1) + \alpha\varepsilon(0) = -\alpha\varepsilon(0) + \alpha\varepsilon(0) = 0$$

$$e^d(2) = \alpha s^d(1) + (1-\alpha)e^d(1) - \varepsilon(2) + (1-\alpha)\varepsilon(1) + \alpha\varepsilon(0) = -\alpha\varepsilon(0) + \alpha\varepsilon(0) = 0$$

Period $t \geq 3$

$$s^d(t) = 0 \; and \; e^d(t) = 0$$

Given the sequence for savings, the capital stock per worker sequence is as follows (See Table 1).

Period 0

$$k^d(0) = s^d(-1) - \varepsilon(-1) + \varepsilon(-2) = 0$$

Period 1

$$k^d(1) = s^d(0) - \varepsilon(0) + \varepsilon(-1) = -\varepsilon(0)$$

Period 2

$$k^d(2) = s^d(1) - \varepsilon(1) + \varepsilon(0) = -\varepsilon(0) + \varepsilon(0) = 0$$

Period $t \geq 3$

$$k^d(t) = 0$$

Given the capital stock per worker sequence and the sequence for human capital, the factor price sequences are as follows (See Table 1).

Period 0

$$w^d(0) = \alpha k^d(0) - \alpha e^d(-1) = 0$$

$$R^d(0) = -(1-\alpha)k^d(0) + (1-\alpha)e^d(-1) = 0$$

Period 1

$$w^d(1) = \alpha k^d(1) - \alpha e^d(0) = -\alpha\varepsilon(0) + \alpha\varepsilon(0) = 0$$

$$R^d(1) = -(1-\alpha)k^d(1) + (1-\alpha)e^d(0) = (1-\alpha)\varepsilon(0) - (1-\alpha)\varepsilon(0) = 0$$

Period 2

$$w^d(2) = \alpha k^d(2) - \alpha e^d(1) = 0$$

$$R^d(2) = -(1-\alpha)k^d(2) + (1-\alpha)e^d(1) = 0$$

Period $t \geq 3$

$$w^d(t) = 0 \; and \; R^d(t) = 0$$

This last result demonstrates the complete factor price smoothing effect when $\gamma = 1$ under policy rule 1. Thus, proving part (a) of Proposition 4. Before determining the consumption profiles for each generation, we need the social security payment sequence (See Table 1).

Period 0

$$SS^d(0) = w^d(0) + e^d(-1) + \varepsilon(-1) - \varepsilon(-2) = 0$$

Period 1

$$SS^d(1) = w^d(1) + e^d(0) + \varepsilon(0) - \varepsilon(-1) = -\varepsilon(0) + \varepsilon(0) = 0$$

Period 2

$$SS^d(2) = w^d(2) + e^d(1) + \varepsilon(1) - \varepsilon(0) = -\varepsilon(0)$$

Period 3

$$SS^d(3) = w^d(3) + e^d(2) + \varepsilon(2) - \varepsilon(1) = 0$$

Period $t \geq 4$

$$SS^d(t) = 0$$

Given the factor price sequences and the social security payment sequence, the consumption sequences are as follows (See Table 1).

Period 0

$$c_w^d(0) = \frac{1}{\lambda_1} w^d(0) + \frac{1}{\lambda_1} e^d(-1) - \frac{1-\lambda_1}{\lambda_1} s^d(0) = 0$$

$$c_R^d(0) = \lambda_4 R^d(0) + \lambda_4 s^d(-1) + (1-\lambda_4) SS^d(0) = 0$$

Period 1

$$c_w^d(1) = \frac{1}{\lambda_1} w^d(1) + \frac{1}{\lambda_1} e^d(0) - \frac{1-\lambda_1}{\lambda_1} s^d(1) = -\frac{1}{\lambda_1}\varepsilon(0) + \frac{1-\lambda_1}{\lambda_1}\varepsilon(0) = -\varepsilon(0)$$

$$c_R^d(1) = \lambda_4 R^d(1) + \lambda_4 s^d(0) + (1-\lambda_4) SS^d(1) = 0$$

Period 2

$$c_w^d(2) = \frac{1}{\lambda_1} w^d(2) + \frac{1}{\lambda_1} e^d(1) - \frac{1-\lambda_1}{\lambda_1} s^d(2) = 0$$

$$c_R^d(2) = \lambda_4 R^d(2) + \lambda_4 s^d(1) + (1-\lambda_4) SS^d(2) = -\lambda_4\varepsilon(0) - (1-\lambda_4)\varepsilon(0) = -\varepsilon(0)$$

Period 3

$$c_w^d(3) = \frac{1}{\lambda_1} w^d(3) + \frac{1}{\lambda_1} e^d(2) - \frac{1-\lambda_1}{\lambda_1} s^d(3) = 0$$

$$c_R^d(3) = \lambda_4 R^d(3) + \lambda_4 s^d(2) + (1-\lambda_4) SS^d(3) = 0$$

Period $t \geq 4$

$$c_w^d(t) = 0 \; and \; c_R^d(t) = 0$$

Thus, the consumption profile for each generation takes the following form:

Initial Old: $c_R^d(0) = 0$

Initial Working Generation (Parents): $c_w^d(0) = 0 \; and \; c_R^d(1) = 0$

Children subject to Demographic Shock: $c_w^d(1) = -\varepsilon(0) \; and \; c_R^d(2) = -\varepsilon(0)$

All Future Generations ($t \geq 2$): $c_w^d(t) = 0 \; and \; c_R^d(t+1) = 0$

Thus, under policy rule 1, the demographic shock is borne by the generation subject to the shock when $\gamma = 1$ and all other generations are left unaffected. This is part (b) in Proposition 4. ■

Appendix D

This appendix solves the system of equations in Table 1 for policy rule 2. In this case, we set the policy variables $h^d(t) = 0$ and $\tau_S^d(t) = 0$ for all t. This implies that education spending per child remains constant for each generation and the education tax rate adjusts to any changes in the demographic composition of society.

From equation (22) and (23) in the body of the paper, we have,

(22') $\quad e^d(t) = \gamma h^d(t) = 0$

(23') $\quad 0 = w^d(t) + \tau_E^d(t) + \varepsilon(t-1) - \varepsilon(t),$

which demonstrates that the education component drops out of the model. To solve the system for the one remaining state variable, savings, we follow the same procedure as in appendix B. First, start with the left-hand side of the Euler equation,

$$c_W^d(t) = \frac{1}{\lambda_1} w^d(t) - \frac{\lambda_2}{\lambda_1}\tau_E^d(t) - \frac{1-\lambda_1}{\lambda_1} s^d(t).$$

Eliminate the education tax rate (23') and collect like terms

$$c_W^d(t) = \frac{1+\lambda_2}{\lambda_1} w^d(t) - \frac{\lambda_2}{\lambda_1}\varepsilon(t) + \frac{\lambda_2}{\lambda_1}\varepsilon(t-1) - \frac{1-\lambda_1}{\lambda_1} s^d(t).$$

Now eliminate the factor payment and collect like terms,

$$c_W^d(t) = \frac{(1+\lambda_2)\alpha}{\lambda_1} k^d(t) - \frac{\lambda_2}{\lambda_1}\varepsilon(t) + \frac{\lambda_2}{\lambda_1}\varepsilon(t-1) - \frac{1-\lambda_1}{\lambda_1} s^d(t).$$

Finally, eliminate capital using the market clearing equation and collecting like terms,

$$c_W^d(t) = -\frac{1-\lambda_1}{\lambda_1} s^d(t) + \frac{(1+\lambda_2)\alpha}{\lambda_1} s^d(t-1) - \frac{\lambda_2}{\lambda_1}\varepsilon(t) + \frac{\lambda_2 - (1+\lambda_2)\alpha}{\lambda_1}\varepsilon(t-1) + \frac{(1+\lambda_2)\alpha}{\lambda_1}\varepsilon(t-2).$$

Now solve for savings on the right-hand side of the Euler equation. Again, the solution process is equivalent to the one found in appendix B (minus the education variable):

$$c_R^d(t+1) - \eta R^d(t+1) = (\alpha(1-\eta)+\eta)s^d(t) + (1-\alpha)(1-\eta)\varepsilon(t) - (1-\alpha)(1-\eta)\varepsilon(t).$$

Equating both sides of Euler equation and solving for savings in period t results in the equation of motion for the savings variable (equation (34)). The coefficients are provided in Table 7.