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On the Mach-Uniformity of Lagrange-Projection Scheme

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ON THE MACH-UNIFORMITY OF LAGRANGE–PROJECTION SCHEME

HAMED ZAKERZADEH

Abstract. In the present work, we show that the Lagrange–projection scheme presented in Coquel et al.'s paper (Math. of Comp. **79**.271 (2010): 1493–1533), is asymptotic preserving for isentropic Euler equations, i.e. at the discrete level it preserves the incompressible limit, satisfies the *div*-free condition as well as the asymptotic expansion for the density in the continuous level. Moreover, we prove that the scheme is positivity-preserving, L_∞-stable and entropy-admissible under some Mach-uniform restrictions. The analysis is similar to what has been presented in the original paper, but with the emphasis on the uniformity regarding the Mach number.

1. Introduction

Studying singular limits of conservation laws (or more generally PDEs), may result in severe difficulties to be treated either in analysis or numerics. The main issue is that the type of the equations changes in the limit [29], e.g. when Mach number approaches zero for Euler equations. This limit is singular, since the sound speed (the characteristic speed) goes to infinity and the PDE changes to be elliptic, in the so-called incompressible limit. So, there are difficulties to show the convergence of the solution of compressible Euler equations to the incompressible one (see [24, 29]). Tackling this problem numerically is more complicated, since as the eigenvalues of the flux Jacobian blow up, the time step should tend to zero due to Courant–Friedrichs–Lewy (CFL) condition, which leads to very small time steps and thus huge computational cost. Also it has been shown that in the general case, the usual numerical schemes, lose their accuracy in the limit for under-resolved mesh size; see [13, 14, 17, 16, 32, 33, 31].

Throughout this paper, we assume that at least in the continuous level, the *solution* of compressible flow equations corresponds to Mach number ϵ, converges to the solution of the limit equation, as $\epsilon \to 0$, and try to show that the counterpart of such convergence also exists (at least formally) in the discrete level. It means that the asymptotic limit for the computed solution should be analogous to the limit of continuous system, e.g. the div-free condition holds for the scheme in the limit, or the zeroth-order term in the asymptotic expansion of the density should be constant at the discrete level, since as has been shown in [30] (for single scale analysis, also see [25] for multiple scale analysis), it is the case for the PDE level. This is so-called Asymptotic Preserving (AP) consistency, which has been

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$$\begin{tikzcd} \mathcal{M}_\Delta^\epsilon \arrow[r, "\epsilon \to 0"] \arrow[d, "\Delta \to 0"'] & \mathcal{M}_\Delta^0 \arrow[d, "\Delta \to 0"] \\ \mathcal{M}^\epsilon \arrow[r, "\epsilon \to 0"'] & \mathcal{M}^0 \end{tikzcd}$$

FIGURE 1. Illustration of Asymptotic Preserving schemes.

introduced by Jin in [21] for relaxation systems; see also [22] for a general review. This can be translated to the consistency of the scheme with the underlying PDE in the limit of the parameter. But to be practically useful, the scheme should also satisfy AP stability, that the stability of the computed solution is guaranteed, i.e. the computed solution is bounded in some suitable norm, under some ϵ-uniform conditions. We denote that a scheme is AP if it is *AP stable* and *AP consistent*.

Figure 1 illustrates this definition; $\mathcal{M}^\epsilon$ stands for a continuous physical model with perturbation parameter ϵ, and $\mathcal{M}_\Delta^\epsilon$ is a discrete-level model which provides a consistent discretization of $\mathcal{M}$. If the stability condition of $\mathcal{M}_\Delta^\epsilon$ is independent of ϵ and its limit $\mathcal{M}_\Delta^0$ provides a consistent discretization of $\mathcal{M}^0$, then the scheme is called AP; see [9] .

This property has been studied widely and several AP schemes have been developed for Euler or shallow water equations; see [9, 12, 18]. The bottom line of these schemes is mixed implicit-explicit idea, stems from the more general *operator splitting* method; to split the flux (or its Jacobian) into two parts and treat one part explicitly in time and the other one implicitly in time (IMEX schemes). This approach is definitely necessary to find schemes with ϵ-uniform CFL conditions. But as mentioned in [11], it is not sufficient at all to obtain AP stability; see for example [1] whence it is shown that for explicit-explicit splitting with Lax–Wendroff scheme, even if both splitted parts are stable in terms of CFL condition, the resulting scheme is unconditionally unstable in L_2-norm using Von Neumann stability analysis. On the other hand, it is shown in [18] that for IMEX splitting if each part is L_2-stable, the overall scheme is L_2-stable as well. So, there is a critical gap between these two cases. Also note that using IMEX splitting schemes, makes the analysis more delicate compared to explicit splittings; see [4, 5] for some results.

In this paper, the main point is to study the issue of stability of IMEX splittings for some specific splitting; the idea comes from Arbitrary Lagrangian-Eulerian (ALE) approach. ALE nowadays is a classic approach in mechanics, trying to benefit from advantages of Eulerian and Lagrangian formulations simultaneously; see [19] for a nice introduction. It has been introduced in the analysis of splitting schemes in [8] for a two-phase model, without any concern about the incompressible limit, and later used in [4, 5] for Euler equations accompanied with a friction term and two-dimensional Euler equations for all-Mach flows. Considering this approach, it has been proved in [8] that the Lagrange–projection scheme is positivity preserving and entropy stable, under some conditions. On the other hand, it is well-known that Godunov-type schemes (of which Lagrange–projection scheme is a member) show no accuracy problem for low-Mach one-dimensional problems as long as the initial condition is *well-prepared* (see Definition 3.3). The reader can consult with [13, 32, 14, 33, 5] for more details. This justifies the motivation to investigate the results of [8] in the low-Mach limit. In fact here, we show that the stability conditions in [8] are uniform in ϵ provided that the initial condition is well-prepared. So, all those stability properties hold without any restriction regarding Mach number. Also we show that the solution is stable in L_∞-norm for well-prepared initial conditions. There are also some stability results in [5] for unstructured grids, with the focus on the accuracy problem of Godunov-type schemes in low Mach numbers but with the modification of Lagrange–projection scheme implied by a careful look at the truncation error.

The plan of the paper is as follows: In section 2 we introduce the ALE splitting, after a brief explanation on ALE formalism and relaxation schemes. Then, in section 3, we launch the numerical analysis of the scheme, starting from AP property, continuing with positivity preserving, stability and entropy stability. We then conclude the discussion with some immediate extensions and future works.

2. Lagrange–projection scheme: Continuous PDE level

In this section, we introduce the splitting to be used, so-called *ALE splitting*, implied by classic Lagrange–projection scheme (see [15]). Let us suppose isentropic Euler equations. One natural way to split the waves, is to split them into acoustic and transport waves. The high speed acoustic waves are formulated in Lagrangian framework and slow transport waves in Eulerian one. The framework has been introduced in [15, Chapter III, section 2.2] as Lagrange–projection scheme, which consists of solving Riemann problems in the Lagrangian formulation, and then projecting the computed solution into the fixed Eulerian grid. Thus there are two steps, Lagrangian and the projection. The former is usual Lagrangian form of isentropic Euler equations, and the latter is actually the projection of the computed solution; see [15, Chapter III, section 2.5].

It has been shown in [8] that such a scheme can also be conceived in ALE framework, to write the equations in *referential* coordinates χ which are necessarily neither spatial (Eulerian) x nor material (Lagrangian) X. Referential frame has a relative velocity v seen from spatial frame, which is arbitrarily chosen. Note that the Lagrange–projection scheme is a special case of ALE, in which the velocity v is chosen such that after completing each step, the domain is the same as fixed Eulerian one. Intending not to be lengthy, we refer the reader to [8, Section 3.3] for more details.

Now, consider the system of isentropic Euler equations:

$$\partial_t\rho + \partial_x(\rho u) = 0, \tag{2.1}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \tag{2.2}$$

when $p(\rho) = \kappa\rho^\gamma$ with $\kappa > 0$ and $\gamma > 1$ is the isentropic pressure law. As an entropy function, we chose the total energy of the solution ρE which can be shown to be strictly convex with respect to the conservative variables. The total energy density is written as $E = \mathcal{E} + \frac{u^2}{2}$ where $\mathcal{E}(\rho) := \frac{\kappa}{\gamma-1}\rho^{\gamma-1}$ is internal energy density (see [27]).

Then, the Lagrange–projection idea is to approximate the solution of the original system, (2.1)-(2.2), using the Lie splitting idea and to solve the following acoustic and transport subsystems, successively:

$$\partial_t \rho + \rho \partial_x u = 0, \tag{2.3}$$

$$\partial_t(\rho u) + \rho u \partial_x u + \partial_x p = 0, \tag{2.4}$$

and

$$\partial_t \rho + u \partial_x \rho = 0, \tag{2.5}$$

$$\partial_t(\rho u) + u \partial_x(\rho u) = 0. \tag{2.6}$$

Simply by using Taylor expansion it can be seen that this splitting is in general (globally) first-order accurate in time. We refer the reader to [20] for more details about the operator splitting methods.

Note that the acoustic part can be rewritten in Lagrangian formulation using the material derivate $\frac{D(\cdot)}{Dt}$. So the equations (2.3) and (2.4) change to $\frac{D\rho}{Dt} = 0$ and $\frac{D(\rho u)}{Dt} + \partial_x p = 0$, respectively . The transport part is simply a transport of conservative variables $(\rho, \rho u)$ with the velocity field u.

2.1. **Lagrangian step.** In the Lagrangian coordinates, the frame moves with the velocity field. So, what the observer sees is the acoustic part, (2.3)-(2.4). It is not difficult to show that they can also be written as

$$\tau_t - u_z = 0, \tag{2.7}$$

$$u_t + p_z = 0, \tag{2.8}$$

where τ is specific volume (the reciprocal of ρ) and $z := \rho \mathrm{d}x$ is the mass coordinate. This is exactly the classical form of isentropic Euler equations in the Lagrangian framework. To obtain *non-dimensionalized* equations, we set

$$\widehat{\tau} := \tau \rho_\circ, \quad \widehat{u} := \frac{u}{u_\circ}, \quad \widehat{p} := \frac{p}{p_\circ}, \quad p_\circ := \rho_\circ c_\circ^2,$$

where $c_\circ$ is the reference sound speed, defined as $c_\circ := \sqrt{\kappa \gamma \rho_\circ^{\gamma-1}}$. Thus, after suppressing hats, the equations become

$$\tau_t - u_z = 0, \tag{2.9}$$

$$u_t + \frac{p_z}{\epsilon^2} = 0. \tag{2.10}$$

Assuming a one-dimensional torus as spatial domain $\Omega := \mathbb{T}$, i.e. periodic boundary, we define the domain of solutions as $\Omega_T := \mathbb{T} \times \mathbb{R}_+$ in space and time. So, we are left with a Cauchy initial value problem which needs solution of the Riemann problems. To ease the situation, we relax the system so that all characteristic fields would be linearly degenerate, which is easy to solve the Riemann problem for. We actually substitute the source of genuine nonlinearity $p(\rho)$ with some variable π, called relaxation pressure and add another equation for π. This is the heart of so-called *relaxation schemes*; we refer the reader to [2, 28, 6, 23] for more details.

In the non-dimensionalized form, the Suliciu relaxation system [2, 7] reads as

$$\tau_t - u_z = 0, \tag{2.11}$$

$$u_t + \Pi_z = 0, \tag{2.12}$$

$$\Pi_t + \alpha^2 u_z = \Lambda(p - \pi), \tag{2.13}$$

with the definitions

$$\Pi := \frac{\pi}{\epsilon^2}, \qquad \alpha := \frac{a}{\epsilon}, \qquad \Lambda := \frac{\lambda}{\epsilon^2},$$

where a is a constant to be specified and λ is the relaxation parameter. At least formally, one can observe that in the asymptotic regime $\lambda \to \infty$, π tends to p and the original system would be recovered. Now, one can easily check that the relaxation system only has linearly-degenerate characteristic fields. To use the feature of linear degeneracy, at first we solve the problem out of equilibrium, setting $\lambda = 0$, and then we project the out-of-equilibrium solution to the equilibrium manifold, cf. [8]. Interestingly, it is also shown in [8] that the choice of $\lambda = 0$ maps the solution onto the equilibrium manifold, due to the Lagrangian coordinates.

In order to prevent this relaxation system from instabilities (to enforce dissipativity of Chapman–Enskog expansion, see [6, 28]), a must be chosen sufficiently large, according to the so-called *sub-characteristic* or *Whitham stability* condition

$$a^2 > \max(-p_\tau), \tag{2.14}$$

see [3] for the proof.

Since the relaxation system with $\lambda = 0$ is strictly hyperbolic with eigenvalues given by $0, \pm a$—compared to exact eigenvalues $0, \pm c$ for original system—the sub-characteristic condition means that information propagates faster in the relaxation model. Also linear degeneracy of the fields allows us to analytically solve the Riemann problem when $\lambda = 0$. This property justifies by itself the introduction of the proposed relaxation model and its simplicity [5].

So, for $\lambda = 0$, one can simply put the relaxation system (2.11)-(2.13) into an equivalent form like [4, eq. (12)]

$$\tau_t - u_z = 0, \tag{2.15}$$

$$\overrightarrow{w}_t + \alpha \overrightarrow{w}_z = 0, \quad \overrightarrow{w} := \Pi + \alpha u = \frac{\pi}{\epsilon^2} + \frac{a}{\epsilon} u, \tag{2.16}$$

$$\overleftarrow{w}_t - \alpha \overleftarrow{w}_z = 0, \quad \overleftarrow{w} := \Pi - \alpha u = \frac{\pi}{\epsilon^2} - \frac{a}{\epsilon} u. \tag{2.17}$$

Note that $\overrightarrow{w}$ and $\overleftarrow{w}$ are two of Riemann invariants of the relaxation system; the third one is $\mathscr{I} := \Pi + \alpha^2 \tau$. So, instead of the (2.15) one can use $\mathscr{I}_t = 0$.

Remark 2.1. Note that the naturally-splitted systems (2.3)-(2.4) and (2.5)-(2.6) is not conservative in Eulerian coordinates, so to avoid the complications coming with non-conservative products (see [10] for example) and also for solving Riemann problems with more ease and efficiency, we have changed the coordinates to Lagrangian, which provides a conservative formulation.

2.2. **Projection step.** Like the acoustic part, (2.5)-(2.6) can be written in Lagrangian coordinates which provides a conservative form. In fact, since we remap the values into the Eulerian grid, at the end of each step, the referential and spatial (Eulerian) coordinate should coincide. So following the notation in [8], the projection step can be summarized as

$$\partial_t \mathbb{U} + u \partial_x \mathbb{U} = 0, \tag{2.18}$$

where $\mathbb{U} := \big(\rho, \rho u\big)^T$ stands for conservative variables. For further details on the derivation of the splitted systems, the reader can consult with [8]

3. Lagrange–projection scheme: Discrete numerical level

As mentioned above, for linearly-degenerate systems, it is easy to solve the Riemann problem. Moreover in this case, after writing the equations in terms of Riemann invariants, it would be in fact trivial since along each characteristic line, one of the Riemann invariants remains constant. In this way, there are just a set of two symmetric scalar linear advection equations to be solved for $\overrightarrow{w}$ and $\overleftarrow{w}$, $\mathscr{I}$ does not change at all.

At the beginning of Lagrange (acoustic) step from n to $n\dagger$, the Eulerian and Lagrangian coordinates coincide with each other, also $p_j^n = \pi_j^n$. The Lagrange step reads as

$$\tau_j^{n\dagger} = \tau_j^n + \frac{\Delta t}{\Delta z_j}\big(\tilde{u}_{j+1/2}^{n\dagger} - \tilde{u}_{j-1/2}^{n\dagger}\big). \tag{3.1}$$

$$\overrightarrow{w}_j^{n\dagger} = \overrightarrow{w}_j^{n} - \frac{a\Delta t}{\epsilon\Delta z_j}\big(\overrightarrow{w}_j^{n\dagger} - \overrightarrow{w}_{j-1}^{n\dagger}\big), \tag{3.2}$$

$$\overleftarrow{w}_j^{n\dagger} = \overleftarrow{w}_j^{n} + \frac{a\Delta t}{\epsilon\Delta z_j}\big(\overleftarrow{w}_{j+1}^{n\dagger} - \overleftarrow{w}_{j}^{n\dagger}\big), \tag{3.3}$$

where $\Delta z := \rho_j^n \Delta x$ and $j \in \mathbb{S}$ denotes cell indices when $\mathbb{S}$ is a periodic set (the discretization of Ω). Also $\tilde{u}^{n\dagger}$ comes from solving a simple Riemann problem for the relaxation system with characteristic $0, \pm\frac{a}{\epsilon}$ (see [8]), and it is

$$\tilde{u}^{n\dagger} := \frac{1}{2\frac{a}{\epsilon}}\Big(\frac{a}{\epsilon}(u_L + u_R) - \frac{\pi_R - \pi_L}{\epsilon^2}\Big) = \frac{1}{2a\epsilon}\Big(a\epsilon(u_L + u_R) - (\pi_R - \pi_L)\Big).$$

So, the interface velocity would be defined as

$$\tilde{u}_{j+1/2}^{n\dagger} = \frac{u_j^n + u_{j+1}^n}{2} - \frac{1}{2a\epsilon}\big(\pi_{j+1}^n - \pi_j^n\big). \tag{3.4}$$

Note that there are several (equivalent) variants of the scheme (3.1)-(3.3), in different coordinates or with/without using the Riemann invariants; see [8] for further details.

In the next step, the projection step from $n\dagger$ to $n+1$, we map updated values onto the fixed Eulerian gird. There are 4 cases based on upwind direction [4, eq. (34)]:

- $\tilde{u}_{j-1/2}^{n\dagger} < 0, \tilde{u}_{j+1/2}^{n\dagger} < 0 : \mathbb{U}_j^{n+1} = \mathbb{U}_j^{n\dagger} + \frac{\Delta t}{\Delta x}\tilde{u}_{j+1/2}^{n\dagger}\big(\mathbb{U}_j^{n\dagger} - \mathbb{U}_{j+1}^{n\dagger}\big)$
- $\tilde{u}_{j-1/2}^{n\dagger} < 0, \tilde{u}_{j+1/2}^{n\dagger} > 0 : \mathbb{U}_j^{n+1} = \mathbb{U}_j^{n\dagger}$
- $\tilde{u}_{j-1/2}^{n\dagger} > 0, \tilde{u}_{j+1/2}^{n\dagger} > 0 : \mathbb{U}_j^{n+1} = \mathbb{U}_j^{n\dagger} + \frac{\Delta t}{\Delta x}\tilde{u}_{j-1/2}^{n\dagger}\big(\mathbb{U}_{j-1}^{n\dagger} - \mathbb{U}_j^{n\dagger}\big)$
- $\tilde{u}_{j-1/2}^{n\dagger} > 0, \tilde{u}_{j+1/2}^{n\dagger} < 0 : \mathbb{U}_j^{n+1} = \mathbb{U}_j^{n\dagger} + \frac{\Delta t}{\Delta x}\Big(\tilde{u}_{j+1/2}^{n\dagger}\big(\mathbb{U}_j^{n\dagger} - \mathbb{U}_{j+1}^{n\dagger}\big) + \tilde{u}_{j-1/2}^{n\dagger}\big(\mathbb{U}_{j-1}^{n\dagger} - \mathbb{U}_j^{n\dagger}\big)\Big)$

This step, projects the values in updated grid to the fixed Eulerian one. Adding these two steps to each other is what one calls Lagrange–projection scheme.

3.1. Numerical analysis of the scheme.

Considering the Lagrange–projection scheme introduced in the previous section, one can obtain the main theorem of this article, which includes the stability results.

Theorem 3.1. *The Lagrange–projection scheme satisfies the following properties.*

(i) *It can be expressed in the locally conservative form.*

(ii) *The scheme is AP consistent, which means that it preserves the div-free condition for zero-Mach limit and correct asymptotic expansion for the computed solution in terms of* [30].

(iii) *For well-prepared initial data and under some ϵ-uniform CFL constraint* (3.20) *the scheme is positivity preserving, i.e. $\rho_j^{n+1} > 0$ provided that $\rho_j^n > 0$ for all $j \in \mathbb{S}$. Moreover the density is bounded away from zero for finite time, i.e. there exists some $\varrho > 0$ such that $\rho_j^{n+1} \geq \varrho$ for all $j \in \mathbb{S}$.*

(iv) *For well-prepared initial data and under constraint* (3.20)*, the solution (density and velocity) is stable, i.e. it is bounded in L_∞-norm, uniformly in ϵ.*

(v) *Under* (3.20) *and sub-characteristic condition* (3.36)*, the solution fulfills the local (cell) entropy (energy) inequality, i.e.*

$$\frac{\big(\rho E\big)_j^{n+1} - \big(\rho E\big)_j^n}{\Delta t} + \frac{\big(\rho E\tilde{u} + \tilde{\pi}\tilde{u}\big)_{j+1/2}^{n\dagger} - \big(\rho E\tilde{u} + \tilde{\pi}\tilde{u}\big)_{j-1/2}^{n\dagger}}{\Delta x} \leq 0, \tag{3.5}$$

which is consistent with

$$\partial_t\big(\rho E\big) + \partial_x\big(\rho E u + pu\big) \leq 0.$$

Now, let us analyze the properties of this scheme in the subsequent subsections. Note that the locally conservative form of the scheme is proved in [8] and it is not that difficult; so we skip it here.

3.1.1. *Proof of AP consistency (ii).* Now, we show that the scheme is consistent with the PDE in the limit $\epsilon \to 0$, i.e. the both have constant density up to the second order of asymptotic expansion, and the zeroth-order velocity component is divergence free (solenoidal). This coincides with the asymptotic analysis results of [30]. Considering [30], assume for the step n that

$$\rho_j^n(x) = \rho_{0c}^n + \epsilon^2 \rho_{(2)j}^n, \tag{3.6}$$

$$p_j^n(x) = \pi_{0c}^n + \epsilon^2 \pi_{(2)j}^n, \tag{3.7}$$

$$u_j^n(x) = u_{0c}^n + \epsilon u_{(1)j}^n, \tag{3.8}$$

where ρ_{0c}^n, π_{0c}^n and u_{0c}^n are constant values. Here, we want to show that the scheme (3.1)–(3.3) preserves these properties from step n to the intermediate step $n\dagger$ and then to the next time step $n+1$; to show that the acoustic and its remapped solutions are consistent with the PDE as $\epsilon \to 0$.

Lagrange step. We start with the mass equation:

$$\frac{1}{\rho_j^{n\dagger}} = \frac{1}{\rho_j^n} + \frac{\Delta t}{\rho_j^n \Delta x}\big(\tilde{u}_{j+1/2}^{n\dagger} - \tilde{u}_{j-1/2}^{n\dagger}\big), \tag{3.9}$$

and

$$\begin{aligned}\rho_j^n &= \rho_j^{n\dagger}\Big(1 + \frac{\Delta t}{\Delta x}\big(\tilde{u}_{j+1/2}^{n\dagger} - \tilde{u}_{j-1/2}^{n\dagger}\big)\Big) \\ &= \rho_j^{n\dagger}\bigg(1 + \frac{\Delta t}{2a\Delta x}\Big(\frac{\pi_{j+1}^{n\dagger} - 2\pi_j^{n\dagger} + \pi_{j-1}^{n\dagger}}{\epsilon} + a\big(u_{j+1}^{n\dagger} - u_{j-1}^{n\dagger}\big)\Big)\bigg).\end{aligned}$$

So,

$$\mathcal{O}(1/\epsilon): \qquad \pi_{(0)j+1}^{n\dagger} - 2\pi_{(0)j}^{n\dagger} + \pi_{(0)j-1}^{n\dagger} = 0 \Longrightarrow \pi_{(0)j}^{n\dagger} \text{ is a linear function over } j \in \mathbb{S},$$

and due to periodic B.C. $\pi_{(0)j}^{n\dagger} = \pi_{(0)}^{n\dagger}$ which is constant in space.

Since π and ρ are two independent variables at this level, we cannot conclude immediately that the same is true for the density. It is shown in [8, eq. (4.7)] that one can find another equation from (3.1)–(3.3), that is

$$\rho_j^n \frac{\pi_j^{n\dagger} - \pi_j^n}{\Delta t} + \frac{a^2}{\Delta x}\big(\tilde{u}_{j+1/2}^{n\dagger} - \tilde{u}_{j-1/2}^{n\dagger}\big) = 0. \tag{3.10}$$

If one combines it with continuity, it yields

$$a^2\big(\tau_j^{n\dagger} - \tau_j^n\big) + \big(\pi_j^{n\dagger} - \pi_j^n\big) = 0, \tag{3.11}$$

and

$$a^2\big(\rho_j^{n\dagger} - \rho_j^n\big) = \rho_j^n \rho_j^{n\dagger}\big(\pi_j^{n\dagger} - \pi_j^n\big).$$

So,

$$\mathcal{O}(1): \qquad a^2\big(\rho_{(0)j}^{n\dagger} - \rho_{(0)j}^n\big) = \rho_{(0)j}^n \rho_{(0)j}^{n\dagger}\big(\pi_{(0)j}^{n\dagger} - \pi_{(0)j}^n\big),$$

which gives that

$$\rho_{(0)j}^{n\dagger}\Big(a^2 - \rho_{0c}^n\big(\pi_{(0)}^{n\dagger} - \pi_{0c}^n\big)\Big) = a^2 \rho_{0c}^n \Longrightarrow \rho_{(0)j}^{n\dagger} = \rho_{(0)}^{n\dagger} \text{ const. in space}$$

Then, due to periodic B.C. and (3.9), by a spatial summation it can be found out that $\rho_{(0)j}^{n\dagger}$ is constant in time as well, i.e. $\rho_{(0)j}^{n\dagger} = \rho_{0c}^n$. Also from conservation equation for relaxation pressure π, (3.10), and again periodic B.C. and spatial summation, the numerical fluxes cancel out with each other and it turns out that $\pi_{(0)j}^{n\dagger} = \pi_{0c}^n$, constant in both time and space.

Next, let us continue with momentum equation (there is no difference between $\overrightarrow{w}_j$ and $\overleftarrow{w}_j$ in this regard).

$$\rho_j^n\big(\frac{\pi_j^{n\dagger}}{\epsilon^2} + \frac{a}{\epsilon}u_j^{n\dagger}\big) = \rho_j^n\big(\frac{\pi_j^n}{\epsilon^2} + \frac{a}{\epsilon}u_j^n\big) - \frac{a\Delta t}{\epsilon^2 \Delta x}\Big(\frac{\pi_j^{n\dagger} - \pi_{j-1}^{n\dagger}}{\epsilon} + a\big(u_j^{n\dagger} - u_{j-1}^l\big)\Big).$$

So,

$$\mathcal{O}(1/\epsilon^2): \qquad \rho_{0c}^n \pi_{(0)j}^{n\dagger} = \rho_{0c}^n \pi_{0c}^n - \frac{a\Delta t}{\Delta x}\Big(\pi_{(1)j}^{n\dagger} - \pi_{(1)j-1}^{n\dagger} + a\big(u_{(0)j}^{n\dagger} - u_{(0)j-1}^{n\dagger}\big)\Big),$$

which yields

$$\pi_{(1)j}^{n\dagger} - \pi_{(1)j-1}^{n\dagger} + a\big(u_{(0)j}^{n\dagger} - u_{(0)j-1}^{n\dagger}\big) = 0. \tag{3.12}$$

So, there is the possibility that both $\pi_{(1)j}^{n\dagger}$ and $u_{(0)j}^{n\dagger}$ be constant in space. To show it, note that from $\mathcal{O}(1)$ terms in continuity equation, one gets

$$\begin{aligned}\rho_{(0)j}^n =& \rho_{(0)j}^{n\dagger}\Bigg(1 + \frac{\Delta t}{2a\Delta x}\Big(a\big(u_{(0)j+1}^{n\dagger} - u_{(0)j-1}^{n\dagger}\big) - \big(\pi_{(1)j-1}^{n\dagger} - 2\pi_{(1)j}^{n\dagger} + \pi_{(1)j+1}^{n\dagger}\big)\Big)\Bigg) \\ &- \frac{\Delta t}{2a\Delta x}\rho_{(1)j}^{n\dagger}\big(\pi_{(0)j-1}^{n\dagger} - 2\pi_{(0)j}^{n\dagger} + \pi_{(0)j+1}^{n\dagger}\big).\end{aligned}$$

So,

$$a\big(u_{(0)j+1}^{n\dagger} - u_{(0)j-1}^{n\dagger}\big) - \big(\pi_{(1)j-1}^{n\dagger} - 2\pi_{(1)j}^{n\dagger} + \pi_{(1)j+1}^{n\dagger}\big) = 0 \tag{3.13}$$

Combining (3.13) and (3.12) yields that $\pi_{(1)j}^{n\dagger} = \pi_{(1)}^{n\dagger}$ and $u_{(0)j}^{n\dagger} = u_{(0)}^{n\dagger}$. So,

$$\operatorname{div} u_{(0)}^{n\dagger} = 0. \tag{3.14}$$

Again, similar to the zeroth order, one can show that $\rho^{n\dagger}_{(1)j}$ is constant in space and even $\pi^{n\dagger}_{(1)j}$ and $\rho^{n\dagger}_{(1)j}$ are constant in time, i.e.

$$\pi^{n\dagger}_{(1)j} = \pi^{n}_{1c}, \qquad \rho^{n\dagger}_{(1)j} = \rho^{n}_{1c}. \tag{3.15}$$

Hence, it turns out that the Lagrangian step is AP consistent.
Projection step. Now, we move on to the projection step. We show AP property for the first case, $\tilde{u}^{n\dagger}_{j-1/2} < 0$ and $\tilde{u}^{n\dagger}_{j+1/2} < 0$. The other cases can be done in a very similar way.

$$\rho^{n+1}_{j} = \rho^{n\dagger}_{j} - \frac{\Delta t}{2a\Delta x}\big(\rho^{n\dagger}_{j+1} - \rho^{n\dagger}_{j}\big)\Big(- \frac{\pi^{n\dagger}_{j+1} - \pi^{n\dagger}_{j}}{\epsilon} + a\big(u^{n\dagger}_{j+1} - u^{n\dagger}_{j}\big)\Big).$$

So,

$$\begin{aligned}\mathcal{O}(1): \qquad \rho^{n+1}_{(0)j} = \rho^{n\dagger}_{(0)j} - \frac{\Delta t}{2a\Delta x}\Big[&- \big(\rho^{n\dagger}_{(0)j+1} - \rho^{n\dagger}_{(0)j}\big)\big(\pi^{n\dagger}_{(1)j+1} - \pi^{n\dagger}_{(1)j}\big) \\ &- \big(\rho^{n\dagger}_{(1)j+1} - \rho^{n\dagger}_{(1)j}\big)\big(\pi^{n\dagger}_{(0)j+1} - \pi^{n\dagger}_{(0)j}\big) \\ &+ a\big(\rho^{n\dagger}_{(0)j+1} - \rho^{n\dagger}_{(0)j}\big)\big(u^{n\dagger}_{(0)j+1} - u^{n\dagger}_{(0)j}\big)\Big],\end{aligned}$$

and thus

$$\rho^{n+1}_{(0)j} = \rho^{n\dagger}_{(0)j} = \rho^{n}_{0c} \tag{3.16}$$

and as a result $p^{n+1}_{(0)j} = p^{n}_{0c}$; it is constant as well. Similarly, one can find that the first order components are also constant in time and space; if they do not exist in the initial condition, so at the time t_{n+1} there is no pressure fluctuation of order ϵ:

$$\begin{aligned}\mathcal{O}(\epsilon): \qquad \rho^{n+1}_{(1)j} = \rho^{n\dagger}_{(1)j} - \frac{\Delta t}{2a\Delta x}\Big[&- \big(\rho^{n\dagger}_{(0)j+1} - \rho^{n\dagger}_{(0)j}\big)\big(\pi^{n\dagger}_{(2)j+1} - \pi^{n\dagger}_{(2)j}\big) \\ &- \big(\rho^{n\dagger}_{(1)j+1} - \rho^{n\dagger}_{(1)j}\big)\big(\pi^{n\dagger}_{(1)j+1} - \pi^{n\dagger}_{(1)j}\big) \\ &- \big(\rho^{n\dagger}_{(2)j+1} - \rho^{n\dagger}_{(2)j}\big)\big(\pi^{n\dagger}_{(0)j+1} - \pi^{n\dagger}_{(0)j}\big) \\ &+ a\big(\rho^{n\dagger}_{(0)j+1} - \rho^{n\dagger}_{(0)j}\big)\big(u^{n\dagger}_{(1)j+1} - u^{n\dagger}_{(1)j}\big) \\ &+ a\big(\rho^{n\dagger}_{(1)j+1} - \rho^{n\dagger}_{(1)j}\big)\big(u^{n\dagger}_{(0)j+1} - u^{n\dagger}_{(0)j}\big)\Big],\end{aligned}$$

and

$$\rho^{n+1}_{(1)j} = \rho^{n\dagger}_{(1)j} = \rho^{n}_{0c} = 0. \tag{3.17}$$

To show the div-free condition, one can consider $\mathcal{O}(1)$ terms of the momentum equation:

$$\begin{aligned}\rho^{n+1}_{(0)j}u^{n+1}_{(0)j} = \rho^{n\dagger}_{(0)j}u^{n\dagger}_{(0)j} - \frac{\Delta t}{2a\Delta x}\Big[&- \big(\rho^{n\dagger}_{(0)j+1}u^{n\dagger}_{(0)j+1} - \rho^{n\dagger}_{(0)j}u^{n\dagger}_{(0)j}\big)\big(\pi^{n\dagger}_{(1)j+1} - \pi^{n\dagger}_{(1)j}\big) \\ &- \big(\rho^{n\dagger}_{(1)j+1}u^{n\dagger}_{(0)j+1} - \rho^{n\dagger}_{(1)j}u^{n\dagger}_{(0)j}\big)\big(\pi^{n\dagger}_{(0)j+1} - \pi^{n\dagger}_{(0)j}\big) \\ &- \big(\rho^{n\dagger}_{(0)j+1}u^{n\dagger}_{(1)j+1} - \rho^{n\dagger}_{(0)j}u^{n\dagger}_{(1)j}\big)\big(\pi^{n\dagger}_{(0)j+1} - \pi^{n\dagger}_{(0)j}\big) \\ &+ a\big(\rho^{n\dagger}_{(0)j+1}u^{n\dagger}_{(0)j+1} - \rho^{n\dagger}_{(0)j}u^{n\dagger}_{(0)j}\big)\big(u^{n\dagger}_{(0)j+1} - u^{n\dagger}_{(0)j}\big)\Big].\end{aligned}$$

Thus,

$$u^{n+1}_{(0)j} = u^{n\dagger}_{(0)j} = u^{n}_{0c}, \tag{3.18}$$

and the zeroth order component of velocity filed is solenoidal. Hence, combining the AP results for Lagrange and projection steps together, it is obvious that the limit properties is satisfied, and the scheme is AP.

3.1.2. *Proof of density positivity, and stability (iii,iv).* In this section, we show that for well-prepared initial data the density is positive due to an ϵ-independent time restriction. Also we show that the solution vector $\big(\rho, \rho u\big)^T$ has ϵ-uniform upper-bounds, i.e. it lies in $\big(L_\infty(\Omega_T)\big)^2$. Finally we show that the density is bounded away from zero, i.e. there exists some $\varrho > 0$ such that $\rho \geq \varrho$.

From [8, eq. (2.25a)] we first define *local acoustic CFL ratios* μ_j, and *local apparent propagation factor* e_j as

$$\mu_j := \frac{a\Delta t}{\Delta z_j}, \qquad e_j := \frac{\mu_j/\epsilon}{1+\mu_j/\epsilon}.$$

Then, one can write (3.2) as

$$\overrightharpoon{w}_j^{n\dagger} = e_j \overrightharpoon{w}_{j-1}^{n\dagger} + (1-e^j)\overrightharpoon{w}_j^{n}.$$

So, it is not so difficult to show that as soon as $0 \leq e_j \leq 1$ (which can be satisfied for all ϵ uniformly), the scheme satisfies maximum principle, i.e. no new extremum can be generated, as follows.

- **Bound from above:** Assume that i is the index of maximum value of $\overrightharpoon{w}_j^{n\dagger}$, that is $\overrightharpoon{w}_i^{n\dagger} \geq \overrightharpoon{w}_j^{n\dagger}$, $\forall j \in \mathbb{S}$. So,
$$\overrightharpoon{w}_i^{n\dagger} \leq e_i \overrightharpoon{w}_i^{n\dagger} + (1-e_i)\overrightharpoon{w}_i^{n} \Longrightarrow \overrightharpoon{w}_i^{n\dagger} \leq \overrightharpoon{w}_i^{n} \Longrightarrow \max_j \overrightharpoon{w}_j^{n\dagger} \leq \max_j \overrightharpoon{w}_j^{n}.$$
So, it is bounded from above.
- **Bound from below:** Assume that k is the index of minimum value of $\overrightharpoon{w}_j^{n\dagger}$, that is $\overrightharpoon{w}_k^{n\dagger} \leq \overrightharpoon{w}_j^{n\dagger}$, $\forall j \in \mathbb{S}$. So,
$$\overrightharpoon{w}_k^{n\dagger} \geq e_k \overrightharpoon{w}_k^{n\dagger} + (1-e_k)\overrightharpoon{w}_k^{n} \Longrightarrow \overrightharpoon{w}_k^{n\dagger} \geq \overrightharpoon{w}_k^{n} \Longrightarrow \min_j \overrightharpoon{w}_j^{n\dagger} \geq \max_j \overrightharpoon{w}_j^{n}.$$
So, it is bounded from below.

Hence, the values of $\overrightharpoon{w}_j^{n\dagger}$ and $\overleftharpoon{w}_j^{n\dagger}$ are bounded as $\overrightharpoon{m}_j^{n} \leq \overrightharpoon{w}_j^{n} \leq \overrightharpoon{M}_j^{n\dagger}$ and similarly for $\overleftharpoon{w}_j^{n\dagger}$, and this bound *at the end* only depends on the initial condition, i.e.

$$\min_j \overrightharpoon{w}_j^{0} =: \overrightharpoon{m}^{0} \leq \overrightharpoon{w}_j^{n\dagger} \leq \overrightharpoon{M}^{0} := \max_j \overrightharpoon{w}_j^{0}, \tag{3.19}$$

and Similarly for $\overleftharpoon{w}_j^{n\dagger}$.

Now, denoting $\cdot^+ := \frac{\cdot+|\cdot|}{2}$ and $\cdot^- := \frac{\cdot-|\cdot|}{2}$, we claim the following theorem.

Theorem 3.2. *For some Δt satisfying*

$$\begin{aligned}\frac{\Delta t}{\Delta x} &\leq \frac{2a/\epsilon}{\max_j \Big\{ \big(\overrightharpoon{M}_{j-1}^{n\dagger} - \overleftharpoon{m}_j^{n\dagger}\big)^+ - \big(\overrightharpoon{m}_j^{n\dagger} - \overleftharpoon{M}_{j-1}^{n\dagger}\big)^- \Big\}} \\ &\leq \frac{2a/\epsilon}{\big(\overrightharpoon{M}^{0} - \overleftharpoon{m}^{0}\big)^+ - \big(\overrightharpoon{m}^{0} - \overleftharpoon{M}^{0}\big)^-},\end{aligned} \tag{3.20}$$

the Lagrange–projection scheme preserves the positivity of density provided that $\rho_j^0 > 0$ *for all* $j \in \mathbb{S}$.

Proof. In lines of [8], for the Lagrange step to satisfy positivity, one gets from Piola's identity that

$$\frac{\Delta t}{\Delta x}\big(\tilde{u}^{n\dagger}_{j-1/2} - \tilde{u}^{n\dagger}_{j+1/2}\big) < 1, \tag{3.21}$$

ensures $\rho_j^{n\dagger} > 0$ for all $j \in \mathbb{S}$. But on the other hand, Δt should be such that the projection step is a convex combination, thus

$$\frac{\Delta t}{\Delta x}\Big(\big(\tilde{u}^{n\dagger}_{j+1/2}\big)^- - \big(\tilde{u}^{n\dagger}_{j-1/2}\big)^+\Big) < 1, \tag{3.22}$$

Thus the stronger condition should be chosen, which is (3.22). Then, based on the definition of $\tilde{u}^{n\dagger}$, we express Δt in terms of $\overset{\rightharpoonup}{M}$, $\overset{\leftharpoonup}{M}$, $\overset{\rightharpoonup}{m}$ and $\overset{\leftharpoonup}{m}$. □

The next goal is to show that this bound for time step, is uniform in ϵ and it does not vanish as the Mach number goes to zero. It is possible to show this, for a well-prepared initial data, with the following definition.

Definition 3.3. For isentropic Euler equation, the well-prepared initial condition is defined as (see [24, 14])

$$\pi^{(0)}_{WP}(x) = p^{(0)}_{WP}(x) = p_0 + \mathcal{O}(\epsilon^2)p_2(x), \quad p_{2,min} \le p_2(x) \le p_{2,max}, \tag{3.23}$$

$$u^{(0)}_{WP}(x) = u_0 + \mathcal{O}(\epsilon)u_1(x), \quad u_{1,min} \le u_1(x) \le u_{1,max}, \tag{3.24}$$

with constant p_0 and u_0.

Then, one can pose the following corollary.

Corollary 3.4. *For well-prepared initial data, the time restriction bound* (3.20) *is uniform in* ϵ.

Proof. From definition, it is easy to see

$$\begin{aligned}
\overset{\rightharpoonup}{M}^0 &= \frac{p_0}{\epsilon^2} + \mathcal{O}(1)p_{2,max} + a\frac{u_0}{\epsilon} + \mathcal{O}(1)a\max\big(|u_{1,min}|,|u_{1,max}|\big),\\
\overset{\rightharpoonup}{m}^0 &= \frac{p_0}{\epsilon^2} + \mathcal{O}(1)p_{2,min} + a\frac{u_0}{\epsilon} + \mathcal{O}(1)a\min\big(|u_{1,min}|,|u_{1,max}|\big),\\
\overset{\leftharpoonup}{M}^0 &= \frac{p_0}{\epsilon^2} + \mathcal{O}(1)p_{2,max} - a\frac{u_0}{\epsilon} + \mathcal{O}(1)a\max\big(|-u_{1,min}|,|-u_{1,max}|\big),\\
\overset{\leftharpoonup}{m}^0 &= \frac{p_0}{\epsilon^2} + \mathcal{O}(1)p_{2,min} - a\frac{u_0}{\epsilon} + \mathcal{O}(1)a\min\big(|-u_{1,min}|,|-u_{1,max}|\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\overset{\rightharpoonup}{M}^0 - \overset{\leftharpoonup}{m}^0 &= \frac{2au_0}{\epsilon} + \mathcal{O}(1)\big(p_{2,max} - p_{2,min}\big),\\
\overset{\rightharpoonup}{m}^0 - \overset{\leftharpoonup}{M}^0 &= \frac{2au_0}{\epsilon} - \mathcal{O}(1)\big(p_{2,max} - p_{2,min}\big),
\end{aligned}$$

and one gets

$$\frac{\Delta t}{\Delta x} \le \frac{2a/\epsilon}{\mathcal{O}(\frac{1}{\epsilon}) + \mathcal{O}(1)} \le C. \tag{3.25}$$

Hence, the condition (3.20), which provides positivity of the density, is uniform in ϵ. □

Now, let us continue to find a bound for the computed solution to prove stability of the solution. At first we pose the following lemma.

Lemma 3.5. *For the well-prepared initial data, the computed velocity $u_j^{n\dagger}$ is L_∞-bounded uniformly in ϵ.*

Proof. It is enough to use $u_j^{n\dagger} = \frac{\epsilon}{2a}\big(\overrightarrow{w}_j^n - \overleftarrow{w}_j^n\big)$ combined with bounds of $\overrightarrow{w}_j^n$ in (3.19). With straightforward arguments it comes out that $u^{n\dagger}$ is bounded in $L_\infty(\Omega_T)$ (uniformly in ϵ) for well-prepared initial data. □

Next, we continue with the stability of density.

Lemma 3.6. *For the well-prepared initial data, the computed density $\rho_j^{n\dagger}$ is L_∞-bounded uniformly in ϵ. Moreover, it is bounded away from zero in finite time.*

Proof. From the continuity equation one has

$$\rho_j^n\big(\tau_j^{n\dagger} - \tau_j^n\big) = \frac{\Delta t}{\Delta x}\big(\tilde{u}_{j+1/2}^{n\dagger} - \tilde{u}_{j-1/2}^{n\dagger}\big), \tag{3.26}$$

whose right-hand side is bounded by 1 for a time restriction (3.20) which implies (3.21). It yields

$$\tau_j^{n\dagger} = \tau_j^n\Big(1 + \frac{\Delta t}{\Delta x}\big(\tilde{u}_{j+1/2}^{n\dagger} - \tilde{u}_{j-1/2}^{n\dagger}\big)\Big). \tag{3.27}$$

Thus (3.21) combined with (3.27) shows clearly that the density would be positive. But the condition (3.22) also provides us with an ϵ-uniform upper-bound for the density as

$$0 < \rho_j^{n\dagger} < \infty, \tag{3.28}$$

in combination with (3.27). Thus, the density is L_∞-bounded. But one can do a bit better, if one replaces < 1 in (3.21) with $< \varsigma$, i.e. $\frac{\Delta t}{\Delta x}\big(\tilde{u}_{j-1/2}^{n\dagger} - \tilde{u}_{j+1/2}^{n\dagger}\big) < \varsigma$, with $\varsigma \le 1$. Moreover since $\big|\frac{\Delta t}{\Delta x}\big(\tilde{u}_{j+1/2}^{n\dagger}\big)\big|$ should be less than 1, one gets $-1 < \frac{\Delta t}{\Delta x}\big(\tilde{u}_{j+1/2}^{n\dagger}\big) < 1$. So,

$$-\varsigma < \frac{\Delta t}{\Delta x}\Big(\big(\tilde{u}_{j+1/2}^{n\dagger}\big)^- - \big(\tilde{u}_{j-1/2}^{n\dagger}\big)^+\Big) < 2.$$

Thus, for each time step one has

$$\frac{\rho_j^n}{3} < \rho_j^{n\dagger} < \frac{\rho_j^n}{1-\varsigma}, \tag{3.29}$$

and for the finite time $N\Delta t = T$, the density is bounded away from zero by $\varrho := \Big(\frac{\rho_{j,\min}^0}{3}\Big)^N$. □

Remark 3.7. Notice that this gives us the boundedness of the density ρ_j^{n+1} since the projection step is designed as a convex combination due to (3.20). This is similarly the case for the velocity and momentum.

3.1.3. *Proof of local energy inequality (v).* We show that the solution of the scheme satisfies the energy inequality under an ϵ-independent time restriction, as long as the initial condition is well-prepared.

Lagrangian step. Based on [8, Theorem 2.3], we define the entropy function for symmetric advections problem, (2.16)-(2.17), as

$$\eta(\overrightarrow{w}, \overleftarrow{w}) := s(\overrightarrow{w}) + s(\overleftarrow{w}), \quad s(w) := \frac{\epsilon^2 w^2}{4a^2}.$$

So

$$\eta(\overrightarrow{w}, \overleftarrow{w}) = \frac{1}{2}\Big(u^2 + \frac{\pi^2}{\epsilon^2 a^2}\Big) = \Big(E - \mathcal{E} + \frac{\pi^2}{2a^2}\Big)/\epsilon^2, \tag{3.30}$$

since after non-dimensionalization, one gets $E = \frac{\mathcal{E}}{\epsilon^2} + \frac{u^2}{2}$ where $\mathcal{E}(\rho) = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}$. For later use, we should mention that such a definition of internal energy fulfills the Weyl's assumptions as defined below.

Definition 3.8. The Weyl's assumption for the internal energy function are defined as (see [8, 34])

$$\mathcal{E} > 0, \quad \mathcal{E}_\tau = -p < 0, \quad \mathcal{E}_{\tau\tau} > 0, \quad \mathcal{E}_{\tau\tau\tau} < 0.$$

We also define entropy flux function $\psi(\overrightarrow{w}, \overleftarrow{w})$ as

$$\psi(\overrightarrow{w}, \overleftarrow{w}) := \frac{a}{\epsilon}\big(s(\overrightarrow{w}) - s(\overleftarrow{w})\big) = \frac{\pi u}{\epsilon^2}. \tag{3.31}$$

Then, the cell entropy inequality reads as

$$\frac{\eta_j^{n\dagger} - \eta_j^n}{\Delta t} + \frac{\psi_{j+1/2}^{n\dagger} - \psi_{j-1/2}^{n\dagger}}{\Delta z_j} \leq 0. \tag{3.32}$$

Substituting (3.30) and (3.31), one can relate the entropy inequality for symmetric advections problem, to energy inequality for the isentropic Euler equations, i.e.

(3.33)

$$\rho_j^n \frac{E_j^{n\dagger} - E_j^n}{\Delta t} + \frac{\big(\pi u\big)_{j+1/2}^{n\dagger} - \big(\pi u\big)_{j-1/2}^{n\dagger}}{\Delta x} \leq \rho_j^n \underbrace{\left[\mathcal{E}_j^{n\dagger} - \mathcal{E}_j^n - \frac{\big(\pi_j^{n\dagger}\big)^2 - \big(\pi_j^n\big)^2}{2a^2}\right]}_{=:\mathcal{R}_j^{n\dagger}}.$$

Then, to prove entropy stability of the scheme, one should show that the entropy residual $\mathcal{R}_j^{n\dagger}$ is non-positive. Considering $\pi_j^n = p_j^n$, let's rewrite $\mathcal{R}_j^{n\dagger}$ as

$$\begin{aligned} \mathcal{R}_j^{n\dagger} :=& \mathcal{E}_j^{n\dagger} - \mathcal{E}_j^n - \frac{p_j^n}{a^2}\big(\pi_j^{n\dagger} - p_j^n\big) - \frac{\big(\pi_j^{n\dagger} - p_j^n\big)^2}{2a^2} \\ \text{(due to (3.11))} =& \mathcal{E}_j^{n\dagger} - \mathcal{E}_j^n + p_j^n\big(\tau_j^{n\dagger} - \tau_j^n\big) - \frac{a^2}{2}\big(\tau_j^{n\dagger} - \tau_j^n\big)^2. \end{aligned}$$

On the other hand from Taylor expansion with integral remainder, one gets

$$\mathcal{E}_j^{n\dagger} = \mathcal{E}_j^n + \mathcal{E}_\tau|_{x_j, t_n}\big(\tau_j^{n\dagger} - \tau_j^n\big) + \int_{\tau_j^n}^{\tau_j^{n\dagger}} \mathcal{E}_{\tau\tau}(\xi)\big(\tau_j^{n\dagger} - \xi\big)\mathrm{d}\xi.$$

Then, Weyl's assumptions and change of variables in the integral (re-parameterization) yield that

$$\mathcal{E}_j^{n\dagger} = \mathcal{E}_j^n - p_j^n\big(\tau_j^{n\dagger} - \tau_j^n\big) + \big(\tau_j^{n\dagger} - \tau_j^n\big)^2 \int_0^1 \mathcal{E}_{\tau\tau}(\tau_j^{n+1/2})(1 - \zeta)\mathrm{d}\zeta, \tag{3.34}$$

where $\tau_j^{n+1/2} := \zeta\tau_j^{n\dagger} + (1-\zeta)\tau_j^n$. So, for the entropy residual to be non-positive, one gets

$$\begin{aligned}\mathcal{R}_j^{n\dagger} &= \left(\tau_j^{n\dagger} - \tau_j^n\right)^2 \int_0^1 \Big(\mathcal{E}_{\tau\tau}(\tau_j^{n+1/2}) - a^2\Big)(1-\zeta)\mathrm{d}\zeta \\ &= \left(\tau_j^{n\dagger} - \tau_j^n\right)^2 \int_0^1 \Big(-p_\tau(\tau_j^{n+1/2}) - a^2\Big)(1-\zeta)\mathrm{d}\zeta \le 0,\end{aligned}$$

and a sufficient condition would be to set the integrand to be negative. Since $p_\tau = \kappa\gamma\rho^{1+\gamma}$ it yields

$$a^2 \ge \ \kappa\gamma \max_j \max_\zeta \Big(\big(\rho_j^{n+1/2}\big)^{\gamma+1}\Big) = \max_j \Big(\big(\rho_j^{n\dagger}\big)^{\gamma+1}, \big(\rho_j^n\big)^{\gamma+1}\Big). \tag{3.35}$$

Thus, based on the previous section on stability, the bound does not depend on ϵ, and hence a can be chosen as

$$a > \left(\frac{\|\rho^0\|_{\ell_\infty}}{(1-\varsigma)}\right)^{\gamma+1}, \tag{3.36}$$

to satisfy the sub-characteristic condition as well as the energy inequality for the Lagrangian step.

Projection step. In this part, from projection step, it is clear that due to Jensen's inequality the energy inequality holds as

$$(\rho E)_j^{n+1} \le \rho_j^n E_j^{n\dagger} - \frac{\Delta t}{\Delta x}\Big(\big(\rho E\tilde{u}\big)_{j+1/2}^{n\dagger} - \big(\rho E\tilde{u}\big)_{j-1/2}^{n\dagger}\Big). \tag{3.37}$$

Combining (3.33) and (3.37) we get the energy inequality

$$\frac{\big(\rho E\big)_j^{n+1} - \big(\rho E\big)_j^n}{\Delta t} + \frac{\big(\rho E\tilde{u} + \tilde{\pi}\tilde{u}\big)_{j+1/2}^{n\dagger} - \big(\rho E\tilde{u} + \tilde{\pi}\tilde{u}\big)_{j-1/2}^{n\dagger}}{\Delta x} \le 0, \tag{3.38}$$

under a ϵ-uniform time restrictions (3.20) and sub-characteristic condition (3.36).

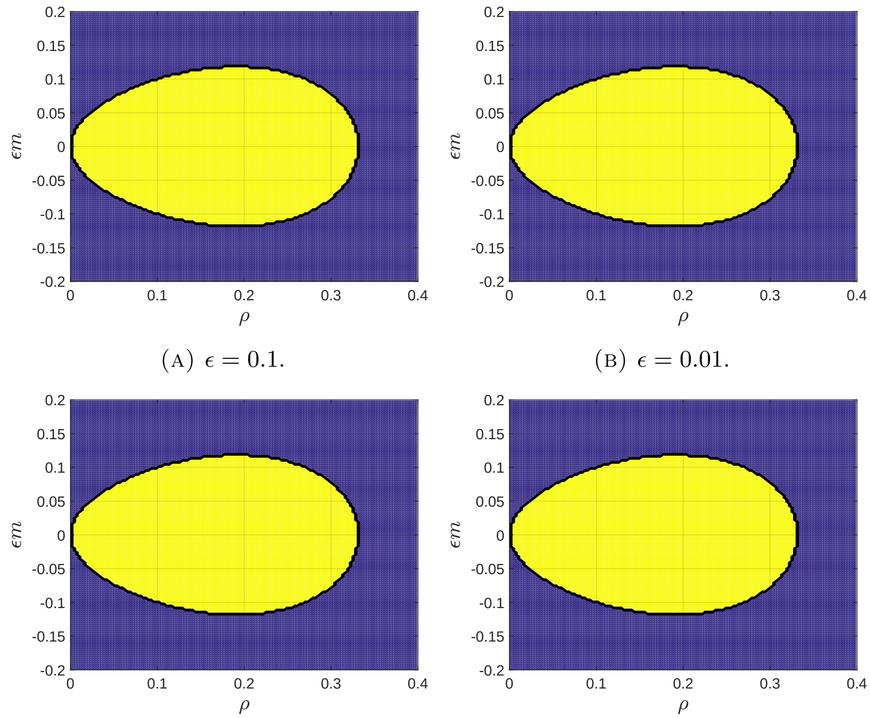
3.2. **On the relation between entropy inequality and L_∞-boundedness.** In this part, we show that energy inequality alone is not enough in order to conclude stability of the solution, and try to find the suitable further assumption to use. Let us denote entropy function $\mathcal{J} := \rho E$ and make a spatial summation on (3.5) to get

$$\sum_j \mathcal{J}(\underline{\mathbb{U}}_j^{n+1}) \le \sum_j \mathcal{J}(\underline{\mathbb{U}}_j^n) \Longrightarrow \sum_j \mathcal{J}(\underline{\mathbb{U}}_j^{n+1}) \le \sum_j \mathcal{J}(\underline{\mathbb{U}}_j^0) \le C_\epsilon < \infty.$$

If in addition one assumes positivity, then since $\mathcal{J}(\mathbb{U}) = \frac{1}{2}\frac{(\rho u)^2}{\rho} + \frac{\kappa/\epsilon^2}{\gamma-1}\rho^\gamma$ is always positive, it yields

$$\begin{aligned}&\mathcal{J}(\underline{\mathbb{U}}_i^{n+1}) + \sum_{j\neq i} \mathcal{J}(\underline{\mathbb{U}}_j^{n+1}) \le C_\epsilon \\ &\Longrightarrow 0 < \mathcal{J}(\underline{\mathbb{U}}_i^{n+1}) \le C_\epsilon - \sum_{j\neq i} \mathcal{J}(\underline{\mathbb{U}}_j^{n+1}) =: \mathcal{C}_{\epsilon,i}.\end{aligned}$$

Thus, the entropy is bounded from below by zero and from above by $\mathcal{C}_{\epsilon,i}$. This clearly shows that both ρ and u cannot approach infinitely large values simultaneously. As ρ approaches zero, entropy blows up, so this case is excluded for fixed ϵ unless m also approaches zero. However since $\mathcal{C}_{\epsilon,i}$ blows up as $\epsilon \to 0$, for every ϵ this entropy stability analysis provides us with a stability region Ξ_ϵ^0 which depends on the initial condition as well as ϵ.



(A) $\epsilon = 0.1$. (B) $\epsilon = 0.01$.

(C) $\epsilon = 0.001$. (D) $\epsilon = 0.0001$.

FIGURE 2. Scaled stability region Ξ^0 for different ϵ (yellow area) with $h_0 = 0.1$ and $m_0 = \frac{0.1}{\epsilon}$ as initial data.

To study the dependence of Ξ^0_ϵ on ϵ, let us assume that the initial density is bounded $0 < \underline{\varrho}_0 \le \rho_0 \le \overline{\varrho}_0$, and similarly for the initial velocity $\underline{u}_0 \le u_0 \le \overline{u}_0$. So, from entropy boundedness one can suppose that for the arbitrary time

$$\rho u^2 + \frac{\rho^2}{\epsilon^2} \le \mathrm{mean}_{j\in\mathbb{S}}\big(\rho_0 u_0^2 + \frac{\rho_0^2}{\epsilon^2}\big) \le \overline{\varrho}_0 \overline{u}_0^2 + \frac{\overline{\varrho}_0^2}{\epsilon^2}$$

and so

$$\rho u^2 + \frac{\rho^2 - \overline{\varrho}_0^2}{\epsilon^2} \le \overline{\varrho}_0 \overline{u}_0^2,$$

which poses a condition on the domain which ρ and u can live in; so, determines the stability region Ξ^0_ϵ. In the incompressible limit $\epsilon \to 0$, it is not possible for the density to get larger that $\overline{\varrho}_0$, and we have $\rho \le \overline{\varrho}_0$ in the limit, which is bounded. But one can see that as the density goes to zero, the momentum blows up to the infinity for $\epsilon \ll 1$; so, this region expands as ϵ shrinks. One can simply see that in this case u is of order $\mathcal{O}(\frac{1}{\epsilon})$.

Figure 2 shows the level set of the entropy function $\mathcal{J}$ with the value C_ϵ (but with the scaled momentum), which is the boundary between yellow and blue regions. So due to above discussion, if one scales the momentum with ϵ, then the scaled region of stability (denoted by Ξ^0) does not expand for the limit $\epsilon \to 0$ anymore; in Figure 2 all the cases have the same *scaled* stability region.

Nonetheless notice that this instability does not occur for the Lagrange–projection scheme due to time restriction (3.20) (see [8, p. 1515]), since we know that the velocity is bounded for the well-prepared initial data. On the other hand, we know that density is also bounded away from zero in the finite time[1]. So again it is reasonable to expect that such a blow-up does not occur.

Summing up, this discussion implies that the entropy stability along with positivity can give us ϵ-dependent stability, i.e. for each ϵ there is a corresponding domain Ξ^0_ϵ for the solution to live in. But this domain is going to expand infinitely as $\epsilon \to 0$. So, positivity and energy inequality are not enough to imply ϵ-uniform stability of $\underline{\mathbb{U}}^{n+1}$. For that, one needs to find a positive lower-bound for the density as well.

4. Conclusion and future works

We have extended the stability results of the Lagrange–projection scheme presented in [8], to all-Mach one-dimensional isentropic Euler equations flows. We have shown that the scheme is AP consistent, i.e. the scheme satisfies the div-free condition in the zero-Mach number limit, and satisfies the discrete counterpart of continuous asymptotic expansion. Also we have shown that for the well-prepared initial data, there exists a Mach-uniform time step which satisfies entropy inequality, density positivity, as well as stability of the solution in L_∞-norm. In other words, we have extended the results of [8] for all-Mach regime, in the price of forcing the initial condition to be well-prepared.

The natural next step would be to extend this analysis to full Euler equations, or to balance laws (with an additional source term like topogrphy for the shallow water equations), which is a formidable task. Also as done in [26] by compensated compactness approach, it is of interest to prove the convergence of the scheme to the unique entropy solution.

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[1] Also note that for low Mach number flows, it is a reasonable to assume that the density is bounded away from zero provided that the initial density is, even for infinite time.

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