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A new stable splitting for singularly perturbed ODEs

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In this publication, we consider IMEX methods applied to singularly perturbed ordinary differential equations. We introduce a new splitting into stiff and non-stiff parts that has a direct extension to systems of conservation laws, and investigate analytically and numerically its performance. We show that this splitting can in some cases improve the order of convergence, showing that the phenomenon of order reduction is not only a consequence of the method but also of the splitting.

1. Introduction

The computation of extremely stiff ordinary differential equations has been the subject of extensive research over the last decades, see for example the standard textbook [19]. Our interest in stiff ordinary differential equations stems from the approximation of compressible fluid flows [1, 35] using appropriate spatial discretization, such as for example the finite volume [16, 17, 28, 27] or the discontinuous Galerkin [10, 9, 8, 7] method. In particular, we are interested in the approximation of flows in the nearly incompressible regime, i.e., at very low Mach numbers ε [26]. This does not only lead to stiffness induced by the discretization parameter (Δx, say), but also to stiffness induced by the singular perturbation problem that constitutes the transition from compressible to incompressible flows [24, 32].

It has been recognized that in many cases, it might be beneficial to separate stiff (w.r.t. ε) from non-stiff terms and treat them implicitly and explicitly, respectively. Examples in the context of computational fluid dynamics can be found, e.g., in [25, 11, 14, 18, 29]; other examples, for instance from linear or elliptic equations can be found in [13, 33]. It has been recognized in [34] that decomposing the equations into stiff and non-stiff terms is not trivial. Even if both parts are stable independently, this does not necessarily mean that the overall algorithm is stable. For linear equations, the authors in [34] have found a uniformly stable scheme based on characteristic decomposition. This, however, can not easily be extended to nonlinear equations. To this end, we investigate a new, more general splitting based on the solution of the unperturbed ('incompressible') solution in this paper. This splitting has a direct extension to systems of conservation laws.

Splitting methods lead to implicit / explicit (IMEX) time integration routines. Famous integrators include IMEX multistep methods, see, e.g., [12, 2, 20] and IMEX Runge-Kutta methods [3, 4, 5, 31]. We focus on IMEX-BDF methods (to be explained in Sec. 3) and IMEX-RK methods (to be explained in Sec. 4).

In this work, we consider the ordinary differential equation

$$w'(t) = f(w(t)), \qquad w(0) = w_0, \tag{1}$$

where, for $\varepsilon > 0$,

$$w := (y, z), \qquad f(w) := \begin{pmatrix} z \\ \frac{g(y,z)}{\varepsilon} \end{pmatrix}.$$

We assume that $\partial_z g(y, z)$ is different from zero in the vicinity of the solution and that it is (at least) in class $C^2(\mathbb{R}^2)$. This guarantees that the limit equation for $\varepsilon \to 0$ is a differential algebraic equation (DAE) of index one. One particular instance of (1) is van der Pol equation, defined by

$$g(y, z) := \left(1 - y^2\right) z - y. \tag{2}$$

Our interest is in the case as $\varepsilon \to 0$. Expanding w in terms of ε formally as

$$w(t) = w_{(0)}(t) + \varepsilon w_{(1)}(t) + \varepsilon^2 w_{(2)}(t) + \mathcal{O}(\varepsilon^3)$$

reveals that $w_{(0)} = (y_{(0)}, z_{(0)})$ fulfills the DAE

$$y'_{(0)}(t) = z_{(0)}(t), \qquad g(y_{(0)}, z_{(0)}) = 0. \tag{3}$$

(Please note that a subscript "0" refers to initial conditions, while a subscript "(0)" refers to asymptotic expansion. We refer to $w_{(0)}$ as the *reference solution.*) Obviously, only carefully crafted initial conditions induce a well-posed DAE for $w_{(0)}$, and the same holds true for any $w_{(i)}$. Initial conditions w_0 that 'survive' the limit as $\varepsilon \to 0$ are called *well-prepared.* One particular set of initial conditions for van der Pol equation from literature [19], that we are going to use in the sequel, is given by

$$w_0 = \left(2, -\frac{2}{3} + \frac{10}{81}\varepsilon - \frac{292}{2187}\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)^T.$$

We note that, unlike for systems of conservation laws, the question of a suitable splitting for ordinary differential equations has usually not been discussed in literature. This is probably because for ordinary differential equations, there is a 'naive' splitting that can be applied in a stable way, namely,

$$f(w) = \begin{pmatrix} z \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{g(y,z)}{\varepsilon} \end{pmatrix}, \tag{4}$$

where the second part is treated implicitly. In this sense, this paper can be understood as a proof of concept, with obvious extension to more complex problems.

It is well-known that zero-stable IMEX-BDF methods (up to six stages) are both uniformly stable and uniformly consistent, i.e., independent of the relation between the perturbation parameter ε and the time step Δt, the error converges with the correct order in Δt to zero. The same is not true for general IMEX-RK methods [4] (with the exception of the specifically designed Runge-Kutta method by Boscarino [5]). This means that for $\Delta t \gg \varepsilon$, the error seems to exhibit some kind of degradation in convergence in general. In this work, we present some comparisons between the naive splitting and the newly developed splitting, showing that order reduction is also a phenomenon of the splitting and not only the temporal integration.

The paper is organized as follows: In Sec. 2, we introduce the new splitting based on what we call the *reference solution* $w_{(0)}$. In Sec. 3, we combine this splitting with the IMEX-BDF method and show that this constitutes a reasonable discretization in the sense that for $\varepsilon \to 0$ and Δt fixed, it converges toward a scheme for the limit DAE. This property is frequently called *asymptotic preserving*, see, e.g., [21, 23, 22]. This analysis is equipped with numerical results. In Sec. 4, we apply the splitting to IMEX-RK schemes and show numerical results. It seems that for a certain sub-class of those methods, there is *no* (or less) order reduction for the new splitting, see Sec. 4.

2. Splitting

In this section, we define the newly-developed splitting. More formally, we introduce $\widehat{f}$ and $\widetilde{f}$, such that the right-hand side f of the ODE (1) can be split into

$$f(w) = \widehat{f}(w) + \widetilde{f}(w),$$

and we think about $\widetilde{f}(w)$ as a 'stiff' contribution to the flux. In (4), we already showed the standard way to do it. In this publication, we propose a splitting that has an extension to other types of ODEs and is related to the reference solution (RS) $w_{(0)}$, i.e., the (formal) limit solution for $\varepsilon \to 0$, see also (3). The splitting is consequently called RS-IMEX splitting:

Definition 1. *RS-IMEX splitting: We define the following splitting for the right-hand side f of (1):*

$$\widetilde{f}(w) = f(w_{(0)}) + f'(w_{(0)})(w - w_{(0)}), \qquad \widehat{f}(w) = f(w) - \widetilde{f}(w). \tag{5}$$

There are a couple of remarks in order here:

Remark 1.

1. *The motivation of this splitting is that for ε close to zero, the term $w - w_{(0)}$ is supposed to be $\mathcal{O}(\varepsilon)$, i.e., small. In particular, $\widehat{f}$ is supposed to be small. Therefore, we can have the hope that stiffness is reduced.*
2. *Note that $f(w_{(0)}) = (z_{(0)}, 0)^T$ and*

$$f'(w_{(0)}) = \begin{pmatrix} 0 & 1 \\ \frac{1}{\varepsilon}\partial_1 g(w_{(0)}) & \frac{1}{\varepsilon}\partial_2 g(w_{(0)}) \end{pmatrix}.$$

From this, one can conclude that for $\Delta w := w - w_{(0)}$ (Δy, Δz accordingly), there holds

$$\widetilde{f}(w) = \begin{pmatrix} z \\ \frac{1}{\varepsilon}\left(\partial_1 g(w_{(0)})\Delta y + \partial_2 g(w_{(0)})\Delta z\right) \end{pmatrix},$$
$$\widehat{f}(w) = \begin{pmatrix} 0 \\ \frac{1}{\varepsilon}\left(g(w) - \partial_1 g(w_{(0)})\Delta y - \partial_2 g(w_{(0)})\Delta z\right) \end{pmatrix}.$$

3. *In practice, $w_{(0)}$ is replaced by an approximation $w_{(0)}^{app}$.*
4. *Because $w_{(0)}$ depends on t, both $\widehat{f}$ and $\widetilde{f}$ depend on t as well. Most of the time, we will omit this dependence for the sake of simplicity.*

We note once again that this splitting is universal in the sense that for any singular perturbation problem, such a splitting is - at least formally - possible. Based on this splitting, we can introduce IMEX schemes in the next sections.

3. IMEX-BDF

In this section, we couple the splitting defined in Section 2 to an IMEX-BDF scheme [20]. It is well-known that BDF schemes belong to the class of linear multistep schemes and are constructed in such a way that the update $w_{\Delta t}^{n+1}$ is given by the expression

$$\sum_{j=-1}^{s} \alpha_j w_{\Delta t}^{n-j} = \Delta t f(w_{\Delta t}^{n+1}).$$

Remark 2. *1. BDF schemes are zero-stable up to $s = 5$.*

2. Computing the coefficients is easily possible using the relation $A\vec{\alpha} = (0, 1, 0, \dots, 0)^T$ for $\vec{\alpha} = (\alpha_{-1}, \dots, \alpha_s)^T$ and matrix A with $A_{ij} = -\frac{(j-1)^{i-1}}{(i-1)!}$.

The so-called extrapolated BDF scheme can be construced to be

$$\sum_{j=-1}^{s} \alpha_j w_{\Delta t}^{n-j} = \sum_{j=0}^{s} \Delta t \beta_j f(w_{\Delta t}^{n-j}),$$

obviously, they are explicit.

Remark 3. *Again, the β_j fulfill a linear system of equations: $\vec{\beta} = (\beta_0, \dots, \beta_s)^T$ fulfills $B\vec{\beta} = (1, 0, 0, \dots, 0)^T$ for matrix B with $B_{ij} = (-1)^{i-1} \frac{j^{i-1}}{(i-1)!}$.*

Based on a splitting of f as in (5) it is obvious to construct the IMEX-BDF scheme as

$$\sum_{j=-1}^{s} \alpha_j w_{\Delta t}^{n-j} = \Delta t \widetilde{f}(w_{\Delta t}^{n+1}) + \Delta t \sum_{j=0}^{s} \beta_j \widehat{f}(w_{\Delta t}^{n-j}). \tag{6}$$

Those schemes are usually indexed by their convergence order $s + 1$. We are now ready to show the main theorem, namely, that the algorithm is *asymptotic preserving*, to be explained in the sequel.

3.1. Asymptotic Preserving Property

We start this section with a definition:

Definition 2. *An algorithm for the computation of a solution to (1) is called* asymptotic preserving *(AP) if the* discrete *limit (w.r.t. $\varepsilon \to 0$) algorithm is a stable approximation to (3).*

An illustration that is frequently shown in this context [22] can be seen in Fig. 1. If the diagram commutes (i.e., the order of limits $\Delta t \to 0$ and $\varepsilon \to 0$ can be changed) the algorithm is asymptotic preserving.

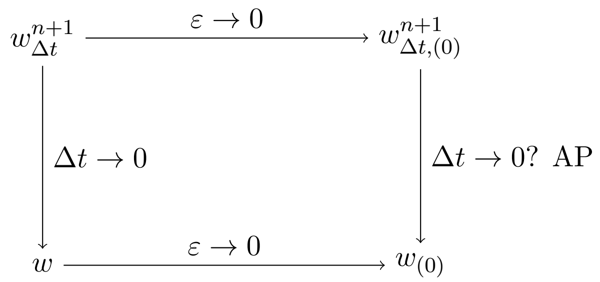

Figure 1: Illustration of the AP property. If $w_{\Delta t,(0)}^{n+1}$ converges toward $w_{(0)}$ for $\Delta t \to 0$, the algorithm is asymptotic preserving (AP).

Theorem 1. *The algorithm (6) is asymptotic preserving with correct order $s + 1$.*

Proof. Let $w_{\Delta t}^n$ be expanded in terms of ε as

$$w_{\Delta t}^n = w_{\Delta t,(0)}^n + \varepsilon w_{\Delta t,(1)}^n + \mathcal{O}(\varepsilon^2)$$

for all n. We assume that start values $w^j_{\Delta t,(0)}$, $0 \le j \le s$ are consistent to the right order, i.e., $w^j_{\Delta t,(0)} - w_{(0)}(t^j) = \mathcal{O}(\Delta t^{s+1})$.

The (formal) limit algorithm for $\varepsilon \to 0$ is given by

$$\sum_{j=-1}^{s} \alpha_j y^{n-j}_{\Delta t,(0)} = \Delta t z^{n+1}_{\Delta t,(0)}, \tag{7a}$$

$$0 = \sum_{j=0}^{s} \beta_j \left(g(w^{n-j}_{\Delta t,(0)}) - \partial_1 g(w_{(0)}) \Delta y^{n-j}_{\Delta t,(0)} - \partial_2 g(w_{(0)}) \Delta z^{n-j}_{\Delta t,(0)} \right) \tag{7b}$$

$$+ \partial_1 g(w_{(0)}) \Delta y^{n+1}_{\Delta t,(0)} + \partial_2 g(w_{(0)}) \Delta z^{n+1}_{\Delta t,(0)}. \tag{7c}$$

The first equation can be rewritten as

$$\sum_{j=-1}^{s} \alpha_j y^{n-j}_{\Delta t,(0)} = \Delta t z_{(0)}(t^{n+1}) + \Delta t \Delta z^{n+1}_{\Delta t,(0)}.$$

This means that $y^{n+1}_{\Delta t,(0)} = y_{(0)}(t^{n+1}) + \mathcal{O}(\Delta t^{s+1}) + \mathcal{O}(\Delta t \Delta z^{n+1}_{\Delta t,(0)})$, which implies that $\Delta y^{n+1}_{\Delta t,(0)} = \mathcal{O}(\Delta t \Delta z^{n+1}_{\Delta t,(0)}) + \mathcal{O}(\Delta t^{s+1})$ for all n. Furthermore, the algebraic equation (7b)-(7c) then implies recursively that $\Delta z^{n+1}_{\Delta t,(0)} = \mathcal{O}(\Delta t^{s+1})$. □

3.2. Numerical Results

Validation of the scheme In this section, we show numerical results based on van der Pol equation (see eqs. (1) and (2)) to show that the performed algorithm works as expected. We employ both IMEX-BDF 2 and IMEX-BDF 4, given by

$$\frac{3}{2} w^{n+1} - 2w^n + \frac{1}{2} w^{n-1} = \Delta t \left(\widetilde{f}(w^{n+1}) + 2\widehat{f}(w^n) - \widehat{f}(w^{n-1}) \right) \text{ and}$$

$$\frac{25}{12} w^{n+1} - 4w^n + 3w^{n-1} - \frac{4}{3} w^{n-2} + \frac{1}{4} w^{n-3} = \Delta t \left(\widetilde{f}(w^{n+1}) + 4\widehat{f}(w^n) - 6\widehat{f}(w^{n-1}) + 4\widehat{f}(w^{n-2}) - \widehat{f}(w^{n-3}) \right)$$

respectively.

In Fig. 2, numerical results are shown for IMEX-BDF 2 (left figure) and IMEX-BDF 4 (right figure). The splitting employed is the one given in Def. 1, the reference solution $w_{(0)}$ is computed exactly. Note that this is possible for van der Pol's example analytically. Initial steps needed for this multistep scheme are computed with a stiff integrator to extremely high precision. From the figures, one can clearly see that there is no order degradation as $\varepsilon \to 0$, i.e., both IMEX-BDF schemes converge with their respective order of two and four uniformly in ε until they hit machine zero. Furthermore, and this is obviously a consequence of the AP property, error curves lie nearly on top of each other, because they only differ by $\mathcal{O}(\varepsilon)$.

Comparison to standard and approximate splitting Obviously, the most straightforward splitting (which, indeed, is usually employed in literature) is to split as in (4) and treat the ε-dependent part implicitly, i.e., take the stiff part to be $\widetilde{f}(w) = \begin{pmatrix} 0 \\ \frac{g(y,z)}{\varepsilon} \end{pmatrix}$. We denote this splitting in short form by Std.

Furthermore, in practical cases, $w_{(0)}$ (needed in (5)) is not readily available, so one has to compute it numerically. This approach, which we call RS-Approximate (or RSApp, for short), has also been implemented using a BDF discretization (always with corresponding order) of the limit differential algebraic equation. In Fig. 3, we show numerical results for these approaches in comparison to the splitting given in Def. 1. One can observe that the RS-IMEX always behaves a little bit better than the other two (which is not surprising,

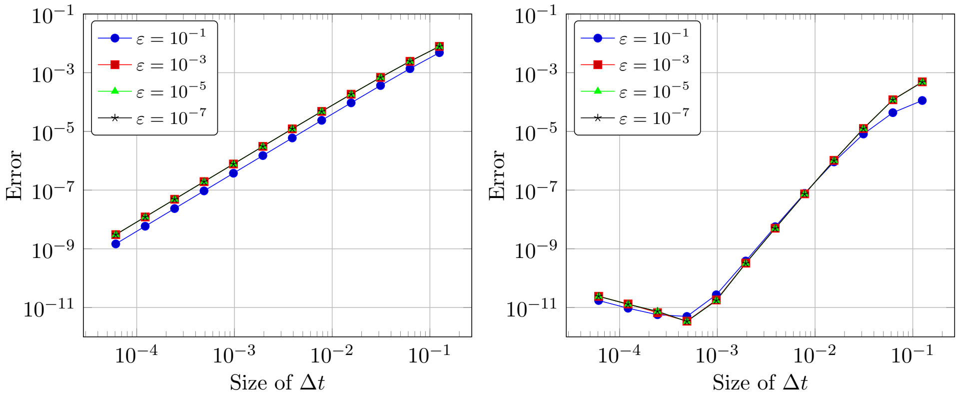

Figure 2: Convergence results for van der Pol equation for different values of ε, using the RS-IMEX splitting coupled with IMEX-BDF 2 (left) and IMEX-BDF 4 (right). $w_{(0)}$ has been computed analytically; initial values for multistep scheme stem from a highly resolved ('exact') numerical approximation.

because there is actually much more information about the limit solution than in, e.g., the naive splitting). Furthermore, it seems that the RSApp scheme behaves somewhat worse than the standard splitting. Overall, however, it is interesting to see that quantitatively and qualitatively, they all behave very similarly.

4. IMEX-RK

Similar to BDF schemes, one can also couple implicit and explicit Runge-Kutta methods with the splitting defined in Def. 1. The following definition gives those schemes, for a more thorough discussion we refer to [4] and the references therein. We only use diagonally implicit methods.

Definition 3 (IMEX Runge-Kutta Scheme)**.** *For every $t^{n+1} = t^n + \Delta t$ do the following:*

1. *(Stages) For $i = 1, \ldots, s$ solve*

$$w_i = w^n + \Delta t \sum_{j=1}^{i} \widetilde{A}_{i,j} k_j + \Delta t \sum_{j=1}^{i-1} \hat{A}_{i,j} l_j.$$

   with $k_i = \tilde{f}(w_i, t^n + \widetilde{c}_i \Delta t)$ and $l_i = \hat{f}(w_i, t^n + \widehat{c}_i \Delta t)$. (Note that here, the dependence of $\tilde{f}$ and $\hat{f}$ on t is crucial, see also Rem. 1, which is why we make it explicit.)

2. *(Update) Finally evaluate*

$$w^{n+1} = w^n + \Delta t \sum_{j=1}^{s} \widetilde{b}_j k_j + \Delta t \sum_{j=1}^{s} \hat{b}_j l_j$$

The coefficients of the IMEX-RK method are given by two Butcher tableaux, the one with overhats referring to the explicit, the other to the implicit method. See also Tbl. 1.

Remark 4. *Example tableaux used in the numerical results can be found in A.*

As already pointed out in the introduction, solving singularly perturbed equations can actually lead to the phenomenon of order reduction. This means that for common schemes and $\Delta t \gg \varepsilon$, the formal convergence order is not achieved. In the numerical results section, we compare different IMEX-RK schemes for standard and IMEX-RS splitting. Again, we use van der Pol equation with both an 'exact' reference solution $w_{(0)}$ and an approximate reference solution $w_{(0)}^{app}$. The computation of this approximation is explained in the sequel.

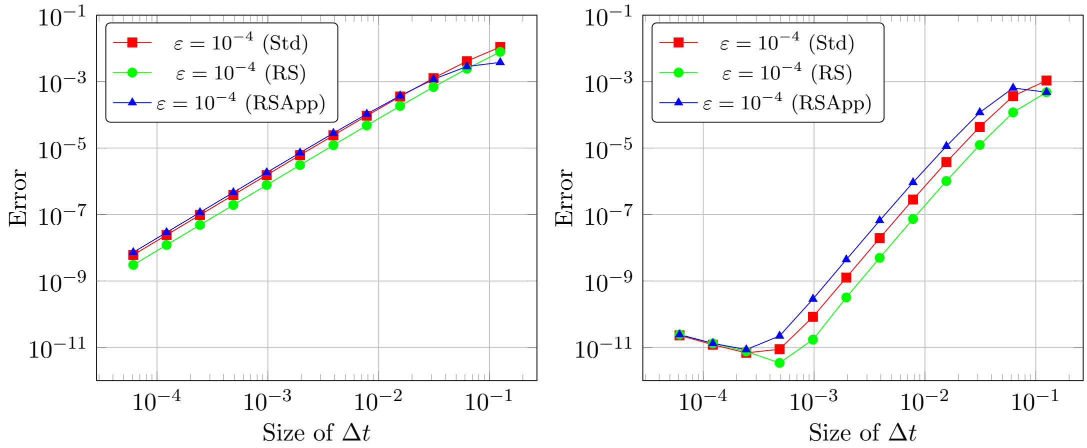

Figure 3: Convergence results for van der Pol equation and $\varepsilon = 10^{-4}$ with IMEX-BDF 2 (left) and IMEX-BDF 4 (right). Three different methods are investigated: (Std) Standard method from literature with splitting as in (4). (RS) RS-IMEX splitting from Def. 1. (RSApp) RS-IMEX splitting from Def. 1, but $w_{(0)}$ has been computed using a BDF scheme of order 2 (left) and order 4 (right).

$\hat{c}$	$\hat{A}$	$\widetilde{c}$	$\widetilde{A}$
	$\hat{b}^T$		$\widetilde{b}^T$

Table 1: Butcher tableaux for an IMEX-RK scheme. Left explicit, right implicit. Actual tableaux can be found in A.

Approximation of $w_{(0)}^{app}$ Using the IMEX scheme as in Def. 3 with the standard splitting given in (4) allows to take the formal limit as $\varepsilon \to 0$. This yields a Runge-Kutta discretization for the differential-algebraic equation (3). In our numerical experiments, we compute the approximation $w_{(0)}^{app}$ using the same Runge-Kutta integrator as in the example considered, with exactly the same time steps. The stages of the computation of $w_{(0)}^{app}$ are saved and used in the computation of k_j and l_j needed in Def. 3. Note that it does not make sense to use the RS-IMEX splitting for this computation, because its use necessitates the knowledge of $w_{(0)}$.

4.1. Numerical Results

In this section we show the numerical results for van der Pol equation with different IMEX-RK discretizations, and show some interesting observation which we have not found any explication for so far.

Again, in all the results, we compare standard splitting versus the new RS-IMEX splitting versus a splitting that uses an approximate version of $w_{(0)}$ (RSApp, for short). We want to mention already at this point that there is hardly any difference between RS-IMEX and RSApp splitting in all our numerical experiments.

Error has been computed as the two-norm of the difference to the solution at end-time $T = 0.5$

Works as expected: ARS-443 with RS-IMEX Our point of departure is the ARS-443 scheme given in [2], see also the augmented Butcher tableau in Tbl. 3. (443 refers to 4 stages explicit, 4 stages implicit, and third order convergence.) In Fig. 4, we plot results for the standard splitting (4), the new RS-IMEX splitting and its approximate version for various ε. It can be observed that results are nearly non-distinguishable. In particular, the order reduction (the non-uniform convergence as ε gets increasingly small) can be observed

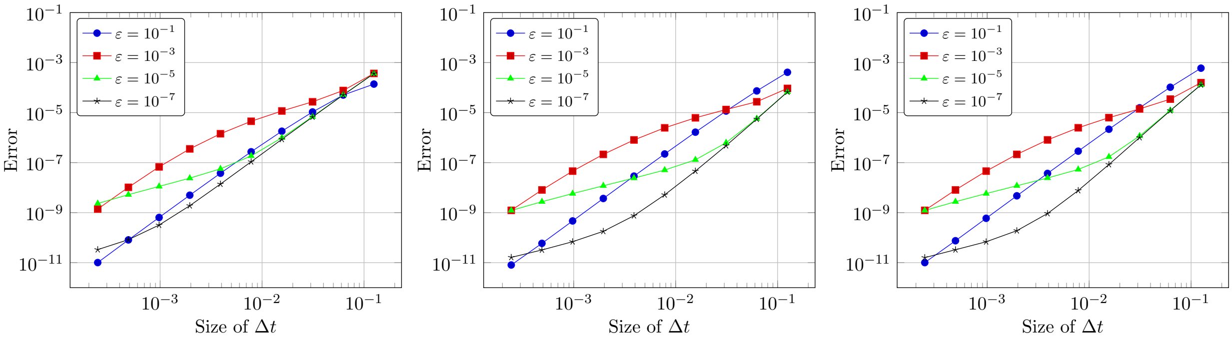

Figure 4: Convergence results for van der Pol equation for different values of ε, using the ARS-443 IMEX-RK scheme coupled with the standard splitting (left), the RS-IMEX splitting with analytical $w_{(0)}$ (middle) and the RS-IMEX splitting with approximate $w_{(0)}$ (right).

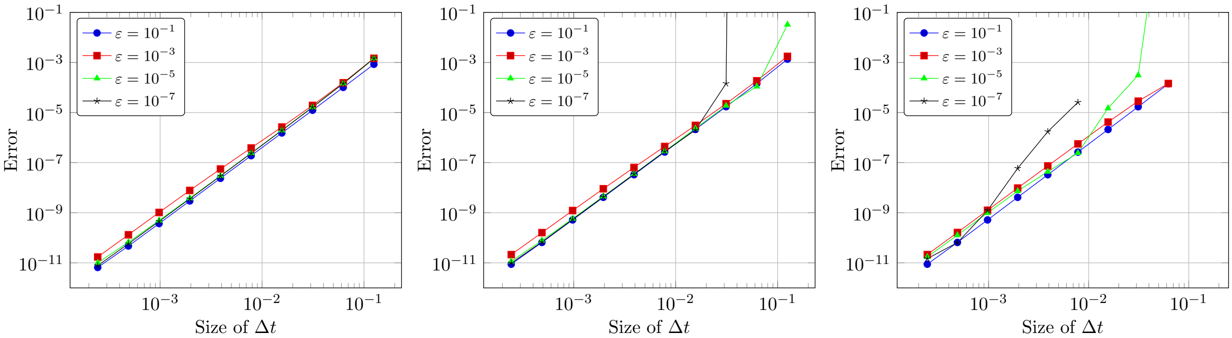

Figure 5: Convergence results for van der Pol equation for different values of ε, using the BHR-553 IMEX-RK scheme coupled with the standard splitting (left), the RS-IMEX splitting with analytical $w_{(0)}$ (middle) and the RS-IMEX splitting with approximate $w_{(0)}$ (right).

for both schemes. So far, this is what one would actually expect from our experiences with the IMEX-BDF scheme, where the choice of splittings did hardly have an influence.

Works better without update step: BHR-553 with RS-IMEX The BHR-553 (5 stages, third order) scheme has been explicitly designed in [5] to circumvent the order reduction. See Tbl. 5 for the corresponding Butcher tableau.

In our numerical experiments, see Fig. 5, we can observe uniform third order convergence for both standard and RS-IMEX splitting. However, it seems that for extremely small values of ε, and 'large' values of Δt, the algorithm becomes unstable for the RS-IMEX, see Fig. 6. We were not able to figure out whether this is because of a conceptual problem, or due to cancellation errors. Curiously, this effect does not show up if one chooses w^{n+1} to be w_5, i.e., instead of taking an update, the last stage of the Runge-Kutta method is taken as new update, see Fig. 6. Additionally, third order is recovered, which is peculiar because the last stage only has a formal consistency of two.

Improved convergence: BPR-353 and DPA-242 with RS-IMEX The last paragraph is devoted to BPR-353 and DPA-242 schemes, presented in [6] and [15], respectively. Corresponding Butcher tableaux can be found in Tbl. 4 and Tbl. 2. Coupled to the standard splitting (4), it is well-known that these methods

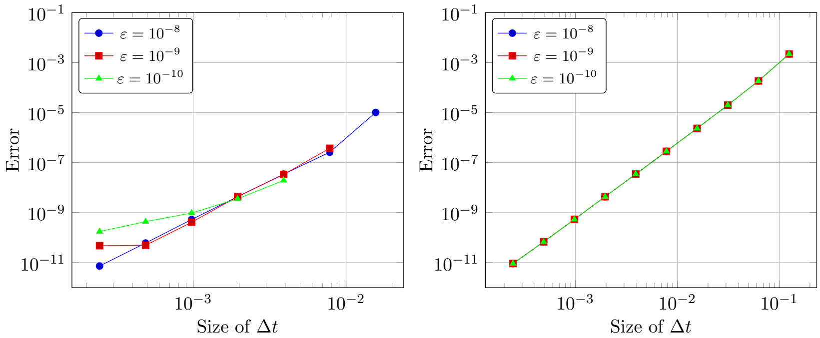

Figure 6: Convergence results for van der Pol equation for different values of ε, using the BHR-553 IMEX-RK scheme coupled with the RS-IMEX splitting. Left: Full BHR-553 scheme which obviously exhibits instabilities (unplotted values are NaN). Right: Neglecting the update step. $w_{(0)}$ has been computed analytically.

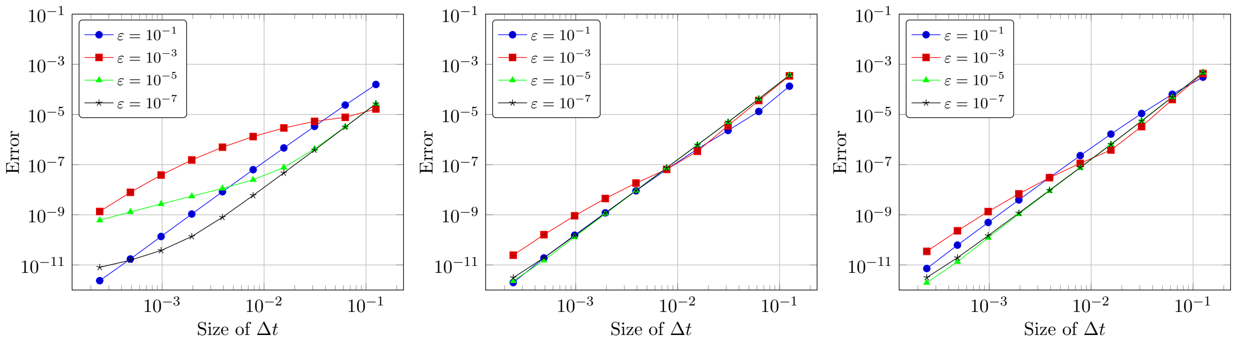

Figure 7: Convergence results for van der Pol equation for different values of ε, using the BPR-353 IMEX-RK scheme coupled with the standard splitting (left), the RS-IMEX splitting with analytical $w_{(0)}$ (middle) and the RS-IMEX splitting with approximate $w_{(0)}$ (right).

exhibit quite severe order loss as can be seen in Figs. 7 and 9 on the left sides. In particular, for the DPA scheme, this order loss is quite significant. Coupling both methods to the newly developed RS-IMEX splitting seems to yield uniform convergence in ε, see Figs. 7 and 9 on the right.

Remark 5. *We note that we did also test this on the Pareschi-Russo equation [30]*

$$y' = -z, \qquad z' = y + \frac{\sin(y) - z}{\varepsilon} \tag{8}$$

see also Fig. 8, showing that the observed phenomenon is not a feature of van der Pol equation.

Up to now, we do not have a solid explanation for this effect, but we conjecture that it is because of the Taylor series approach employed in (5), that enforces the numerical solution to be close to $w_{(0)}$.

5. Conclusion and Outlook

In this publication, we have developed a new splitting based on the reference solution and shown numerical comparison with the more established standard splitting. We have shown that for IMEX schemes of BDF

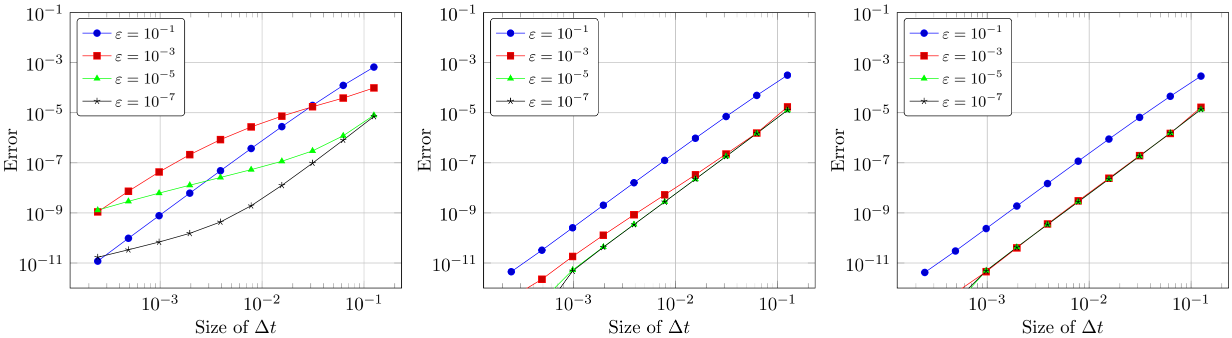

Figure 8: Convergence results for the *Pareschi-Russo* (8) equation for different values of ε, using the BPR-353 IMEX-RK scheme coupled with the standard splitting (left), the RS-IMEX splitting with analytical $w_{(0)}$ (middle) and the RS-IMEX splitting with approximate $w_{(0)}$ (right).

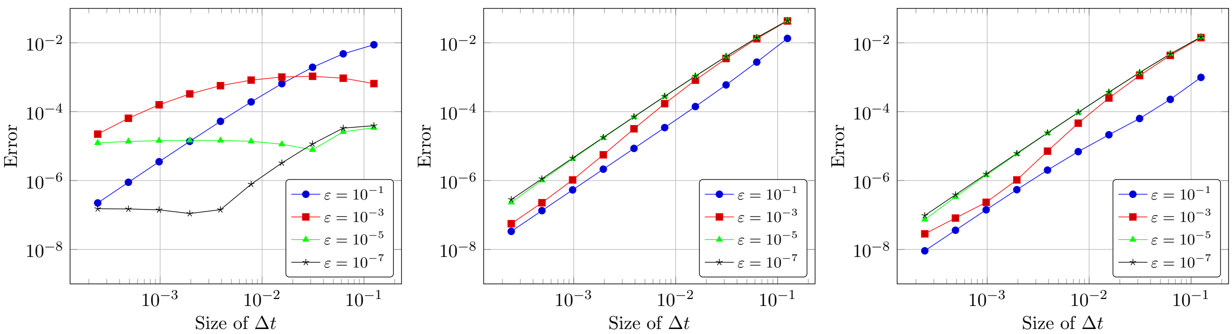

Figure 9: Convergence results for van der Pol equation for different values of ε, using the DPA-242 IMEX-RK scheme coupled with the standard splitting (left), the RS-IMEX splitting with analytical $w_{(0)}$ (middle) and the RS-IMEX splitting with approximate $w_{(0)}$ (right).

type, the influence of the splitting is marginal, and we could also show the asymptotic preserving property. In contrast, for IMEX schemes of Runge-Kutta type, the splitting has indeed a broad influence. We have shown numerical results demonstrating that it is even possible to obtain faster-converging schemes.

Obviously, in this context, van der Pol equation is not necessarily the most interesting context. Our interest is in high-order approximation of singularly perturbed conservation laws. The next key step is therefore to apply the RS-IMEX splitting to the compressible Euler / Navier-Stokes equations. We anticipate that this will impose more severe problems - although preliminary results already show a satisfactory behavior - as the discretization of the limit equation will most likely have a much higher influence on the quality and stability of an overall algorithm. Currently, this is work in progress.

A. Butcher tableaux

For the sake of completeness, we show the augmented Butcher tableaux of the employed IMEX-RK schemes.

0	0	0	0	0
1/3	1/3	0	0	0
1	1	0	0	0
1	1/2	0	1/2	0
	1/2	0	1/2	0

1/2	1/2	0	0	0
2/3	1/6	1/2	0	0
1/2	-1/2	1/2	1/2	0
1	3/2	-3/2	1/2	1/2
	3/2	-3/2	1/2	1/2

Table 2: DPA-242 [15]

0	0	0	0	0	0
1/2	1/2	0	0	0	0
2/3	11/18	1/18	0	0	0
1/2	5/6	-5/6	1/2	0	0
1	1/4	7/4	3/4	-7/4	0
	1/4	7/4	3/4	-7/4	0

0	0	0	0	0	0
1/2	0	1/2	0	0	0
2/3	0	1/6	1/2	0	0
1/2	0	-1/2	1/2	1/2	0
1	0	3/2	-3/2	1/2	1/2
	0	3/2	-3/2	1/2	1/2

Table 3: ARS-443 [2]

0	0	0	0	0	0
1	1	0	0	0	0
2/3	4/9	2/9	0	0	0
1	1/4	0	3/4	0	0
1	1/4	0	3/4	0	0
	1/4	0	3/4	0	0

0	0	0	0	0	0
1	1/2	1/2	0	0	0
2/3	5/18	-1/9	1/2	0	0
1	1/2	0	0	1/2	0
1	1/4	0	3/4	-1/2	1/2
	1/4	0	3/4	-1/2	1/2

Table 4: BPR-353[6]

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0	0	0	0	0	0
0.871733	0.871733	0	0	0	0
0.871733	0.435867	0.435867	0	0	0
2.34021	-0.800998	0	3.14121	0	0
1	0.356753	-0.19734	0.881949	-0.0413622	0
	0.412898	0	0.19734	-0.0461045	0.435867

0	0	0	0	0	0
0.871733	0.435867	0.435867	0	0	0
0.871733	0.435867	0	0.435867	0	0
2.34021	-0.0667587	0	1.9711	0.435867	0
1	0.412898	0	0.19734	-0.0461045	0.435867
	0.412898	0	0.19734	-0.0461045	0.435867

Table 5: BHR-553 [5]

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