A Certified Reduced Basis Method for Parametrized Quadratically Nonlinear Diffusion Equations

Mohammad Rasty[a] **and Martin A. Grepl**[a,*]

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Institut für Geometrie und Praktische Mathematik
RWTH Aachen

Templergraben 55, D-52056 Aachen, (Germany)

[a] RWTH Aachen University, Numerical Mathematics, Templergraben 55, 52056 Aachen, Germany

[*] Corresponding author. Tel.: +49 241 8096470, Fax: +49 241 80696470
Email addresses: rasty@igpm.rwth-aachen.de (Mohammad Rasty), grepl@igpm.rwth-aachen.de (Martin A. Grepl)

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A Certified Reduced Basis Method for Parametrized Quadratically Nonlinear Diffusion Equations

Mohammad Rasty[a], Martin A. Grepl[a,*]

[a]RWTH Aachen University, Numerical Mathematics, Templergraben 55, 52056 Aachen, Germany

Abstract

We present a certified reduced basis method for a steady-state quadratically nonlinear diffusion equation. We employ the standard Galerkin recipe for the reduced basis approximation and derive associated *a posteriori* error estimation procedures based on the Brezzi-Rappaz-Raviart (BRR) framework. We show that all necessary ingredients, i.e., the dual norm of the residual, the Sobolev embedding constant, and a lower bound of the inf-sup constant, can be decomposed in an offline-online computational decompostion. Numerical results are presented to confirm the rapid convergence of the reduced basis approximation and the rigor and sharpness of the associated *a posteriori* error bound.

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1. Introduction

Many problems in computational science and engineering require real-time and/or numerous evaluations of input-output relationships induced by parametrized partial differential equations (PDEs). For example, the iterative solution of inverse and parameter optimization problems involves the evaluation of the PDE at each iteration, whereas the online solution of model-based control problems necessitates the real-time evaluation of the PDE-model. Despite increased computing capabilities and tremendous advances in numerical analysis, this task can easily become computationally infeasible. Thus, fast and reliable reduced computational models are a necessity.

The reduced basis method is a model order reduction technique which provides efficient yet reliable approximations to solutions of parameterized PDEs; see e.g. the review [25] and the references therein for reduced basis methods and [2, 4, 26] for other model order reduction techniques. The method is thus ideally suited for the real-time and many-query contexts and has been successfully applied to various problems, e.g. parameter optimization [20] and estimation [13], multiscale analysis [5], stochastic problems [6], uncertainty quantification [19], and optimal control [9, 16].

Starting from linear coercive elliptic PDEs with affine parameter dependence [22], the reduced basis method has been extended to a large class of parametrized PDEs over the last decade. The essential ingredients of the reduced basis method are: Galerkin projection onto a subspace spanned by solutions of the parametrized PDE at (greedily) selected parameter values; rigorous *a posteriori* error estimation procedures; and offline-online decompositions for the computation of the approximation and associated error bound. Nonlinear problems, however, pose a special challenge in terms of rigorous *a posteriori* error estimation procedures and a full decoupling of the offline and online computations.

In this paper we present a certified reduced basis method for quadratically nonlinear diffusion equations of the form

$$\operatorname{div}\left(G(u;\mu)\,\nabla u\right) = \mathrm{f}, \tag{1}$$

*Corresponding author. Tel.: +49 241 8096470, Fax: +49 241 80696470
Email addresses: `rasty@igpm.rwth-aachen.de` (Mohammad Rasty), `grepl@igpm.rwth-aachen.de` (Martin A. Grepl)

where $G(u;\mu)$ is a μ-parametrized function which depends linearly on the field variable u and f denotes a source term; see Section 2 for a detailed problem description. The Brezzi-Rappaz-Raviart (BRR) framework [7, 8] allows us to derive rigorous and efficiently evaluable *a posteriori* error bounds for the reduced basis approximation of (1). We show that for this type of nonlinearity the computation of all necessary ingredients of the BRR framework can be decomposed in an offline-online fashion, i.e., the dual norm of the residual, the Sobolev embedding constant, and a lower bound of the solution-dependent inf-sup stability factor. To obtain the latter, we employ the successive constraint method (SCM) [15]. Nonlinear diffusion problems like (1) appear for example in the area of nonlinear heat transfer, where the thermal conductivity of the material is not just assumed to be constant — an often used simplification to obtain the linear heat equation — but is more accurately modeled as temperature dependent; see e.g. [14, 27] for a reference article and book. In an inverse heat transfer setting, the goal would be to estimate the parametrized function $G(u;\mu)$ from temperature measurements on the boundary, see e.g. [1, 10]. Furthermore, the model may also be considered as a first order approximation to the higher-order nonlinear porous medium equation [28, 29].

There are essentially two approaches that have been pursued in the reduced basis literature for nonlinear problems. Problems involving at most quadratically nonlinear terms — like the Burgers and Navier Stokes equations — have been successfully treated with the standard Galerkin recipe, which still allows to obtain an efficient offline-online decomposition. Furthermore, rigorous *a posteriori* error bounds can be derived for such problems based on the BRR framework. For a certified reduced basis method of the Burgers equation we refer to [31, 34] and for the steady and unsteady incompressible Navier-Stokes equations to [30, 33]; also see [17] for an alternative approach not involving the BRR framework. For problems involving higher-order or nonpolynomial nonlinearities, on the other side, the Empirical Interpolation Method [3] is typically used to approximate the nonlinear terms. The reason lies in the Galerkin recipe: an N-dimensional reduced basis approximation of a problem involving a nonlinearity of order q results in an online cost of $O(N^{2q})$ and is thus prohibitive for large q; nonpolynomial nonlinearities do not even allow a full offline-online decomposition [24]. The Empirical Interpolation Method recovers the online efficiency — see [12, 18, 11] for applications to nonlinear elliptic and parabolic problems — but the *a posteriori* error bounds are rigorous only under certain conditions on the nonlinear function approximation [11].

The rest of this paper is organized as follows: In Section 2 we introduce the problem statement as well as necessary definitions and assumptions and present a model problem. In Section 3 we discuss the reduced basis approximation before turning to the *a posteriori* error estimation in Section 4. The offline-online decomposition of all necessary quantities is presented in Section 4.3. Finally, in Section 5 we present numerical results to confirm the rapid convergence of the presented method and the rigor and the sharpness of the associated *a posteriori* error bound.

2. Problem Statement

2.1. Abstract formulation

We first define the Hilbert space $X^{\mathrm{e}} \equiv H_0^1(\Omega)$ — or, more generally, $H_0^1(\Omega) \subset X^{\mathrm{e}} \subset H^1(\Omega)$ — where $H^1(\Omega) = \left\{v \mid v \in L^2(\Omega), \nabla v \in \left(L^2(\Omega)\right)^d\right\}$, $H_0^1(\Omega) = \left\{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\right\}$, and $L^2(\Omega)$ is the space of square integrable functions over Ω [23]. Here, the superscript e denotes "exact," and Ω is a bounded domain in $\mathbb{R}^d, d = 1, 2, 3$, with Lipschitz continuous boundary $\partial\Omega$, a typical point in which shall be denoted by x. The inner product and norm associated with X^{e} are given by $(\cdot\,,\cdot)_{X^{\mathrm{e}}}$ and $\|\cdot\|_{X^{\mathrm{e}}} = \sqrt{(\cdot\,,\cdot)_{X^{\mathrm{e}}}}$ respectively; for example $(w, v)_{X^{\mathrm{e}}} \equiv \int_\Omega \nabla w \nabla v + \int_\Omega wv,\ \forall w, v \in X^{\mathrm{e}}$. We also introduce the parameter domain $\mathcal{D} \subset \mathbb{R}^2$ in which our input parameter $\mu = (\mu_0, \mu_1)$ resides.

The weak formulation of (1) can then be stated as follows: Given any parameter value $\mu \in \mathcal{D}$, evaluate $u^{\mathrm{e}}(\mu) \in X^{\mathrm{e}}$ from

$$a\left(u^{\mathrm{e}}(\mu), v, G(u^{\mathrm{e}}(\mu);\mu)\right) = f(v), \qquad \forall v \in X^{\mathrm{e}}; \tag{2}$$

where $f(v)$ is a X^{e}-continuous linear form and

$$a\left(w, v, G(w;\mu)\right) = \int_\Omega G(w;\mu)\, \nabla w \nabla v, \qquad \forall w, v \in X^{\mathrm{e}}. \tag{3}$$

We consider the parametrized function $G : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ given by

$$G(w;\mu) = \mu_0 + \mu_1 w. \tag{4}$$

We can thus write

$$
\begin{aligned}
a\,(w, v, G(w;\mu)) &= \mu_0 \int_\Omega \nabla w \nabla v + \mu_1 \int_\Omega w \nabla w \nabla v \\
&= \mu_0\, a_0(w, v) + \mu_1\, a_1(w, w, v), \qquad (5)
\end{aligned}
$$

where $a_0(w, v) = \int_\Omega \nabla w \nabla v$ (bilinear) and $a_1(z, w, v) = \int_\Omega z \nabla w \nabla v$ (trilinear). We also define an output of interest $s^e : \mathcal{D} \mapsto \mathbb{R}$ as

$$
s^e(\mu) = \ell\,(u^e(\mu)), \qquad (6)
$$

where $\ell(v)$ is a X^e-continuous linear form.

Finally, we assume that $\mu_0 > 0$ and $\mu_1 \geq 0$ and restrict ourselves to nonnegative solutions $u(x) \geq 0$ defined on Ω such that (2) is well-posed and does not degenerate [29, 8]. Under these conditions $G(u;\mu)$ remains positive, which is reasonable from a physical point of view.

2.2. *Truth Approximation*

In actual practice we do not have access to the exact solution. We thus introduce a "truth" finite element approximation subspace $X \subset X^e$ and replace $u^e(\mu) \in X^e$ with a "truth" approximation $u(\mu) \in X$. Here, X is a suitably fine piecewise linear finite element approximation space with large dimension $\mathcal{N}$. X shall inherit the inner product and norm from X^e. Our truth approximation is thus: Given any $\mu \in \mathcal{D}$, find $u(\mu) \in X$ such that

$$
a\,(u(\mu), v, G(u(\mu);\mu)) = f(v), \qquad \forall v \in X, \qquad (7)
$$

and evaluate the output $s : \mathcal{D} \mapsto \mathbb{R}$ from

$$
s(\mu) = \ell\,(u(\mu)). \qquad (8)
$$

We shall assume that the discretization is sufficiently rich such that $u(\mu)$ and $u^e(\mu)$ are practically indistinguishable. The reduced basis approximation shall be built upon this truth finite element approximation and the reduced basis error will thus be evaluated with respect to $u(\mu) \in X$.

In order to formulate conditions for existence and uniqueness of the solution, for given $z \in X$ and every $w, v \in X$, we define the Frechet derivative form $dg : X^3 \times \mathcal{D} \mapsto \mathbb{R}$ as

$$
dg(w, v; z; \mu) = \mu_0 a_0(w, v) + \mu_1 a_1(z, w, v) + \mu_1 a_1(w, z, v). \qquad (9)
$$

We also define the family of inf-sup constants

$$
\beta_z(\mu) = \inf_{w \in X} \sup_{v \in X} \frac{dg(w, v; z; \mu)}{||w||_X ||v||_X}, \qquad z \in X, \qquad (10)
$$

and the family of continuity constants

$$
\gamma_z(\mu) = \sup_{w \in X} \sup_{v \in X} \frac{dg(w, v; z; \mu)}{||w||_X ||v||_X}, \qquad z \in X. \qquad (11)
$$

We further assume that a_0 and a_1 satisfy

$$
\begin{aligned}
|a_0(w, v)| &\leq ||w||_X ||v||_X, & \forall w, v \in X, \qquad (12)\\
|a_1(z, w, v)| &\leq \rho ||z||_X ||w||_X ||v||_X, & \forall w, v, z \in X, \qquad (13)
\end{aligned}
$$

where ρ is the Sobolev embedding constant; see Lemma 1 for the proof. Assumptions (12) and (13) immediately imply boundedness of dg. We also assume that there exists a constant $\beta_0 > 0$, such that

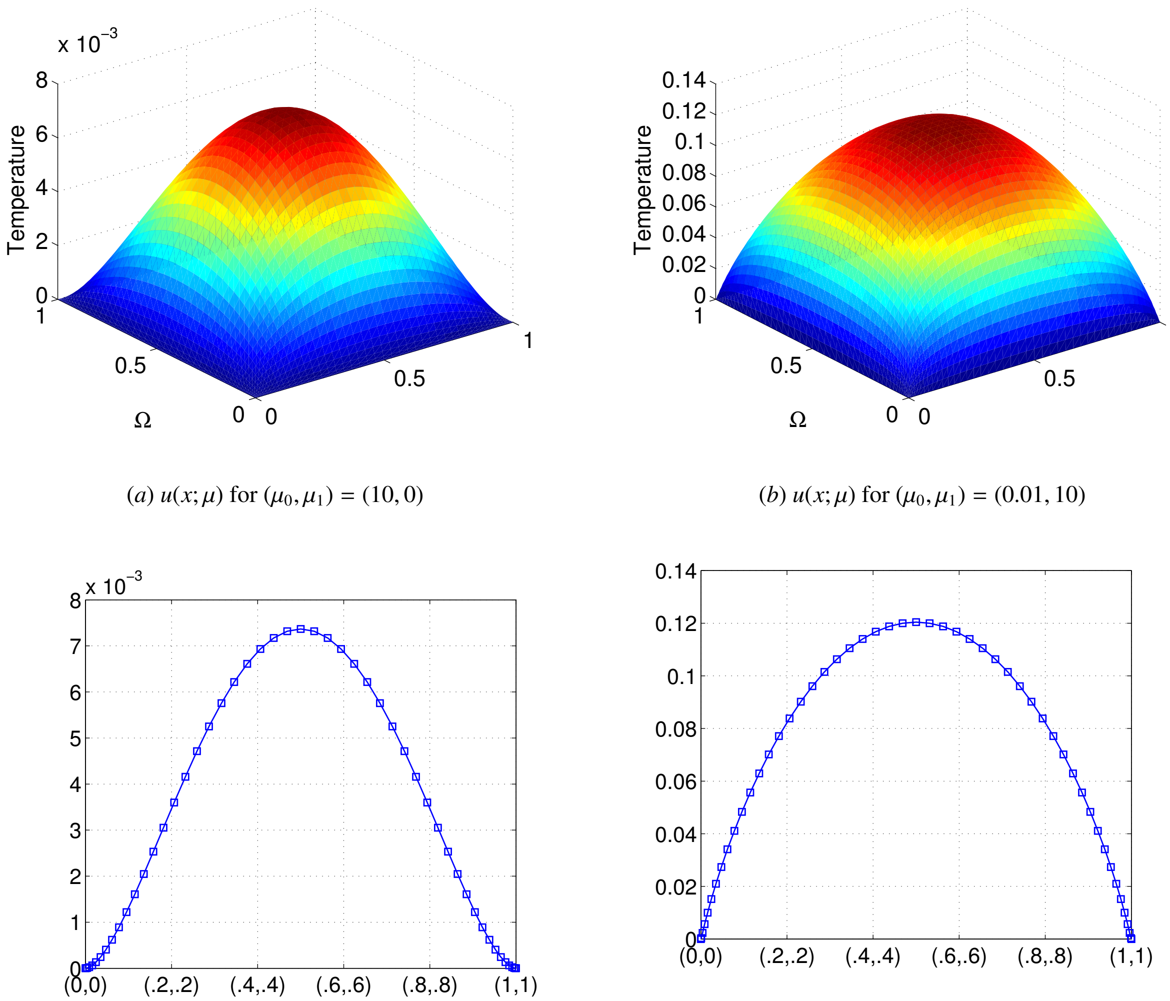
$$
\beta_z(\mu) \geq \beta_0, \quad \forall z \in X,\ \forall \mu \in \mathcal{D}. \qquad (14)
$$

We can verify this hypothesis *a posteriori*.

We solve (7) using a Newton iterative scheme where we take the solution to the corresponding linear problem ($\mu_0 \neq 0, \mu_1 = 0$) as the initial guess.

2.3. *Model problem*

We introduce a "thermal block" model problem defined on the unit square $\Omega = [0, 1]^2$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$. We define the linear functional $f(v) = \int_\Omega v \, d\Omega$ and output functional $\ell(v) = \frac{1}{|\Omega|} \int_\Omega v \, d\Omega$; we also specify the inner product $(\cdot, \cdot)_X \equiv a_0 (\cdot, \cdot)$. We consider the parameter domain $\mathcal{D} \subset \mathbb{R}^2$ given by $\mathcal{D} \equiv [10^{-2}, 10] \times [0, 10]$. The non-dimensional temperature $u(\mu) \in X$ then satisfies (7), where $X \subset X^{\mathrm{e}} \equiv H^1(\Omega)$ is a linear finite element approximation subspace of dimension $\mathcal{N} = 2601$.



(*a*) $u(x; \mu)$ for $(\mu_0, \mu_1) = (10, 0)$

(*b*) $u(x; \mu)$ for $(\mu_0, \mu_1) = (0.01, 10)$

(*c*) Cut of $u(x; \mu)$ for $(\mu_0, \mu_1) = (10, 0)$ along x_1-x_2 axis.

(*d*) Cut of $u(x; \mu)$ for $(\mu_0, \mu_1) = (0.01, 10)$ along x_1-x_2 axis.

Figure 1: Truth solution of the model problem at two different representative parameter values.

In Figure 1 we present the truth solutions for two "corners" of the parameter domain: at $(\mu_0, \mu_1) = (10, 0)$ the problem simply reduces to the linear heat equation whereas at $(\mu_0, \mu_1) = (0.01, 10)$ the nonlinearity has the strongest influence, i.e., μ_1 represents the strength of the nonlinearity. On the top row we show a plot of the temperature distribution over Ω, on the bottom row we present a cut along the diagonal of Ω (the $x_1 = x_2$ axis). We note that the shape of the linear solution resembles the Gaussian profile and the shape of the nonlinear solution is closer to the Barenblatt profile [28].

3. Reduced Basis Method

3.1. Approximation

We first introduce a nested set of parameter samples $S_1 \equiv \{\mu^1 \in \mathcal{D}\} \subset \cdots \subset S_{N^{\max}} \equiv \{\mu^1, \mu^2, \cdots, \mu^{N^{\max}} \in \mathcal{D}\}$ and associated reduced basis spaces $X_N \subset X$, $1 \le N \le N^{\max}$, as

$$X_N \equiv \text{span}\left\{\xi_j,\ 1 \le j \le N\right\} = \text{span}\left\{u(\mu^j),\ 1 \le j \le N\right\}, \quad 1 \le N \le N^{\max},$$

where the $\xi_j,\ 1 \le j \le N$, are mutually $(\cdot,\cdot)_X$-orthogonal basis functions. We construct the samples using the weak greedy algorithm discussed in Section 4.4.

The reduced basis approximation is then clear: Given any $\mu \in \mathcal{D}$, evaluate $u_N(\mu) \in X_N$ from

$$a\left(u_N(\mu), v, G(u_N(\mu); \mu)\right) = f(v), \qquad \forall v \in X_N, \tag{15}$$

and subsequently evaluate the reduced basis output approximation $s_N(\mu)$ from

$$s_N(\mu) = \ell\left(u_N(\mu)\right). \tag{16}$$

We solve (15) using a Newton iterative scheme. We briefly outline the computational procedure in the next section.

3.2. Computational Procedure

We first express $u_N(\mu)$ as $u_N(\mu) = \sum_{j=1}^{N} u_{Nj}(\mu)\xi_j$ and choose as test functions $v = \xi_i,\ 1 \le i \le N$ in (15). It then follows that $\underline{u}_N(\mu) = [u_{N1}(\mu), u_{N2}(\mu), \cdots, u_{NN}(\mu)]^{\mathrm{T}} \in \mathbb{R}^N$ satisfies

$$\left\{\mu_0 A_{0N} + \mu_1 A_{1N}\left(\underline{u}_N(\mu)\right)\right\}\underline{u}_N(\mu) = F_N, \tag{17}$$

where $A_{0N} \in \mathbb{R}^{N\times N}$ and $F_N \in \mathbb{R}^N$ are a parameter-independent matrix and vector with entries $A_{0N,ij} = a_0\left(\xi_j, \xi_i\right),\ 1 \le i, j \le N$, and $F_{N,i} = f(\xi_i),\ 1 \le i \le N$, respectively. Furthermore, $A_{1N}(\underline{u}_N) \in \mathbb{R}^{N\times N}$ has entries $A_{1N,ij}(\underline{u}_N(\mu)) = \sum_{n=1}^{N} u_{Nn}(\mu)A^n_{1N,ij},\ 1 \le i, j \le N$, where $A^n_{1N,ij} = a_1\left(\xi_n, \xi_j, \xi_i\right),\ 1 \le i, j, n \le N$.

We solve for $\underline{u}_N(\mu)$ using a Newton iterative scheme: given the current iterate $\underline{u}^k_N(\mu)$, we find an increment $\delta\underline{u}^k_N$ from

$$\left\{\mu_0 A_{0N} + \mu_1\left(A_{1N}\left(\underline{u}^k_N(\mu)\right) + \widetilde{A}_{1N}(\underline{u}^k_N(\mu))\right)\right\}\delta\underline{u}^k_N = F_N - \left(\mu_0 A_{0N} + \mu_1 A_{1N}\left(\underline{u}^k_N(\mu)\right)\right)\underline{u}^k_N(\mu), \tag{18}$$

and update $\underline{u}^{k+1}_N(\mu) = \underline{u}^k_N(\mu) + \delta\underline{u}^k_N$. Here, the matrix $\widetilde{A}_{1N}(\underline{u}_N) \in \mathbb{R}^{N\times N}$ is given by $\widetilde{A}_{1N}(\underline{u}_N) = \sum_{n=1}^{N} u^k_{N,n}\widetilde{A}^n_{1N}$, where $\widetilde{A}^n_{1N}$ has entries $\widetilde{A}^n_{1N,ij} = a_1\left(\xi_j, \xi_n, \xi_i\right),\ 1 \le i, j, n \le N$. Finally, we evaluate the output $s_N(\mu)$ from

$$s_N(\mu) = L^T_N \underline{u}_N(\mu), \tag{19}$$

where $L_N \in \mathbb{R}^N$ is given by $L_{N,i} = \ell(\xi_i),\ 1 \le i \le N$.

The Reduced Basis offline-online decomposition is now clear. In the offline stage – performed only *once* – we first compute and store the μ-independent quantities A_{0N}, A_{1N}, $\widetilde{A}_{1N}$, F_N, and L_N. In the online stage, we assemble — at each Newton step — the matrices $A_{1N}(\underline{u}_N)$ and $\widetilde{A}_{1N}(\underline{u}_N)$ at cost $O(2N^3)$ and then solve (18) for $\underline{\delta u}_N$ at cost $O(N^3)$ per Newton iteration. Finally, given $\underline{u}_N(\mu)$, we evaluate the output $s_N(\mu)$ from (19) at cost $O(N)$. The online cost is thus *independent* of $\mathcal{N}$ even in the presence of the quadratically nonlinear term.

4. A Posteriori Error Estimation

We develop an *a posteriori* error estimator which helps us to (*i*) assess the error introduced by the reduced basis approximation (relative to the "truth" finite element solution) and (*ii*) devise an efficient procedure for generating the RB space X_N.

4.1. *Preliminaries*

To begin, we define the dual norm of residual

$$\varepsilon_N(\mu) = \sup_{v \in X} \frac{g\,(u_N(\mu), v; \mu)}{||v||_X} = ||\hat{e}_N(\mu)||_X\,, \quad \forall \mu \in \mathcal{D}, \tag{20}$$

where $g\,(u_N(\mu), v; \mu)$ is the residual operator defined as

$$g\,(u_N(\mu), v; \mu) = \mu_0 a_0\,(u_N(\mu), v) + \mu_1 a_1\,(u_N(\mu), u_N(\mu), v) - f(v), \quad \forall v \in X, \tag{21}$$

and the Riesz representer, $\hat{e}_N(\mu) \in X$, is given by

$$(\hat{e}_N(\mu), v)_X = g\,(u_N(\mu), v; \mu)\,, \quad \forall v \in X. \tag{22}$$

We also assume that we are given a positive lower bound $\beta_N^{LB}(\mu)$ of the inf-sup constant $\beta_z(\mu)$ defined in (10) for $z = u_N(\mu)$, i.e.,

$$0 < \beta_N^{LB}(\mu) \leq \beta_N(\mu) \equiv \inf_{w \in X} \sup_{v \in X} \frac{dg(w, v; u_N(\mu); \mu)}{||w||_X ||v||_X}, \tag{23}$$

and define the continuity constant (11) for $z = u_N(\mu)$ by $\gamma_N(\mu)$. Before proceeding with the formulation of the *a posteriori* error bound, we show the boundedness of a_0 and a_1 in (12) and (13), respectively.

Lemma 1. *The bilinear and trilinear forms satisfy*

$$\begin{aligned} |a_0(u, v)| &\leq ||u||_X ||v||_X, \\ |a_1(z, u, v)| &\leq \rho ||z||_X ||u||_X ||v||_X, \end{aligned}$$

where ρ here is the $L^2(\Omega)$-$H^1(\Omega)$ Sobolev embedding constant.

Proof. The boundedness of a_0 directly follows from the Hölder inequality. For a_1 we also obtain from the Hölder inequality that

$$\begin{aligned} |a_1(z, u, v)| &= \int_\Omega z\, \nabla u \cdot \nabla v \\ &\leq \left[\int_\Omega z^4 \right]^{1/4} \left[\int_\Omega \Big(\sum_{j=1}^{2} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} \Big)^{4/3} \right]^{3/4} \\ &\leq ||z||_{L^4} \left[\int_\Omega \sum_{j=1}^{2} \Big(\frac{\partial u}{\partial x_j} \Big)^4 \right]^{1/4} \left[\int_\Omega \sum_{j=1}^{2} \Big(\frac{\partial v}{\partial x_j} \Big)^2 \right]^{1/2}. \end{aligned}$$

It follows from Theorem 5 in [21] that the truth solution satisfies $u \in X \cap W^{2,p}$ for $2 \leq p < \infty$ and thus

$$|a_1(z, u, v)| \leq ||z||_{L^4} ||\nabla u||_{L^4} ||v||_X. \tag{24}$$

Furthermore, since X is finite dimensional we have $|| \cdot ||_{L^4} \leq || \cdot ||_{L^2}$ and we can thus write

$$\begin{aligned} |a_1(z, u, v)| &\leq ||z||_{L^2} ||\nabla u||_{L^2} ||v||_X \\ &\leq \rho ||z||_X ||u||_X ||v||_X, \end{aligned}$$

where $\rho = \sup_{v \in X} \frac{||v||_{L^2}}{||v||_X}$ is the L^2-H^1 Sobolev embedding constant and we used the fact that $||\nabla u||_{L^2} = ||u||_X$. □

We make one important remark here: Our proof relies on the fact that the truth approximation space X is finite dimensional. This also allows us to replace the L^4-H^1 embedding constant — which requires the solution of a nonlinear generalized eigenvalue problem [30] — with the L^2-H^1 embedding constant; also see the discussion in [33] on the mesh-dependence of the L^4-H^1 embedding constant and the comparison with the L^2-H^1 embedding constant.

4.2. BRR Framework

The *a posteriori* error bound for our reduced basis approximation is based on the BRR framework [7], which was originally used in the reduced basis context in [30]. To this end, we first define a proximity indicator

$$\tau_N(\mu) \equiv \frac{4\rho\mu_1\varepsilon_N(\mu)}{\beta_N^{LB}(\mu)^2}, \quad \forall\mu \in \mathcal{D}, \tag{25}$$

and the *a posteriori* error bound

$$\Delta_N^u(\mu) = \begin{cases} \frac{\beta_N^{LB}(\mu)}{2\rho\mu_1}\left(1 - \sqrt{1-\tau_N(\mu)}\right), & \forall\mu \in \mathcal{D} \setminus \mu_1 = 0; \\ \frac{\varepsilon_N(\mu)}{\beta_N^{LB}(\mu)}, & \forall\mu \in \mathcal{D} \cap \mu_1 = 0. \end{cases} \tag{26}$$

We also define the neighbourhood, i.e., *ball* with radius r around $v \in X$, as $\mathcal{B}(v, r) = \{w \in X \,\big|\, \|w - v\| < r\}$. We thus obtain

Proposition 1. *If $\tau_N(\mu) < 1$ for some $\mu \in \mathcal{D}$, then there exists a unique solution $u(\mu) \in \mathcal{B}(u_N(\mu), \beta_N^{\mathrm{LB}}(\mu)/2\mu_1\rho)$ in the neighborhood of $u_N(\mu) \in X_N$. Furthermore, the error in the field variable, $e_N(\mu) = u(\mu) - u_N(\mu)$, and output, $s(\mu) - s_N(\mu)$, satisfy*

$$\|u(\mu) - u_N(\mu)\|_X \le \Delta_N^u(\mu), \quad \forall\mu \in \mathcal{D}. \tag{27}$$

and

$$|s(\mu) - s_N(\mu)| \le \Delta_N^s(\mu) \equiv \|\ell\|_{X'}\Delta_N^u(\mu), \quad \forall\mu \in \mathcal{D}, \tag{28}$$

where the dual norm of the output functional is given by $\|\ell\|_{X'} = \sup_{v\in X} \frac{\ell(v)}{\|v\|_X}$.

We may also bound the effectivity of the *a posteriori* error bound in the following

Proposition 2. *If $\tau_N(\mu) \le \frac{1}{2}$ for some $\mu \in \mathcal{D}$, the effectivity $\eta_N^u(\mu) = \frac{\Delta_N^u(\mu)}{\|e_N(\mu)\|_X}$ satisfies*

$$\eta_N^u(\mu) \le \frac{4\gamma_N(\mu)}{\beta_N^{LB}(\mu)}, \quad \forall\mu \in \mathcal{D}. \tag{29}$$

We remark that (2) reduces to the linear heat equation for $\mu_1 = 0$: using L'Hôpital's rule it is straightforward to show that

$$\lim_{\mu_1\to 0} \frac{\beta_N^{LB}(\mu)}{2\rho\mu_1}\left(1 - \sqrt{1-\tau_N(\mu)}\right) = \frac{\varepsilon_N(\mu)}{\beta_N^{LB}(\mu)}. \tag{30}$$

It thus follows from (9) and (23) that we obtain the standard results for the linear case for $\mu_1 = 0$. Furthermore, existence and uniqueness is then obviously guaranteed for all solutions in X, i.e., $\beta_N^{\mathrm{LB}}(\mu)/2\mu_1\rho \to \infty$ for $\mu_1 \to 0$.

The proofs of Propositions 1 and 2 follow along the lines of [30] and are sketched in Appendix A. We next turn to the efficient online evaluation of the various quantities involved in the *a posteriori* error bound.

4.3. Offline-Online Decomposition

The *a posteriori* error bound (26) is composed of three quantities: the dual norm of the residual, the Sobolev embedding constant, and a lower bound of the inf-sup-constant. We summarize their computational decomposition in an offline and online stage in the next three sections.

4.3.1. Dual Norm of Residual

The offline-online decomposition of the dual norm of the residual in the reduced basis method is fairly standard for linear problems by now. The nonlinear case follows directly, but we incur an N^4 computational cost for a quadratically nonlinear problem. To summarize, it follows from (21) and (22) that

$$(\hat{e}_N(\mu), v)_X = \mu_0 \sum_{j=1}^{N} a_0(\zeta_j, v)u_{Nj} + \mu_1 \sum_{j=1}^{N}\sum_{j'=1}^{N} a_1(\zeta_j, \zeta_{j'}, v)u_{Nj}u_{Nj'} - f(v), \tag{31}$$

and from linearity of linear, bilinear, and trilinear form that we can express

$$\hat{e}_N(\mu) = \sum_{j=1}^{N} \mu_0 z_{a0}^{j} u_{Nj} + \sum_{j=1}^{N}\sum_{j'=1}^{N} \mu_1 z_{a1}^{j,j'} u_{Nj} u_{Nj'} + z_f, \tag{32}$$

where $(z_f, v)_X = -f(v)$, $\forall v \in X$, $(z_{a0}^{j}, v)_X = a_0(\zeta_j, v)$, $\forall v \in X$, and $(z_{a1}^{j,j'}, v)_X = a_1(\zeta_j, \zeta_j', v)$, $\forall v \in X$. We thus obtain

$$\begin{aligned} \|\hat{e}_N(\mu)\|_X^2 \quad = \quad & \left(z_f, z_f\right)_X + \sum_{j=1}^{N} u_{Nj}\Bigg\{2\mu_0\left(z_{a0}^{j}, z_f\right)_X + \sum_{j'=1}^{N} u_{Nj'}\Bigg\{2\mu_1\left(z_{a1}^{j,j'}, z_f\right)_X + \mu_0^2\left(z_{a0}^{j}, z_{a0}^{j'}\right)_X \\ & + \sum_{j''=1}^{N} u_{Nj''}\Big\{2\mu_0\mu_1\left(z_{a0}^{j}, z_{a1}^{j',j''}\right)_X + \sum_{j'''=1}^{N} u_{Nj'''}\mu_1^2\left(z_{a1}^{j,j'}, z_{a1}^{j'',j'''}\right)_X\Big\}\Bigg\}\Bigg\}. \end{aligned}$$

from which the offline-online decomposition directly follows. We precalculate the inner products in the offline stage. Then, given $\mu \in \mathcal{D}$ and the associated solution $u_N(\mu)$ we evaluate the quadruple sum at cost $O(N^4)$. Note that a third-order nonlinearity would involve a sextuple sum which may become prohibitive even for modest N.

4.3.2. Sobolev Embedding Constant

The Sobolev embedding constant is given by

$$\rho = \sup_{v\in X} \frac{\|v\|_{L^2}}{\|v\|_X}, \tag{33}$$

and thus depends only on the spatial domain Ω and our truth approximation subspace X. Since ρ is parameter independent, we simply solve the generalized eigenvalue problem $\lambda = \sup_{v\in X} \|v\|_{L^2}^2/\|v\|_X^2$ once offline and set $\rho = \sqrt{\lambda_{\max}^*}$, where $(\lambda^*, \psi^*)_{\max} \in (\mathbb{R}, X)$ are the maximum eigenvalue and eigenvector.

4.3.3. Inf-sup Lower Bound

We employ the successive constraint method proposed in [15] to obtain an efficiently calculable lower bound for the inf-sup constant, $\beta_N(\mu)$, defined in (23). Since $\beta_N(\mu)$ depends on the reduced basis solution $u_N(\mu) \in X_N$, however, we need to slightly adapt the procedure from [15]. To this end, we first define the supremizing operators $T^\mu : X \mapsto X$ such that, $\forall \mu \in \mathcal{D}$ and any $w \in X$,

$$(T^\mu w, v)_X = dg\left(w, v, u_N(\mu)\right), \qquad \forall v \in X. \tag{34}$$

We then have

$$\beta_N(\mu) = \inf_{v\in X}\sup_{w\in X} \frac{dg\left(w, v; u_N(\mu)\right)}{\|w\|_X\|v\|_X} = \inf_{w\in X} \frac{\left\|T^\mu w\right\|_X}{\|w\|_X}, \tag{35}$$

and can thus define (23) as

$$\alpha_N(\mu) \equiv (\beta_N(\mu))^2 \equiv \inf_{w\in X} \frac{(T^\mu w, T^\mu w)_X}{\|w\|_X^2}. \tag{36}$$

We next note from (9) and (34) and recalling the expansion $u_N(\mu) = \sum_{j=1}^{N} u_{Nj}(\mu)\xi_j$ that

$$(T^\mu w, v) = \mu_0 a_0(w, v) + \mu_1 a_1\Big(\sum_{q=1}^{N} u_{Nq}\xi_q, w, v\Big) + \mu_1 a_1\Big(w, \sum_{q=1}^{N} u_{Nq}\xi_q, v\Big), \quad \forall v \in X. \tag{37}$$

It follows from linearity that we can express $T^\mu w$ as

$$T^\mu w = \sum_{q=1}^{2N+1} \Theta_T^q(\mu) T^q w, \tag{38}$$

where

$$\begin{aligned} \left(T^1 w, v\right) &= a_0(w, v), && \forall v \in X; \\ \left(T^{1+q} w, v\right) &= a_1(\xi_q, w, v), && \forall v \in X, \quad 1 \le q \le N; \\ \left(T^{N+1+q} w, v\right) &= a_1(w, \xi_q, v), && \forall v \in X, \quad 1 \le q \le N; \end{aligned}$$

and the parameter-dependent functions $\Theta_T^q(\mu)$ are given by $\Theta_T^1(\mu) = \mu_0$ and $\Theta_T^{1+q} = \Theta_T^{N+1+q}(\mu) = \mu_1 u_{Nq}(\mu),\ 1 \le q \le N$. Finally, plugging (38) into (36) it follows that

$$\alpha_N(\mu) = \inf_{w \in X} \sum_{q=1}^{\hat{Q}} \hat{\Theta}^q(\mu) \frac{\hat{a}^q(w, w)}{\|w\|_X^2}, \tag{39}$$

where $\hat{Q} = (2N+1)(N+1)$ and we identify $\hat{a}^q(w, v),\ 1 \le q \le \hat{Q}$, with $\frac{1}{2}((T^q w, T^{q'} w) + (T^{q'} w, T^q w)),\ 1 \le q \le q' \le 2N+1$, and $\hat{\Theta}^q(\mu),\ 1 \le q \le \hat{Q}$, with $(1 - \delta_{qq'})\Theta_T^q(\mu)\Theta_T^{q'}(\mu),\ 1 \le q \le q' \le 2N+1$, where $\delta_{qq'}$ is the Kronecker delta. We can now apply the standard SCM procedure [15] to (39) to obtain a lower bound for the inf-sup constant. However, as opposed to the linear coercive or noncoercive case, the coefficients $\hat{\Theta}^q(\mu)$ also depend on the reduced basis solution $u_N(\mu) \in X_N$. We also note that the inf-sup lower bound allows us to confirm the well-posedness of the reduced basis approximation *a posteriori*.

4.4. Greedy Sampling Procedure

We briefly discuss the greedy algorithm used to construct the reduced basis space X_N [32]. One complication arises in our context since the inf-sup constant and hence its lower bound depends on the reduced basis solution and space. Theoretically, we would thus need to perform the SCM offline procedure each time we add a new snapshot to the basis, i.e., after each greedy search over the parameter space. In order to avoid this excessive cost, we follow a different strategy and proceed as follows: We first specify a coarse parameter sample $\Xi_{\text{beta}} \subset \mathcal{D}$ of size n_{beta} and a very large (fine) training parameter sample $\Xi_{\text{train}} \subset \mathcal{D}$ of size n_{train}. We then evaluate the truth inf-sup constant (10) for all $\mu \in \Xi_{\text{beta}}$ and define a surrogate inf-sup constant $\tilde{\beta}(\mu)$ for all $\mu \in \mathcal{D}$ by linearly interpolating between the four closest parameter points in Ξ_{beta}.

We then choose $\mu_1 \in \mathcal{D}$ randomly and $N_{\max}$ (or a desired error tolerance $\varepsilon_{\text{tol,min}}$) and start the standard greedy procedure over Ξ_{train}, where we replace the inf-sup lower bound, $\beta_N^{\text{LB}}(\mu)$ in (25) and (26) by the surrogate $\tilde{\beta}(\mu)$. Once $N_{\max}$ is reached, we run the SCM offline procedure and subsequently perform another search over Ξ_{train} using the actual SCM online inf-sup lower bound in (25) and (26). If the maximum error bound over Ξ_{train} is still acceptable, e.g., by checking that $\Delta_N^u(\mu) \le \varepsilon_{\text{tol,min}}$ holds for all $\mu \in \Xi_{\text{train}}$, the greedy procedure is done. Otherwise we append more basis functions to X_N and perform the SCM offline step again.

5. Numerical Results

We return to the model problem introduced in Section 2.3 and present numerical results for the reduced basis approximation and associated *a posteriori* error estimation. We first construct the reduced basis space X_N following the greedy procedure discussed in Section 4.4. To this end, we evaluate the Sobolev embedding constant defined in (33) to obtain $\rho = 0.225$ and also define the sample set Ξ_{beta} of size $n_{\text{beta}} = 625$ (arguably too fine). We also set $N_{\max} = 16$ and introduce a train sample Ξ_{train} of size $n_{\text{train}} = 2500$ (the union of a linearly and logarithmically distributed grid in each parameter dimension).

Given the reduced basis space X_N, we perform the offline SCM procedure where we use the following parameters [15]: $\Xi_{\text{train}}^{\text{SCM}} = \Xi_{\text{train}}$, a required tolerance of $\epsilon_\alpha = 0.25$, $M_\alpha = \infty$, and $M_+ = 0$. The greedy SCM procedure then selects $K_{\max} = 83$ parameters; note that online we set $M_\alpha = K_{\max}$. We also remark that we require at least $K_{\min} = 45$ to obtain an inf-sup lower bound $\beta_N^{LB}(\mu) > 0$ for all $\mu \in \Xi_{\text{train}}^{\text{SCM}}$. We then perform another greedy search over Ξ_{train} to find that the desired error tolerance is still satisfied. We present in Figure 2 the inf-sup constant $\beta_N(\mu)$ and its lower bound $\beta_N^{\text{LB}}(\mu)$ as a function of μ_0 for $\mu_1 = 1$ and 10 as well as $K = 45$ and 83, respectively. We note that the bound is very sharp for all values of μ_0 and μ_1. We henceforth use $K = 83$ for the numerical results.

In Figure 3 (a) and (b) we plot the sample sets picked by the reduced basis and SCM greedy procedures, respectively. We observe that the samples — especially for the SCM — are clustered at the top left corner corresponding to small values of μ_0 and large values of μ_1. The reason is obviously that the nonlinearity has the largest influence for $\mu_0 = \mu_{0,\min}$ and $\mu_1 = \mu_{1,\max}$.

We next turn to the *a posteriori* error estimation. We plot the maximum relative error $e^u_{N,\max,\mathrm{rel}}(\mu)$ and bound $\Delta^u_{N,\max,\mathrm{rel}}(\mu)$ in Figure 4 (a) and the maximum and average proximity indicator $\tau_{N,\max}$ and $\tau_{N,\mathrm{avg}}$ in Figure 4 (b). Here, $e^u_{N,\max,\mathrm{rel}}(\mu)$ and $\Delta^u_{N,\max,\mathrm{rel}}(\mu)$ are the maxima of $\|e^u_N(\mu)\|_X/\|u(\mu)\|_X$ and $\Delta^u_N(\mu)/\|u(\mu)\|_X$ over Ξ_{test}, respectively; and $\tau_{N,\max}$ and $\tau_{N,\mathrm{avg}}$ is the maximum and average of $\tau_N(\mu)$ over Ξ_{test}, respectively; and $\Xi_{\mathrm{test}} = \Xi_{\mathrm{beta}}$. We observe that the error and bound converge very fast and that the error bound is very sharp for all values of N. From Figure 4 (b) we further note that the maximum of $\tau_N(\mu)$ is less than one only for $N \geq 8$. In fact, for $N = 2$ the average effectivity is even less than one indicating that the error bound is not rigorous. However, the bound is provably rigorous for $N \geq 8$ where the effectivities show that the error bound is also very sharp.

Finally, in Table 1 we present, for different values of N, the maximum proximity indicator, the maximum relative error and bound as well as the average effectivity for the field variable and outputs, and the computational savings in the online stage. Here, $\eta^u_{N,\mathrm{avg}}$ is the average of $\Delta^u_N(\mu)/\|e^u_N(\mu)\|_X$ over Ξ_{test}; the maximum relative output error, $e^s_{N,\max,\mathrm{rel}}$, and bound, $\Delta^s_{N,\max,\mathrm{rel}}$, are the maxima of $|s(\mu) - s_N(\mu)|/|s(\mu)|$ and $\Delta^s_N(\mu)/|s(\mu)|$ over Ξ_{test}, respectively; and $\eta^s_{N,\mathrm{avg}}$ is the average of $\Delta^s_N(\mu)/|s(\mu) - s_N(\mu)|$ over Ξ_{test}. We note that the effectivity of the field variable error bound is very close to 1. We also observe that the output error and bound converge very fast. Furthermore, the average output effectivity is considerably larger than $\eta^u_{N,\mathrm{avg}}$ but — given the fast convergence — is still acceptable for all values of N. Finally, we present the average computational savings in the online stage, i.e., the ratio of the computational time to solve the truth finite element problem and the computational time to solve the reduced basis approximation *and* evaluation of the *a posteriori* error bound. The average is taken over Ξ_{test}. The savings are $O(10^3)$ for all values of N confirming the online-efficiency of the reduced basis method.

N	$e^u_{N,\max,\mathrm{rel}}$	$\tau_{N,\max}$	$\Delta^u_{N,\max,\mathrm{rel}}$	$\eta^u_{N,\mathrm{avg}}$	$e^s_{N,\max,\mathrm{rel}}$	$\Delta^s_{N,\max,\mathrm{rel}}$	$\eta^s_{N,\mathrm{avg}}$	$\partial t_{\mathrm{FEM/RB}}$
2	2.59 E−1	8.92 E+1	5.09 E−1	0.99	1.58 E−2	9.54 E−2	9.78 E+2	1.69 E+4
4	4.87 E−2	3.92 E+1	9.03 E−2	1.18	8.34 E−5	1.69 E−2	2.72 E+3	9.74 E+3
8	2.32 E−3	4.90 E−2	2.85 E−3	1.33	3.27 E−6	5.33 E−4	5.48 E+3	6.65 E+3
12	2.52 E−5	1.71 E−3	2.69 E−5	1.28	1.00 E−9	5.03 E−6	2.75 E+4	4.82 E+3
16	4.69 E−7	7.86 E−4	2.51 E−6	1.90	2.36 E−10	4.71 E−7	2.82 E+3	3.60 E+3

Table 1: Proximity indicator, maximum relative error and bound as well as effectivity for the field variable and the output, and computational online-savings for different values of N.

Appendix A. Proofs

Appendix A.1. Proof of Proposition 1

Given $g(\cdot,\cdot;\mu)$ from (21) and $dg(\cdot,\cdot;\cdot;\mu)$ from (9), we first define the operator $G : X \mapsto X'$ given by

$$\langle G(w;\mu), v\rangle = g(w, v;\mu), \qquad \forall w, v \in X, \tag{A.1}$$

and its Frechet derivative for any $z \in X$ by

$$\langle dG(z;\mu)w, v\rangle = dg(w, v; z;\mu), \qquad \forall w, v \in X, \tag{A.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between X' and X. We also introduce the mapping $H(\cdot;\mu) : X \mapsto X$ as

$$H(w;\mu) = w - dG(u_N(\mu);\mu)^{-1}G(w;\mu), \qquad \forall w \in X. \tag{A.3}$$

Note that a fixed point of H, $w^* = H(w^*;\mu)$, must be a solution of (2) and vice versa. We now consider $H(z^2;\mu) - H(z^1;\mu)$ for $z^1, z^2 \in \mathcal{B}(u_N(\mu), \alpha)$ which from (A.3) can be expressed as

$$H(z^2;\mu) - H(z^1;\mu) = (z^2 - z^1) - dG(u_N(\mu);\mu)^{-1}\{G(z^2;\mu) - G(z^1;\mu)\}. \tag{A.4}$$

Furthermore, from the Taylor series expansion we note that

$$\langle G(z^2;\mu) - G(z^1;\mu), v\rangle = \int_0^1 \langle dG\left(z^1 + t(z^2 - z^1);\mu\right)(z^2 - z^1), v\rangle\, dt, \qquad \forall v \in X. \tag{A.5}$$

Multiplying (A.4) from the left by $dG(u_N(\mu);\mu)$ and invoking (A.5) we obtain

$$\langle dG(u_N(\mu);\mu)\{H(z^2;\mu) - H(z^1;\mu)\}, v\rangle = \int_0^1 \langle \left\{dG(u_N(\mu);\mu) - dG\left(z^1 + t(z^2 - z^1);\mu\right)\right\}(z^2 - z^1), v\rangle\, dt. \tag{A.6}$$

We next note from (9), (A.2) and(13) that

$$\sup_{w\in X}\sup_{v\in X} \frac{\langle\{dG(z^1;\mu) - dG(z^2;\mu)\}w, v\rangle}{\|w\|_X\|v\|_X} \le 2\mu_1\rho\|z^2 - z^1\|_X, \tag{A.7}$$

and from the triangle inequality and the fact that $z^1, z^2 \in \mathcal{B}(u_N(\mu),\alpha)$ we immediately obtain

$$\left\|u_N(\mu) - [z^1 + t(z^2 - z^1)]\right\|_X \le t\left\|(z^2 - u_N(\mu))\right\| + (1-t)\left\|(z^1 - u_N(\mu)\right\|_X \le \alpha. \tag{A.8}$$

It thus follows from (A.6) by inovking (A.7) and (A.8) that

$$\langle dG(u_N(\mu);\mu)\{H(z^2;\mu) - H(z^1;\mu)\}, v\rangle \le 2\,\mu_1\,\rho\,\alpha\,\|z^2 - z^1\|_X\,\|v\|_X. \tag{A.9}$$

From (23), (23), and the definition of the supremizer (34) we furthermore know that

$$\beta_N^{LB}(\mu) \le \frac{\langle dG(u_N(\mu);\mu)\{H(z^2;\mu) - H(z^1;\mu)\}, T^\mu(H(z^2;\mu) - H(z^1;\mu))\rangle}{\left\|H(z^2;\mu) - H(z^1;\mu)\right\|_X\left\|T^\mu(H(z^2;\mu) - H(z^1;\mu))\right\|_X}. \tag{A.10}$$

Combining (A.9) and (A.10) we finally obtain

$$\|H(z^2;\mu) - H(z^1;\mu)\big\|_X \le \frac{2\,\mu_1\,\rho\,\alpha}{\beta_N^{LB}(\mu)}\|z^2 - z^1\|_X, \tag{A.11}$$

which means that $H(\cdot;\mu)$ is a contraction mapping for

$$\alpha < \frac{\beta_N^{LB}(\mu)}{2\,\mu_1\,\rho}. \tag{A.12}$$

We next let $z \in \mathcal{B}\,(u_N(\mu),\alpha)$ and consider $H(z;\mu) - u_N(\mu)$ which from (A.3) can be expressed as

$$H(z;\mu) - u_N(\mu) = z - u_N(\mu) - dG(u_N(\mu);\mu)^{-1}\{G(z;\mu) - G(u_N(\mu);\mu)\} - dG(u_N(\mu);\mu)^{-1}\{G(u_N(\mu);\mu)\}. \tag{A.13}$$

Following the same steps as above starting at (A.6) we finally obtain

$$\|H(z;\mu) - u_N(\mu)\|_X \le \frac{\varepsilon_N(\mu)}{\beta_N(\mu)} + \frac{\rho\,\mu_1\,\alpha^2}{\beta_N^{LB}(\mu)}. \tag{A.14}$$

We now look for the smallest value of α such that

$$\frac{\varepsilon_N(\mu)}{\beta_N^{LB}(\mu)} + \frac{\rho\,\mu_1}{\beta_N^{LB}(\mu)}\alpha^2 < \alpha, \tag{A.15}$$

which is satisfied for $\alpha \in [\alpha_-, \alpha_+]$ where $\alpha_\pm$ are the roots of the quadratic equation

$$\frac{\rho\,\mu_1}{\beta_N^{LB}(\mu)}\alpha^2 - \alpha + \frac{\varepsilon_N(\mu)}{\beta_N^{LB}(\mu)} = 0 \tag{A.16}$$

given by

$$\alpha_{\pm} = \frac{\beta_N^{LB}(\mu)}{2\rho\mu_1}\left(1 \pm \sqrt{1 - \frac{4\rho\mu_1\,\varepsilon_N(\mu)}{\beta_N^{LB}(\mu)^2}}\right). \tag{A.17}$$

If $\tau_N(\mu) = \frac{4\rho\mu_1\varepsilon_N(\mu)}{\beta_N^{LB}(\mu)^2} \leq 1$, then $\alpha_{\pm} \in \mathbb{R}$. For $\alpha \in [\alpha_-, \frac{\beta_N^{LB}(\mu)}{2\mu_1\rho}]$ and $z \in \mathcal{B}(u_N(\mu), \alpha)$ it thus follows that $H(z;\mu)$ maps $\mathcal{B}(u_N(\mu), \alpha)$ into itself and we conclude from the contraction mapping theorem that there exists a unique solution $u(\mu) \in \mathcal{B}\left(u_N(\mu), \beta_N^{LB}(\mu)/2\mu_1\rho\right)$. Furthermore, the error satisfies

$$\|u(\mu) - u_N(\mu)\|_X \leq \Delta_N^u(\mu) \equiv \frac{\beta_N^{LB}(\mu)}{2\rho\mu_1}\left(1 - \sqrt{1 - \tau_N(\mu)}\right). \tag{A.18}$$

The output error bound then directly follows from the linearity and definition of the dual norm of the output functional, i.e.,

$$|s(\mu) - s_N(\mu)| = |\ell(u(\mu)) - \ell(u_N(\mu))| \leq \sup_{v \in X} \frac{\ell(v)}{\|v\|_X} \|u(\mu) - u_N(\mu)\|_X. \tag{A.19}$$

Appendix A.2. Proof of Proposition 2

We first note from from (9) and (21) that

$$g(w + \Delta w, v; \mu) = g(w, v; \mu) + dg(\Delta w, v; w; \mu) + \mu_1 a_1(\Delta w, \Delta w, v). \tag{A.20}$$

Choosing $w = u_N(\mu)$, $\Delta w = e(\mu) = u(\mu) - u_N(\mu)$, and $v = \hat{e}_N(\mu)$ and invoking (11), (13), and (22) it follows that

$$\|\hat{e}_N(\mu)\|_X \leq \gamma_N(\mu)\,\|e(\mu)\|_X + \rho\mu_1\,\left\|e(\mu)\right\|_X^2. \tag{A.21}$$

Since $\|e(\mu)\|_X \leq \Delta_N^u(\mu)$ and, for $\tau_N(\mu) \leq 1$, $\Delta_N^u(\mu) \leq \frac{2\varepsilon_N(\mu)}{\beta_N^{LB}(\mu)}$ we obtain

$$\Delta_N^u(\mu) \leq 2\frac{\gamma_N(\mu)}{\beta_N^{LB}(\mu)}\|e(\mu)\|_X + \frac{4\rho\mu_1\,\varepsilon_N(\mu)}{\beta_N^{LB}(\mu)^2}\Delta_N^u(\mu). \tag{A.22}$$

Finally, invoking $\tau_N(\mu) \leq 1/2$ gives

$$\frac{1}{2}\Delta_N^u(\mu) \leq 2\frac{\gamma_N(\mu)}{\beta_N^{LB}(\mu)}\|e(\mu)\|_X \tag{A.23}$$

which proves the desired result.

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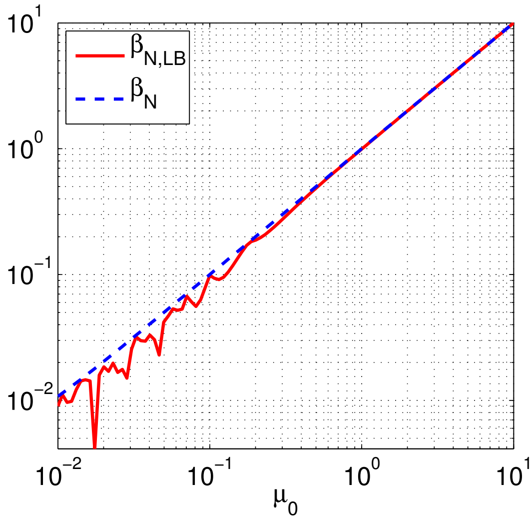
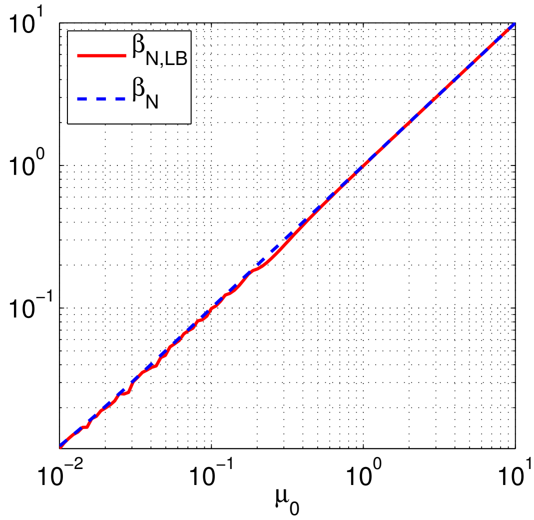
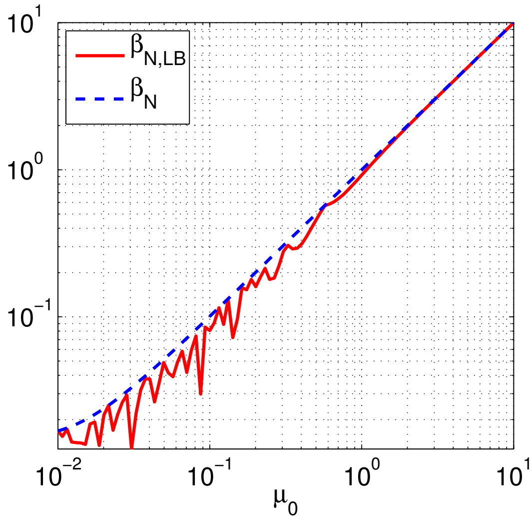


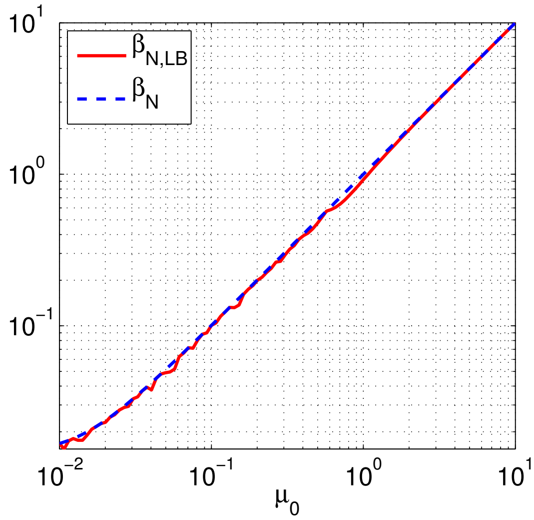
(*a*) $\mu_1 = 1$ and $K = 45$



(*b*) $\mu_1 = 1$ and $K = 83$

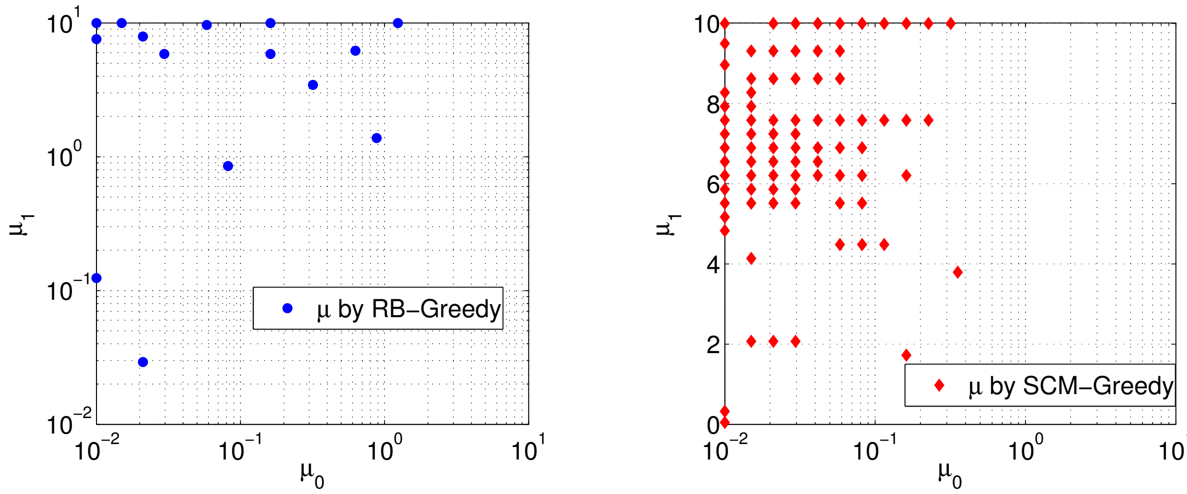


(*c*) $\mu_1 = 10$ and $K = 45$



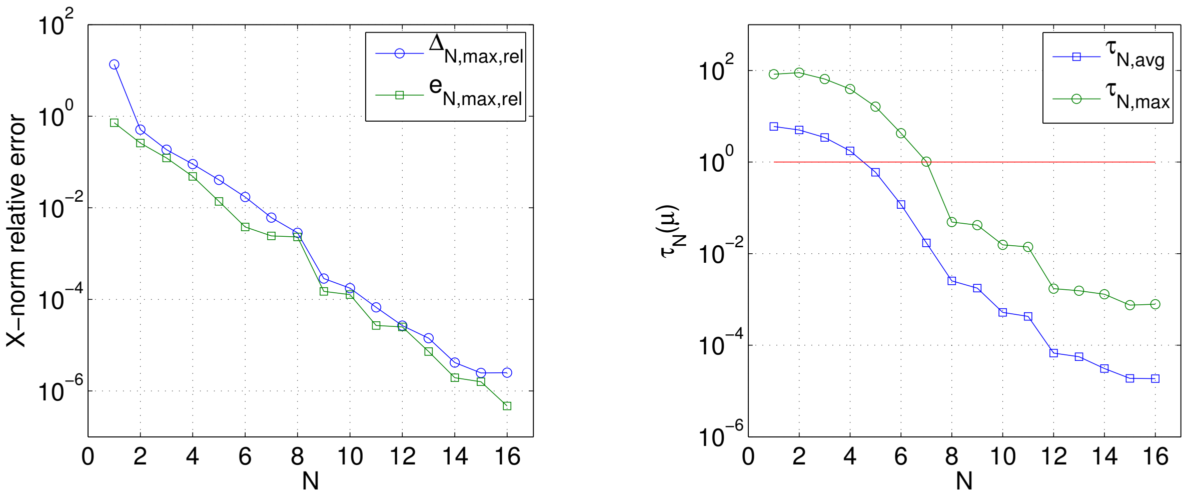
(*d*) $\mu_1 = 10$ and $K = 83$

Figure 2: Inf-sup constant $\beta_N(\mu)$ and lower bound $\beta_N^{\mathrm{LB}}(\mu)$ as a function of μ_0 for $\mu_1 = 1$ and 10 and $K = 45$ and 83.



(*a*) (*b*)

Figure 3: Parameter samples picked by (*a*) the reduced basis and (*b*) SCM greedy procedures.



(*a*) Error $e_{N,\max,\mathrm{rel}}$ and bound $\Delta_{N,\max,\mathrm{rel}}$ (*b*) Proximity indicator τ_N

Figure 4: Maximum relative error, $e_{N,\max,\mathrm{rel}}$, and bound, $\Delta_{N,\max,\mathrm{rel}}$ and maximum and average proximity indicator, τ_N, as a function of N for $K = 83$.