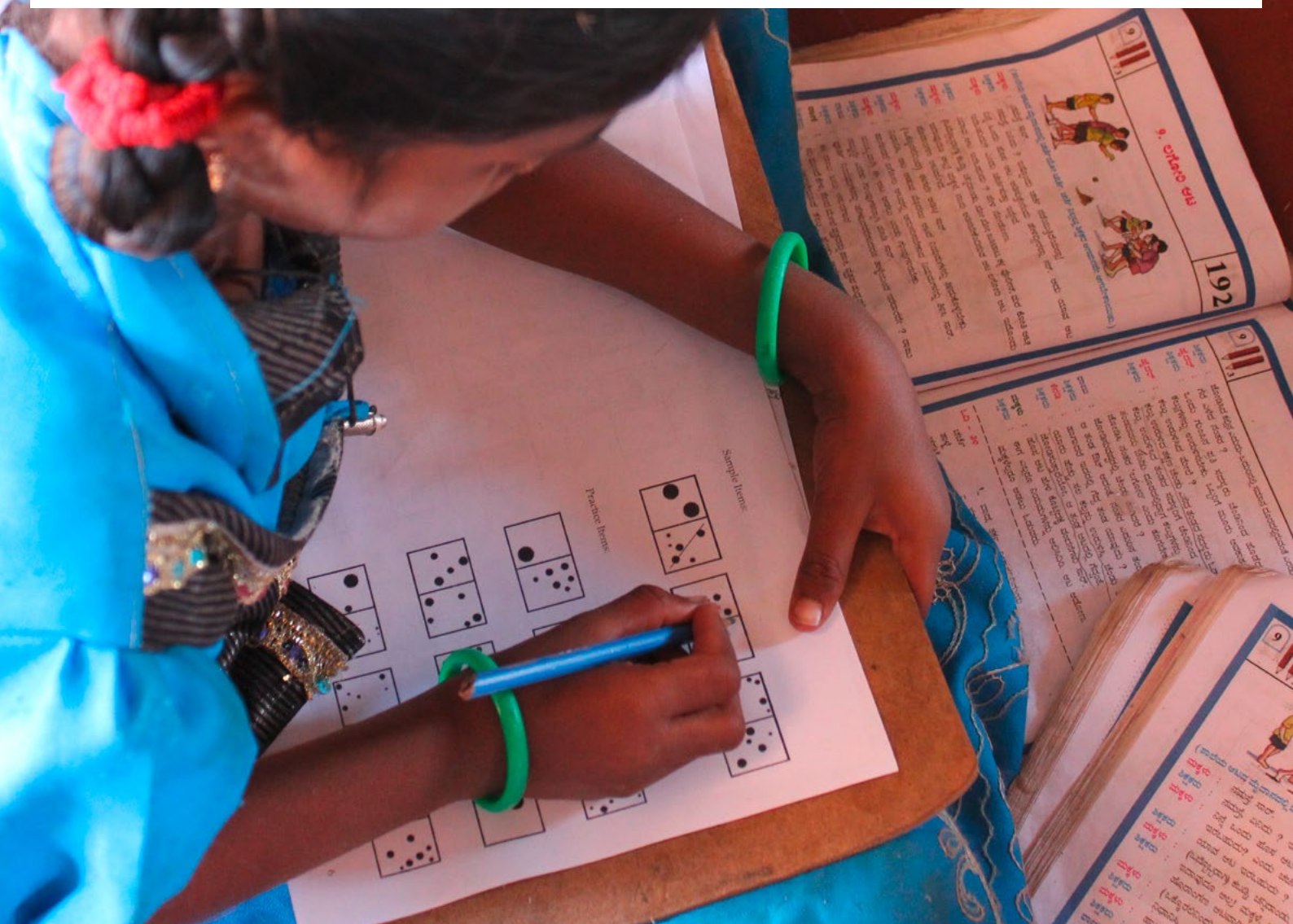




Federal Ministry
for Economic Cooperation
and Development



Early Grade Development and Numeracy

The academic state of knowledge and how it can be applied in project
implementation in low income countries

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Mathematics – The Cheapest Science for Great Minds

Parmenides Foundation

“Mathematics is the cheapest science. Unlike physics or chemistry, it does not require any expensive equipment. All one needs for mathematics is a pencil and paper.”

George Pólya

Table of Contents

Executive Summary.....	4
Introductory Remarks.....	6
Scientific State of the Art.....	8
Prerequisites for Numeracy	8
A Cognitive Model of Number-processing.....	10
Neuro-Cognitive Evidence on Numeracy.....	10
Cognitive Development of Numeracy.....	12
Cultural Differences.....	15
Dyscalculia	15
Individual Error Analysis.....	16
Mathematical Education in Young Children	17
The Parmenides Foundation's Proposal: Learning to Think (L2T) – The Art of Mathematical Thinking.....	20
Motivational Aspects with Special Focus on Low Income Countries.....	26
Measurement Approaches and Implications for Teaching Strategies Concerning Numeracy Education in Low Income Countries	27
Screening and Evaluating Teaching Strategies by Using Cameras.....	27
Screening and Evaluating Teaching Strategies by Using Mobile Learning Applications	28
Technical Requirements for the Usage of Cameras and Applications.....	28
Mobile Learning, E-profiling and Adaptive Training Configurations.....	30
References	31



Executive Summary

In October 2012, the Deutsche Gesellschaft für Internationale Zusammenarbeit (GIZ) GmbH on behalf of the German Federal Ministry for Economic Cooperation and Development (BMZ) commissioned the Parmenides Foundation to conduct a desk study on Early Grade Development and Numeracy, particularly in low income countries, as part of a series of studies on numeracy. Two other studies would examine the Assessment of Early Grade Numeracy and Mobile Learning and Numeracy.

We, the authors, propose a multi-dimensional approach to numeracy. We review the important findings on low income countries, which reveal that there already are significant efforts to improve learning outcomes in numeracy, e.g. in African countries, but that there were only weak positive effects on learning. These findings elicit the question: What other factors play a role in transferring knowledge from teachers to students? We focus especially on motivational aspects that increase the commitment and interest of children, parents and teachers to learn, respectively, teach numeracy.

From a scientific point of view, numeracy requires the interplay of several well distinguishable cognitive systems. There are three quantification processes, which can be distinguished: a) counting, b) subitizing and c) estimation. It is assumed that we represent quantities on an internal number line. In particular, Dehaene proposed a modular concept, which states that three distinct systems (Triple Code Model) are involved when people perform mathematical operations (respectively, a system that represents mathematical knowledge in visual and auditory format, and a more general module that is responsible for the analog magnitude representation and suspected to be closely related to the “number sense”). Neuro-scientific studies provide some evidence for the validity and the applicability of this model to different brain regions.

One key region is the intraparietal sulcus (IPS). It is assumed that the IPS is responsible for the approx-

imate representation of quantities (mental number line). During development, the number line changes from a logarithmic to a more linear representation. There is first evidence that intuitive basic number processing measured by the distance or the ratio effect can predict mathematical abilities, but not automatic processes, such as the size-congruity effect. These findings might have interesting implications for the diagnostics of early number processing in children. Given the scientific evidence, it seems worthwhile to focus more strongly on approximate and less on exact numeracy abilities. Increasing the resolution and representation of the mental number line may positively affect children's achievement.

There is evidence that early mathematical education, starting already in preschool, has a sustainable impact on the subsequent mathematical abilities in school or high school. Fostering mathematical skills has to take into account that children do not only need to learn procedures or algorithms (“malgorithms” and misconceptions), but also comprehend the underlying conceptual and structural knowledge. There are cultural variations in numeracy that either depend on educational priorities or on topicalities of the used number system.

In general, it seems difficult to translate the results of basic research to teaching. There is a strong need for scientific studies that investigate, evaluate and foster exactly this transfer.

The Parmenides Learning to Think Project suggests a dual-process account (analytic and constellatory thinking) and provides detailed thinking operations, which serve as general and cross-cultural base for mathematical thinking. We have developed typical educational modules starting with numeracy education at the age of one up to school age. We emphasize the necessity of a multi-method approach, which takes individual aspects (analysis of misconceptions, motivation) and contextual aspects (teacher, parents, etc.) into account.

We conclude that numeracy has to start at an early age, while being insight-oriented and discovered on one's own, has to be taught and experienced joyfully and motivationally and – last but not least – there has to be a stronger focus on constellatory thinking processes than on pure analytical processes.

The study provides recommendations for the acquisition of numeracy competencies with a special focus on the situation in low income countries. We emphasize that it is crucial to increase the intrinsic motivation to acquire numeracy. Teachers, parents and, most

importantly, children have to realize that mathematics is a science that requires only few resources to gain deep insights.

At the end of the study, we illustrate how to use modern technologies (eLearning) in order to foster, evaluate and improve individual mathematical thinking, and recommend video analysis methods to improve the quality of teaching, evaluating students' performance and providing new insights into the study of children's mathematical development.



Introductory Remarks

Our study focuses on selected scientific evidence concerning the research of mathematical thinking and teaching, fostering numeracy and on particular educational aspects of numeracy in low income countries. We aim at determining the main driving factors, which hold true for a variety of different cultures. According to Grisay & Mahlck (1991) there are three quality dimensions of education, which have to be considered:

1. “input”: human and material resources
2. “process”: teaching practices
3. “output”: performance of the students

Furthermore, we want to highlight a certain phenomenon, which has to be taken seriously and which is an explicitly cultural obstacle to establishing better numeracy achievements in low income countries. The Uwezo East Africa Reports 2011 & 2012 revealed that **despite the availability of appropriate human and material resources students do not seem to learn.**

This is a crucial finding. It demonstrates that providing resources (like schools, teachers, teaching materials) are a necessary pre-condition, but do not inevitably ensure educational success, because there might be mediators that prevent the impact of those means. Such mediators, like individual motivation, or cultural attitudes have to be identified and taken into account when planning educational interventions (see Figure 1).

The two Uwezo reports summarize the problem of numeracy and literacy in the following statements:

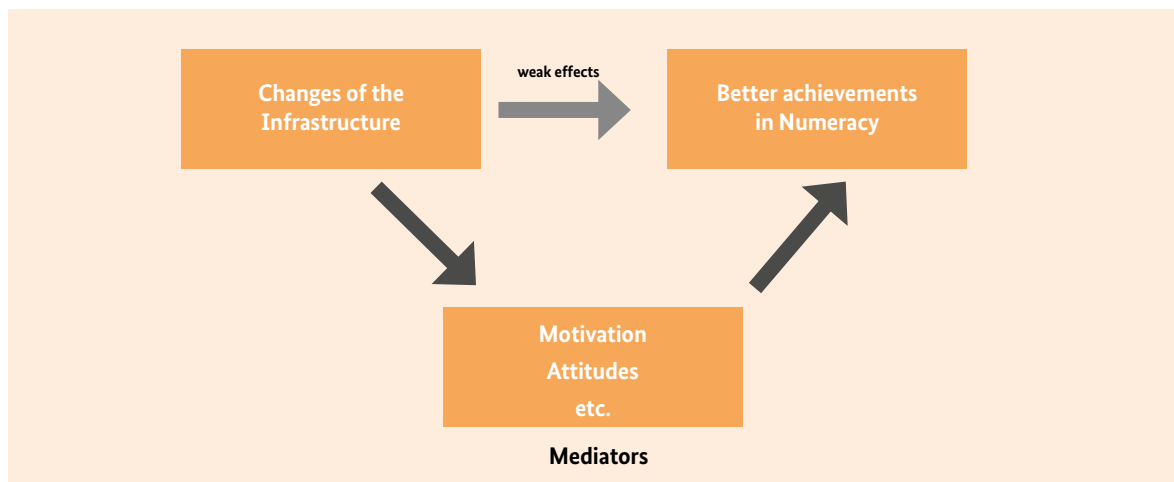
“Bad results inevitably lead to the question, what next? The findings of this report suggest that policy makers would do well to question approaches used so far, instead of doing more of the same. While different investments can have positive effects, some are likely to have greater or more lasting impact than others, or present better value for money. In a context where resources are limited, it is crucial to examine the evidence carefully for what is likely to contribute to greater impact.” (Uwezo, 2011)

“Despite important gains in access to primary schooling throughout the region, evidenced by generally high enrolment rates, large numbers of children are simply not learning.” (Uwezo, 2012)

Even if we believe that it is necessary to give attention to some neglected, crucial aspects of fostering numeracy, there are, of course, more problems to fostering numeracy in low income countries than a positive attitude toward mathematics.

The main body of our desk study begins with detailed information on various aspects of numeracy and briefly summarizes the scientific state of the art in this field.

Figure 1: Essence from the Uwezo East Africa Reports 2011 & 2012.





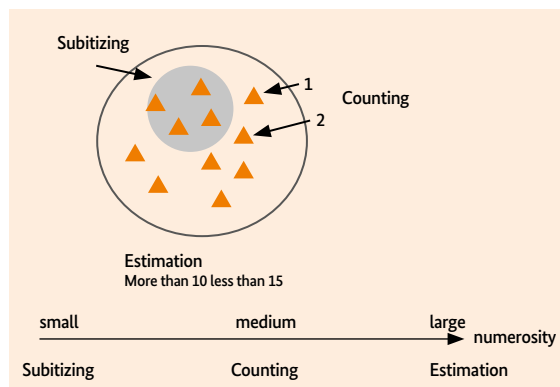
Scientific State of the Art

Prerequisites for Numeracy

Numeracy requires a broad range of cognitive abilities: Seeing and comprehending the written numbers, decoding and understanding the heard numbers, writing down the result of a calculation, activating an abstract representation, activation of the motor system in case we are moving or changing objects in the world, e.g. moving beads on an abacus or putting quantities in a container, mapping the quantity of a set into a mental representation.

The concerted interplay of those systems is necessary and can be disturbed by misconception, or by failure or deficits of parts of the neural system. We propose that for fostering numeracy each of these components have to be taken into account.

Figure 2: Different perspectives and levels on quantification.



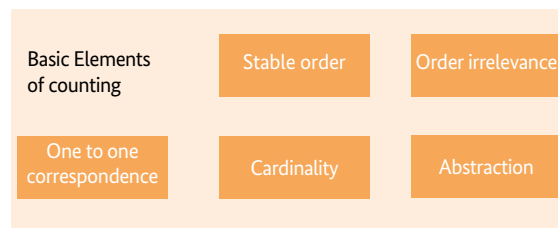
In general, a number can be defined as the property of a set of objects (see Figure 2). In this picture, quantification is the act of mapping the perceived set of elements into a mental number or token. Three quantification processes can be distinguished (Klahr, 1973): a) counting, b) subitizing and c) estimation (see Figure 2).

Counting

A dominant opinion in regard to teaching mathematics is that counting is the basic principle to start with arithmetic.

Characterization: According to Gellman and Gallistel (1978), at least five principles can be defined:

Figure 3: Properties for counting.



The principles illustrated in Figure 3 emphasize the complex act of counting. There are several insights we can derive from the model and abilities children have to learn. For example, when counting a set of objects, e.g. apples, the sequence does not play a role, but it is important to map exactly one apple to one number. The next step is to abstract from the concrete apple to a general understanding of cardinality etc.

There is a long discussion whether children should use fingers to learn counting. It was shown that using finger digits and body parts, as African and New-Guinean tribes do, might support counting. This finding supports our assumption of the interplay of embodied cognition and numeracy substantially (see below). Moreover, it is assumed that the ability to count and the ability to speak are mostly independent (Gellman and Gallistel, 1992).

Phylogenetic: From a phylogenetic point of view, it is interesting that number processing can already be

observed in animals (Verguts et al. 2004, rats, chimpanzees, birds). It is assumed that some animals have a number sense and use a pre-verbal counting mechanism that also is present in infants.

Development: There are two concurrent approaches that try to explain the development of counting in children. Gellman and Gallistel assume that there are innate principles like cardinality, abstraction, order-irrelevance that guide counting abilities – because of the assumption of innate principles, this is called the principle-first account vs. the principle-after account, according to which counting principles are progressively abstracted during children's maturation (e.g. Greeno et al. 1984). For the latter account speaks that there is evidence that understanding cardinality and the ability of a flexible abstraction requires some time in development.

Subitizing

Characterization: Subitizing is the ability to determine the quantity of a set without counting. Generally, between the range of 1 to 4 participants show fairly fast numerosity judgments (Kaufman et al., 1949). Mandler

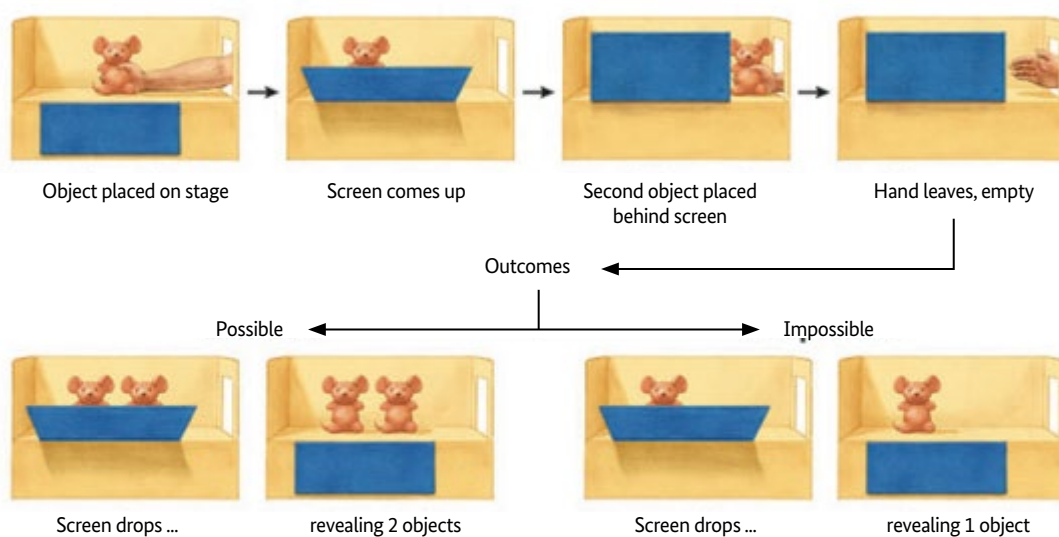
and Shebo (1982) propose a model of canonical configurations (constellations) of visual items assuming that people tend to build perceptual units that can be subitized.

Development: Subitizing can be very impressively studied in young infants: Already 4-day-old babies can differentiate between 1 and 2 object displays. They only failed when they had to compare 4 versus 6 object displays.

In a very famous study Wynn (1992) presents infants with a theater display (see Figure 4). Children see that two puppets wander behind a curtain. When the curtain is lifted and unexpectedly only one puppet appears children older than 5 months do not habituate. This is taken as an indicator that children detect the violation of their expectation, and this requires a rudimentary understanding that the number of objects is incorrect.

Young children are also aware that adding two small quantities augment the size, but they were in general not able to calculate the exact number.

Figure 4: Early quantity judgments.



Estimation and the Number Line

Characterization: Estimation is the attempt to find an approximate quantity of a large set of objects without counting.

Experimental Evidence: Data analysis of number comparisons (same – different, less than – greater than) revealed that numbers were recoded and compared as quantities (Dehaene, 1992) – this is known as the distance effect. According to Dehaene et al. (1990), the digital code is converted into an internal magnitude that is placed on an internal mental number line. Further evidence for the existence of a mental number line were provided by experiments on the SNARC effect (spatial numerical association of response code), which show that participants react faster with their left hand to lower numbers and with their right hands to higher numbers. Shaki et al. (2009) demonstrated that the SNARC effect might result from the reading habits common in a particular culture. They found that Canadian students that read from left to right showed the typical SNARC effect. Palestinian students that read from right to left showed a reversed SNARC effect and Israeli students that read Hebrew text from right to left, but also read Arabic numbers from left to right showed no SNARC effect.

The concept of a number line can also be used to explain arithmetic operations. According to Restle (1970), addition can be interpreted so that the operands represent segments of the number line and by aligning the second segment behind the first, the result can easily be read out.

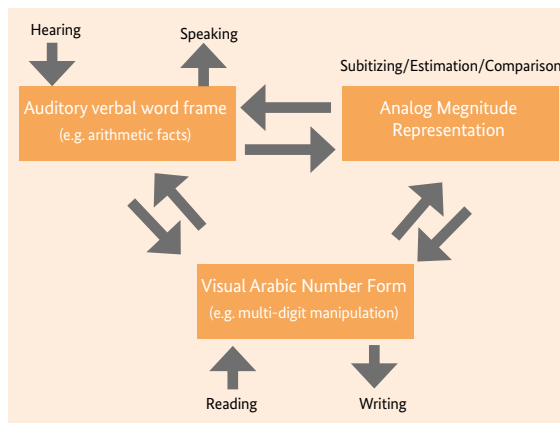
A Cognitive Model of Number-Processing

Given the detailed evidence presented above, it is conceivable that distinct systems are responsible for number processing (Dehaene, 1992). The first system processes numbers as symbols. The second system aligns the numbers on a number line as approximate quantities. Moreover, there is evidence from neuropsychological lesion studies, which suggests that clearly distinguishable brain regions are responsible for different aspects of numeracy (see below).

Dehaene (1992) postulates three different domains that are important for numerical competence: a) transcoding, b) quantification and c) approximation. In his Triple-Code-Model (TCM) he postulates three different representational systems – verbal, visual and analog magnitude representation (see Figure 5). In the auditory, verbal word frame, an analog of a number word is mentally manipulated. In the visual, Arabic number form, Arabic codes are manipulated on a

spatially extended representational medium. In the analog magnitude representation, numbers are represented on an analog mental number line that obeys Weber's laws (the same distance between numbers (e.g. 1-4, 41-44, 5001-5004) is perceived as smaller the larger the size of the number).

Figure 5: Dehaene's Triple-Code model of number processing (adapted from Dehaene, 1992).



Neuro-Cognitive Evidence on Numeracy

In the last 20 years neuro-scientific studies have shed light on the details of brain regions and mental processes involved when humans comprehend and produce numbers or do calculations. Neuropsychological evidence showed that the different numeral abilities, according to Dehaene's model, could be mapped to dissociable brain regions. For example, if number comprehension is impaired after a stroke, number production can still be intact.

Evolutionary Foundations of Number Processing: In an introductory review article, Dehaene et al. (2004) present research on the neural basis of the human ability to do arithmetic to validate the hypothesis that a “number sense” is a basic capacity of the human brain and, thus, “hard-wired via evolution”, and provide neural evidence for his Triple-Code Model. In fact, the authors are able to dissociate a few brain regions that can be attributed to the three core elements of the model.

The Intraparietal Sulcus: One key region pointed out in almost all studies on numerical processing is the intraparietal sulcus (IPS). The IPS might play an important role in the approximate quantity representation (see Figure 6). For example, its activation was recorded when subjects had to estimate the number magnitudes of visual, as well as of auditory objects.

Ansari (2008) reviews the role of the intraparietal sulcus in the processing of numerical magnitude, i.e. quantity.

It was frequently replicated that numerical distance negatively correlates with activation of the IPS. This finding provides evidence that the IPS might play an important role for the mental number line. This distance effect is explained by the tuning curves of neurons with preferred numerosities: The rationale is that if the number distance is small, the neurons overlap more than if the distance is large.

Activation of the IPS region seems to be independent of representation, i.e. independent of mode and language. It is assumed that the angular gyrus is involved in linguistic (verbal) operations (see Figure 6), and the fusiform gyrus is processing the visual number form.

However, whether the IPS is the number module in the brain or not is still a contested issue, with different studies reporting partly contradictory results or at least providing alternative interpretations (Ansari, 2008).

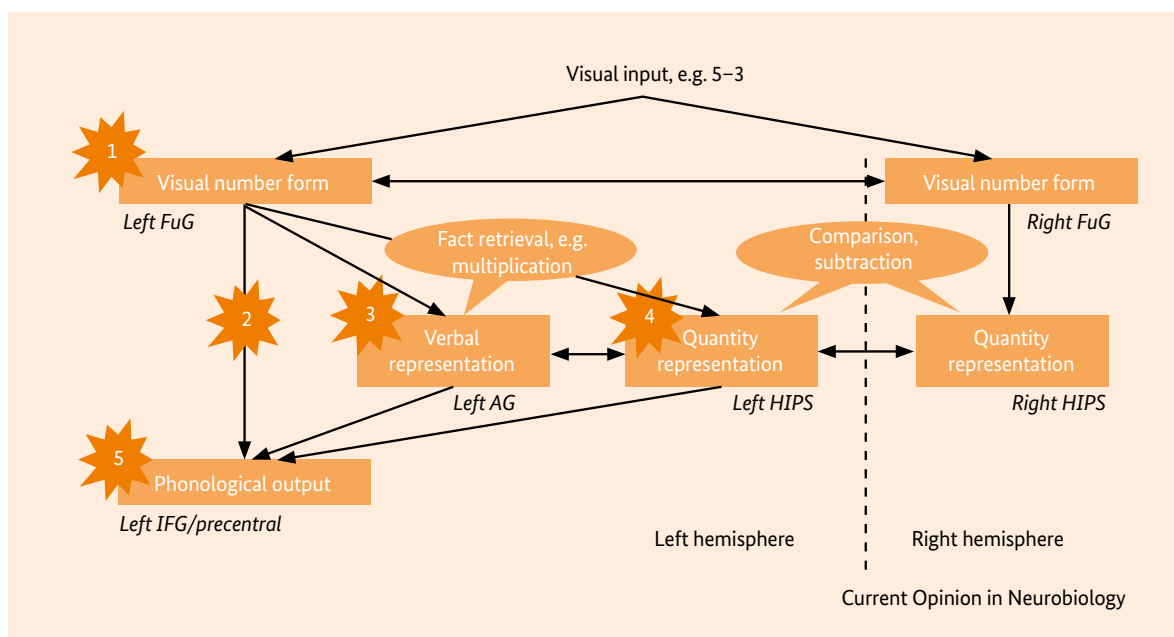
Complex Skills: Maruyama et al. (2012) investigate more complex arithmetic skills in adults who parsed mathematical expressions such as operators via fMRI and MEG. They were interested in understanding whether a mathematical expression relies on language systems or on language independent networks. They selected mathematically trained adults as participants.

They found that parsing mathematical expressions bilaterally activated the occipito-temporal cortices, as well as the right parietal and precentral cortex. A small increase in activation was observed in the left inferior frontal gyrus and the posterior temporal sulcus, which are linguistic regions of interest, when faced with more complex expressions. The authors conclude that trained adults rely on visuo-spatial areas that are mainly language independent.

Animal Models: Animal models help to gain more detailed insights into the neural underpinnings of numeracy (Hubbard et al. 2008). Animal studies revealed rudimentary number skills and a “number sense” in animals. While it is known that several species, such as lions, dolphins or macaques possess some form of number sense, the neural underpinnings still remain unclear. Studies using electrophysiological techniques on trained monkeys have yielded some insights into this area of research. They suggest that numerosity is first computed in the parietal cortex before being transmitted and held by prefrontal activity. The latency in these processes is identical only for the numerosities 1-5 (subitizing), but not for serial counting.

Rhesus monkeys were tested on whether they could carry out a nonverbal addition task and the results were compared to human college students completing the same task. In a number space from 1 to 17, the rhesus monkeys were able to solve these addition problems in accordance with Weber’s law similarly to the human participants. Thus, non-symbolic, non-

Figure 6: Brain regions involved in number processing (Deheane et al. 2004). HIPS: horizontal segment of the IPS (intra parietal sulcus). AG: angular gyrus. IFG: inferior frontal gyrus. FuG: fusiform gyrus.



verbal arithmetic was shown to be a basic cognitive process not restricted to humans. In another study, monkeys were trained to associate numerosity to symbol-like numerical representation. It was found that many neurons in the lateral prefrontal cortex and the intraparietal cortex coded for quantity, regardless of whether the quantities were presented symbolically or non-symbolically. However, when recording single-cell activity, it was found that many neurons in the prefrontal cortex, which responded to the visual shapes of the symbols, reflected the associated numerosity. This is consistent with the fact that human children who are inexperienced with symbolic number representation show increased activity in their prefrontal cortex when faced with symbolic ordinality. This might be evidence that the prefrontal cortex plays an important role in establishing a link between symbolic and non-symbolic number representation.

Dehaene (2009) points out that concerning the arrangement of “numerosity” neurons in the intraparietal cortex, there is experimental evidence from the study of macaque monkeys that showed that there are specialized neurons for number coding, but that these are not highly discretized. Instead they more likely overlap with neurons coding other quantitative parameters. Number-coding neurons have been detected in the intraparietal sulcus of the macaque monkey. These neurons show Gaussian variability on a logarithmic scale and are thought to be analogous to the human neural number sense. Experiments regarding Weber’s law point to preferential treatment of small numerosities: While this law predicts identical treatment of numbers in the range of 1-8 and 10-80, it was clearly violated in the higher ranges. This supports the hypothesis that there is a system dedicated to small numerosities.

Cognitive Development of Numeracy

Posner and Rothbart (2007) analyze the development of numeracy. They detail strategy changes in children’s numeracy processing. In a cross section study that compared children’s and adults’ performance comparing 5-year-olds and adults, they found that the same brain regions were activated in both groups, whereas the overall reaction times were significantly longer for the children. These results suggest that the systems involved in estimation of quantity are developed before schooling systematically fosters higher arithmetic abilities.

It is further assumed that, whereas 5-year-olds possess these number comparison skills even without

any training (see above), attaining more complex arithmetic competencies depends on the acquisition of formal skills, as well as on the development of language. For example, it is well known that young children solve arithmetic problems in a different way than adults regarding addition. Children rely on counting, whereas adults retrieve results from memory.

There are different strategies used by children:

1. Children take the larger operand and count upward as far as the smaller operand indicates, e.g. $5 + 4$. Children start with 5 and count 6, 7, 8, 9 – this is called the “min strategy”.
2. In the “counting all strategy” children count up the number of times the first operand indicates and continue with the number of times the second indicates (1, 2, 3, 4, 5 – 6, 7, 8, 9). Other strategies are guessing, decomposing ($7 + 2 = (5+2) + 2$).

Heine et al. (2010) use eye movement data to investigate the development of basic numerical skills. The study focuses on the mismatch between explicit and implicit knowledge in a number magnitude task in 6 to 9 year old children. The behavioral data suggests that the shift from a logarithmic scale of number representation to a linear model takes place at some point between first and second grade. Furthermore, Heine et al. were able to show that although young children lacked explicit knowledge about numerical magnitude, the eye-movement pattern showed that they had already implicit knowledge, since the eyes shifted more often to the correct response even when the overt behavioral response was false. This provides evidence that body-movements can reveal very early changes in the explicit mathematical knowledge.

Ansari (2008) shows that how and which brain regions are activated in number processing co-varies with age. He illustrates that children more strongly activate prefrontal brain regions than adults (as mentioned above for animals). This might indicate a higher cognitive effort in children, and indicates more automatic and optimized procedures in adults. There is also an age-related increase in activation of the left inferior parietal cortex – termed fronto-parietal shift. Overall, it seems that the recruitment of distinct parietal circuits depends on learning as well as development.

The following studies give a detailed overview on neural correlates observed in the development of numeracy.

Cantlon et al. (2006) compare numerical processing in young children (4 years old), who had no formal

mathematical education, with adults. Again, the key region is the intraparietal sulcus (IPS) in symbolic, as well as non-symbolic numerosity in adults, as well as children. Both groups were presented with non-symbolic numerical values such as displayed sets of visual elements. The two groups show very similar activation patterns in the intraparietal sulcus. The result demonstrates that the IPS is a source of basic numerical abilities and not a consequence of developed numerical abilities observed in adults. A remarkable difference is that, whereas adults display number-related bilateral activation in the intraparietal sulcus, children show a stronger lateralized predominant activation of the right intraparietal sulcus.

As a follow-up Houde et al. (2010) conducted a large meta-analysis including 52 studies and 842 children to answer the same question. They also found that children used the same structures, but also showed a stronger frontal activation. The important outcome was that children probably extend more cognitive effort and attentional resources on numerical tasks than adults.

In summary, the studies show that the IPS plays a role in numeracy very early. Differences appear in the efficiency of the application of mathematical procedure. Children more extensively activated frontal networks.

Accordingly, Dehaene (2009) shows that whenever adults are engaged in tasks of comparison, addition or subtraction, there is a consistent bilateral activation of the horizontal segment of the left intraparietal sulcus (see Figure 6). This holds true for Arabic numerals as well as for spoken number words. It is consistently shown that the parietal cortex already responds to number processing tasks in 4-year-old children. Furthermore, analyzing the ERPs of infants showed that already at the age of seven months they were able to detect errors in simple non-symbolic arithmetic operations.

Dehaene also reviews studies that investigate the development of the understanding of cardinality and ordinality of numbers: Judging which number is larger is a different task than judging whether, say, 9 comes after 7. The cerebral substrate for ordinal knowledge is similar to that for cardinal knowledge. While informal, intuitive mathematics rely on a logarithmic representation, it seems that further learning and formalization leads to a linearization of mathematical intuition. As shown above (Heine et al. 2010), the differentiation might occur between the first and second grade of school.

Dehaene et al. (1999) presented children with arithmetic tasks. They had to verify which one of two sums was either true or approximately true. During the task

the authors recorded the activation of brain regions via EEG. The results showed that for the approximate solution, mainly bilateral parietal areas were activated. Calculation of the exact sum, however, required activation of the language areas of the left hemisphere. This provided further evidence for the Triple-Code model.

Development of internal representations: Ansari (2008) evaluates the hypothesis that cultural numerical symbols acquire their meaning by being mapped onto non-symbolic representations in the brain. Thus, the study implicitly also touches on the important question how symbolic representations evolve. He presents a model introduced by Veguts and Fiat (2004) in which symbolic and non-symbolic inputs are internal differently place-coded representations and processed via different pathways. Non-symbolic inputs are encoded via an intermediate step into a summed representation. This model predicts that children learn symbolic representations by associating non-symbolic inputs with internal representations of numerical magnitude.

At a cellular level, this assumption was specified by the given evidence that neurons react differently according to whether they code symbolic or non-symbolic magnitudes (see also above). Neurons coding symbolic magnitudes do so more precisely than neurons coding non-symbolic magnitudes in the left IPS than in the right IPS. Overall, it seems likely that acquiring symbolic representation either changes pre-existing representations or leads to the construction of format-specific representations.

These differences in symbolic and non-symbolic representations challenge the assumption that symbolic representation rests solely on an innate non-symbolic representation: It seems evident that there are other processes involved and it is conceivable that these two representations rest on different neuro-cognitive processes. There is some experimental evidence to support this claim, for example the independence between understanding the meaning of counting and approximate numerical magnitude representation in children between two and four years old.

It has been found that there is a strong correlation to activity of the left angular gyrus during neural coding and the acquisition of arithmetic knowledge. This is further supported by the fact that subjects scoring low on mathematical ability also displayed a decreased activation of their angular gyrus.

Mathematical Performance: Another interesting point is whether it is possible to predict from basic mathematical processes whether a child has higher or lower mathematical ability. This is particularly interesting when considering early diagnostics that reveal existing

deficits in number processing. It is suggested that the ability of basic number processing correlates positively with the mathematical ability of normally developing children (Bugden & Ansari, 2011). However, the question is: What kind of basic process is actually a good predictor? Number magnitude processing can be investigated by the use of different methods. Bugden & Ansari (2011) test two different number processing tasks and administered how these relate to higher mathematical competence – the ratio effect vs. size congruity effect.

Firstly, the ratio effect refers to intentional number magnitude processing. It measures the judgment time when comparing two numbers. Here Weber's law states that as the ratio between two numbers increases, so does the reaction time, meaning that people can differentiate more quickly between 2 and 4 than between 12 and 14 (see above). The ratio effect is complementary to the distance effect (Holloway & Asari, 2009, see also above) that indicates how accurately the distances on a mental number line can be discriminated. It was shown that the response time decreases over developmental time. Interestingly, it was shown that children with larger distance effect showed lower mathematical abilities.

Secondly, an example of automatic processing is the size congruity effect. This effect can also be referred to as the "Number Stroop" task. Participants were asked to concentrate on either the numerically (8 greater than 2) or the physically larger (2 larger than 8) number and to ignore the irrelevant dimension. Automatic processing refers to the fact that this happens without conscious monitoring on the subject's part.

119 children in 1st and 2nd grade were tested. The ratio effect was measured using single-digit numbers, where children had to find the larger number as fast as possible. The size congruity task also contained only single-digit numbers. Mathematical ability was measured using two standardized mathematical tests in order to assess higher mathematical abilities. In general, the study was able to replicate previous findings insofar performance in the ratio effect test was related to performance in the tests measuring mathematical performance. This implies that the ability to intentionally discriminate between numbers is related to more complex mathematical ability. The automatic size congruity effect was also found, but no correlation was found between this effect and individual performance on the standardized tests. Furthermore, no correlation between the ratio effect and the size congruity effect was found. The authors concluded that higher mathematical abilities are related to intentional, but not to automatic processing of Arabic numerals.

A recent fMRI study conducted by Bugden et al. (2012) investigates the neural correlates that are revealed when children learn arithmetic. The study aims at investigating the relationship between the number ratio effect and arithmetic fluency. 17 children between 8 and 9 years were tested. Brain activation in three different regions of interest was monitored during symbolic number magnitude processing and subsequently correlated with arithmetic achievement scores. The three regions of interest were the inferior frontal gyrus, the left superior frontal gyrus and the left intraparietal sulcus (see Figure 6). A significant correlation was found between mathematical fluency scores and the ratio effect in the left IPS. The ratio effect was not significant in the SFG and the left IFG. Thus, the study could demonstrate that activation of the left intraparietal sulcus during a basic intentional number processing task is related to arithmetic performance.

DeSmedt et al. (2011) are interested in extending the existing neuro-scientific evidence from very basic numeracy skills, such as number magnitude processing, to arithmetic competence in children aged 10-12 years. The main interest was to observe brain activation changes when children become more arithmetically fluent. Firstly, it was analyzed what brain regions were activated for easier, in comparison to more difficult, arithmetic operations. Secondly, the developmental trajectory was investigated. The basic observation was that smaller problems are solved via fact retrieval and, thus, activate the left-lateralized language areas. Larger arithmetic problems are solved using quantity-based procedural strategies and, therefore, involved the intraparietal sulcus more strongly than the smaller problems.

As expected, the study revealed that the activation was driven by the problem size (small or large), the type of operation (addition or subtraction) and the individual level of arithmetical fluency (typical or low). Larger problems and the operation type subtraction more strongly activated a fronto-parietal network including the intraparietal sulci, while the left hippocampus was active during the solution of small problems. The different levels of arithmetical fluency modulated brain activation patterns in the right intraparietal sulcus. The group of children with a typical level of arithmetical fluency showed less activation, when solving small problems in contrast to larger problems, in comparison to children with low levels of arithmetical fluency.

Cultural Differences

In general, we can suppose that across all human cultures and including preschoolers and infants, a non-symbolic number sense is present and intuitive processes such as comparison, addition and subtraction have been confirmed experimentally across all cultures and ages. Nevertheless, there is various evidence showing that differing cultural factors have an impact on numeracy abilities. These differences result on the one hand from different settings of priorities in mathematical education (comprehension vs. learning by heart), on the other hand from cultural particularities, e.g. easier and more regular number systems and languages quantifiers.

It was shown that children (between 9 and 15) from Brazil or Africa, who work as merchants in sidewalk businesses and sell fruits, soft drinks or sweets, showed good applied mathematical competencies. They are excellent in determining problems that are embedded in their familiar context, but fail in solving problems in a more abstract sense.

The TIMSS study (Trends in International Mathematics and Science Study) basically revealed that 8th graders from Japan, Hungary, Hong Kong and the Netherlands showed better comprehension of mathematical concepts than children in the USA (Siegler et al., 2003). It was shown that teaching in the more successful countries is more strongly focused on understanding the underlying concepts and not memorizing mathematical procedures and algorithms.

Learning the ability to count varies strongly between cultures. 5 year-old Chinese children can usually count numbers up to 100. A performance that is much higher than in 5 year-olds in the USA. This might be a consequence of an easier and more regular number system.

Furthermore, there is also some evidence that different learning methods result in different neural activation similar to the effects of enculturation. Ansari (2011) makes the interesting suggestion to study the inter-relationship between culture and education via changes in brain plasticity. Neural plasticity describes how the human brain changes via learning. Cross-cultural brain studies have already shown that cognitive functions differ across cultures. More recently, an interest in neural plasticity as a function of culture has led to increased research in this field and neuro-imaging methods allow scientists to investigate the processes underlying even complex human functions. Cultural effects on brain function with respect to arithmetic processes have been reported. Apart from these effects, a growing body of research focuses on how neural plasticity relates to the process of education and acquiring numeracy or reading skills.

Tang et al. (2006) show that arithmetic processing and neural processes are shaped by culture. Most studies were conducted with participants familiar with the Arabic number system. In a numerical quantity comparison task presented to native English speakers as well as native Chinese speakers (age mid-twenties), it was found that both groups displayed an activation of the inferior parietal cortex. There were differences between the two groups regarding number processing. The native English speakers displayed greater activation of the left perisylvian cortex for mental addition, whereas the native Chinese speakers activated a visuo-premotor association network. Therefore, it seems that numbers are coded depending on the process of language acquisition as well as other cultural factors.

Dyscalculia

Studying the number sense in infants is difficult, since neuro-imaging studies are almost impossible to conduct. Another way to study the human number sense is to investigate its pathologies, such as acalculia or dyscalculia. Children who suffer from developmental dyscalculia have been found to reduce gray matter density of the intraparietal sulcus in comparison to not affected children.

Characterization: In general, dyscalculia can be characterized as learning disability driven by innate difficulties in basic arithmetic skills, such as addition, subtraction, multiplication and/or division. Without the ability to understand or transfer the rules underlying mathematical tasks, affected children and adults tend to remain at a stage where they operate solely by counting. In Germany, the prevalence of dyscalculia has been estimated to range around 4-6%, but there is a far larger amount of people who do not fulfill the criteria of the disorder, but who still show a special educational need (15% in Lorenz & Radatz, 1993).

Kucian et al. (2006) define “developmental dyscalculia” as a deficit in mathematical performance given otherwise normal general intelligence. They tested 18 children with developmental dyscalculia as well as 20 “normal” children (9-12). Both groups were tested on exact, as well as approximate number calculations, using both verbal and quantity-based representations of numbers. Non-symbolic magnitude comparison was also investigated in order to gain data independent from the Arabic number system. Both groups displayed similar neural network activation during number processing. However, there were differences in approximate quantities calculations. Here, dyscalculia children showed weaker activation in almost the entire neural numeracy network, but there were no differences in the domain of exact counting. The results show that dyscalculia impairs exact calculation less

than the approximate representation of the numbers at the mental number line.

Treatment: Consequently, Kucian et al. (2011) investigate whether mental number line training fosters approximate quantity calculation. Importantly, it is postulated that children with developmental dyscalculia have only limited access to this representation. The study investigated changes in neural plasticity caused by improving mental number line skills. The authors compared the dyscalculia group with typically achieving children aged around nine years. After training for five weeks using a computer program, both the experimental and the control group showed improved spatial representation of numbers and an increased ability to correctly solve arithmetic problems.

After the training both groups displayed less activation in the IPS and frontal areas. This decrease can be attributed to successful automatization of the involved mental processes. The authors did not find an increased activation in the parietal lobules of the dyscalculia group immediately after the training. Five weeks after the training, a follow-up test showed an increase in the bilateral parietal regions. It seems that the training effect requires some time for consolidation. This might also be a result of a stronger use of the learned mental representation.

Individual Error Analysis

We pose the hypothesis that typical errors and mathematical misconceptions are an important indicator that children had no insight in the underlying laws, rules, structures, and concepts (see below). We plead for an individual error analysis as the basis for understanding children's thinking and helping to overcome such thinking errors. Generally, we can learn at least two things from cognitive errors: Firstly, general cognitive deficits that prevent the understanding of certain problems by undeveloped brain functions, secondly, biased information processing and constrained problem representations.

Jean Piaget realized the potential of a detailed error analysis of children's thinking ability. Piaget gained insights into the restrictions of cognitive processes during maturation. He deduces the most important evidences for his stage model of cognitive development from errors. We shortly review Piaget's approach and enlist well-known mathematical misconceptions observed in children. Although most of Piaget's theoretical assumptions are criticized, because Piaget generally underestimates children's cognitive abilities and he often used inappropriate and too difficult tests, his work is still an important source for asking the right questions. His approach was strongly influenced by formal logic and mathematical thinking.

A-not-B-Error – Perseverative Error

One of the bases of mathematical thinking is the concept of a container that is either filled with an object or is empty (Lakoff & Nunez, 2001). Piaget examined search behavior in children and discovered that infants at the age of 10 months or younger were only successful in finding an attractive hidden toy, if there was just one location. If in a sequence of trials the children were shown that the object was hidden under box A, they successfully searched in the same place. Then in the test condition the attractive toy was hidden under box B. In general, 10 months-old children continued searching in location A in the first trial. A possible cause of this behavior might consist of insufficient inhibitory control of the representation of and behavior directed toward the first location. What we learn from this task is, firstly, that for young children obviously the history of success plays an important role for their reasoning that completely differs from the behavior of older children (over 12 months) and adults. Secondly, it is not a given that infants can differentiate between locations that are empty or conceal an object.

Misconception in the Understanding of Time

It is often important to take into account the interrelationship of variables regarding mathematical calculations. 4 year-old children show systematic errors when they have to accomplish the interplay of time, distance and speed. Children watched two cars starting and stopping simultaneously, one car travelling faster than the other. When asked why one car had travelled further, 4 year-olds explained the outcome by assuming that this car must have been moving for a longer time – although they had seen the two cars start and stop at the same time. Another misinterpretation of the concept of time in 4 year-olds can be observed in their belief that they will pass their siblings in age when they grow physically.

Conservation Error

For the understanding of numeracy, it is important to understand the invariant of quantities from their arrangement, colors, etc. (abstractness and cardinality, see above). For example, young children (below 5 years) were asked, whether two rows of pellets differ in their quantities. Both rows had exactly 10 elements. In the first row, the pellets lay close together. In the second they were arranged with more space between them. In general, children assumed that the number of pellets was different. Another task presented children with a tall and narrow glass filled with water, which is poured into a broader and shorter one. Young children were not able to understand that the amount of water was still the same and independent from the particular shape of the glass.

Misconceptions in Mathematical Thinking

Since the 1970s, the exploration of children's conceptual and procedural understandings of mathematical concepts received increasing attention. While some studies concentrated on the influence of socio-cultural and affective factors (e.g. Brousseau, 1997; Nunes, Schliemann & Carraher, 1993), others emphasized the value of understanding children's misconceptions (Hart, 1981; Bell, Costello & Kuchemann, 1983) and the need to develop assessments in order to help teachers identify them (Nunes, 2001).

The problem most teachers are faced with in their classes is that many children tend to use algorithms as purely mechanical procedures without thoroughly understanding the underlying processes (Nunes, 2001, see below). This leads to inappropriate adaptation or overgeneralization of these procedures, which Ruthven and Chaplin (1998) refer to as the 'improvisation of malgorithms'.

Many mathematical misconceptions of children can be found in the area of multiplication or division (Anghileri, 2001). Generalizations such as "multiplication makes bigger, division makes smaller" are comprehensible, but lead to severe problems when non-integer numbers are introduced (Greer, 1988). In multiplication, children also show difficulties with handling zeroes (Hart, 1981). Concerning division, Anghileri (1998) obtained results suggesting – besides difficulties with remainders – that although division does not obey the law of commutativity, 10-11 year-olds nevertheless try to use this formerly learned rule. In a 1999 comparison of more than 500 English and Dutch year 5 students, an analysis of strategies yielded that procedures based on counting and chunking led to better results in division problems than procedures based on place value (Anghileri, Beishuizen, Van Putten & Snijders, 1999). There is evidence that multi-digit calculation procedures are based on the arithmetic knowledge acquired in school. Children calculate column-wise, and some children make systematic errors. For example, they "forget" to carry over the 1 to the next column, when the sum of a column is equal or greater than 10, or started with the left column and then go to the right, etc.

In younger children it has often been found that when they learn the typical sequence of number words, two errors occur: 1. Using a number word in place of another ("one", "two", "five", ...). According to Dehaene's model (1992), this is due to incomplete lexical knowledge. 2. They invent new number words like twenty-ten, indicating that they did not recognize the change of the quantifier after twenty-nine was reached. This might reflect an over-generalized syntactic mathematical rule – the amount of those errors depends strongly on the complexity of the cultural number system. Error prim-

ing describes the tendency that the activation of the same calculation increases the likelihood that similar equations are affected by the prior equation: Generally, facilitating priming effects are found. A problem like $4 + 11 = 15$ is classified faster the second time it is presented (Ashcraft, 1992). But, processing $4 \times 6 = 24$ and then 3×7 increased the probability that the response is also 24 for the latter (Dehaene, 1992).

What we can learn from systematic errors children make is the mechanism that tries to "repair" acquired misconceptions of arithmetic knowledge. Children invent their own rules to align the mathematical deficit. Due to the fact that many children mechanically apply the rules they were taught, promotion of mathematical skills in children should focus on understanding the basic principles and the meaning underlying the rules or algorithm. In order to counter misconceptions, a suggestion is engaging children in discussions about their reasoning in specific mathematical situations to assess the reasons for the underpinning errors (Ryan & Williams, 2007). Moreover, investigating the mathematical knowledge of pre-service teachers revealed some misconceptions in their thinking (Ryan, J. & McCrae, B. (2005/2006)). Therefore, it might be important to test and correct the teachers' mathematical misconceptions.

Mathematical Education in Young Children

In general, there is one intriguing, still open and hardly addressed question: To what extent can the evidence from neuro-scientific studies be used to improve numeracy education? We conclude from the given evidence, which provides interesting insights into brain processes, that it might be helpful to develop better and more accurate diagnostic means to assess children's number sense. Moreover, it might be interesting to increase the education of neglected mathematical competencies, such as rough approximation, in a more pronounced way to establish a detailed and reliable mental representation of numbers (e.g. a linear number line). The transfer of basic research results into direct classroom educational methods seems fairly difficult. Methodologically, it is crucial to be able to conduct experimental work in the schools that systematically and empirically test the efficiency of instructions and teaching strategies.

In sum, mathematical education is predominantly settled within the domain of educational sciences that often study and develop certain instructional methods. The anchored instructions approach (Young, 1993) constitutes one of these and is preferentially applied in natural sciences and mathematics classes. Learners are involved in a narrative story (e.g. "The rescue of the

eagle”) containing a problem they have to solve. All the information needed is contained in the story and the children’s task is to develop proposals for a proper solution. This contributes to developing self-initiated learning processes for the acquisition and application of mathematical skills (e.g. Pythagoras’ theorem). Accordingly, in a Swedish study of seventh-graders, Samuelsson et al. (2008) found that traditional instruction and a more interactive approach focusing on the introduction of different solvable problems as basis for group discussions had a positive effect on quantitative concepts like knowledge of shapes, sequences, series of numbers and the ability to provide missing numbers in patterns of numbers, whereas a positive effect of independent learning without instruction or interaction could not be observed.

Early childhood education aiming at teaching children (aged 3-6 years) mathematical concepts typically takes place in pre-school settings. There is a general agreement that more research, particularly for this age, is necessary (see Ginsburg et al. 2008). It was shown that preschool mathematical knowledge predicts later success in school and even in high school, and correlates with a variety of higher cognitive skills (Clements & Sarama, 2011).

Notably, even without formal teaching, young children typically develop informal or intuitive mathematical skills such as basic addition or subtraction. Ginsburg et al. (2008) also pointed out that intuitive mathematics includes rudimentary comprehension of the notion of space, shape, pattern, as well as numbers and operations. There are also indicators that children display interest in mathematical ideas, for example by spontaneous counting, by creating patterns with building blocks or comparing the height between two building block towers. This does not mean, however, that they can grasp abstract concepts in the same way adults can without formal instructions: Their difficulties in understanding that an oddly shaped triangle is as much a triangle as an even-sided one may be an indicator of their flawed abstract concept of a triangle.

In general, specific programs are necessary that draw the attention to explicit mathematical concepts. Educational goals in teaching young children mathematics should not only include the basic concepts such as numbers, operations, geometry of shapes and pattern recognition. Ginsburg & Amit (2008) instead suggest the inclusion of more challenging topics, such as enumeration and understanding the cardinality of a set.

Mulligan and Mitchelmore (2009) introduced an educational concept called “Awareness of Pattern and Structure” (AMPS), which focuses on pattern and structures

across mathematical concepts such as algebra or multiplication. The authors operate under the presumption that knowledge and understanding of the underlying structure is crucial to understanding mathematical concepts. According to the authors, a mathematical pattern is any predictable regularity, such as periodically arranged rows and columns of the same size in a rectangle. Adults can effortlessly recognize those patterns; young children cannot. Pattern recognition and manipulation are crucial across a wide range of mathematical fields, such as numbers, measurement and space, and early algebra or modeling. The authors propose AMPS as a way of integrating these divergent fields. The research deals with the question, “whether it is possible to consistently construct structural categories across a range of tasks. The main question is whether “an individual student’s general level of structural development [is] related to his/her mathematical achievement”. The investigation was conducted with 103 first graders solving a wide range of age-appropriate tasks, whose solution relied on students’ structural development. Examples included asking the children to draw features on an empty ruler to reconstruct triangular patterns using six dots or to find all possible combinations of a number multiplication task and to explain their strategies. It was subsequently found that children achieving high scores also showed a strong achievement in mathematics, whereas low-scoring children also performed poorly in mathematics. The authors propose four different stages of structural development: The pre-structural stage, the emergent stage, the partial structural stage and the stage of structural development. About 90% of the children’s responses could be assigned to one of those categories.

Environmental Influences on Mathematical Abilities

The impact of socioeconomic variations on mathematical competences is reviewed by Jordan & Levine (2009). It is stated that how well a child does in mathematics is often related to his or her socioeconomic status; overall, it was often found that children from low-income families perform worse in this subject than their counterparts from families with a higher income. Performance in mathematics does not only depend on what a child is taught at school, but also on the number sense she acquires before the start of a formal education. There are several components to its development, such as verbal subitizing, counting, numerical magnitude comparison, estimation and arithmetic operations. It is with this symbolic representation that children with mathematic difficulties grapple with. They might display these difficulties in counting procedures, such as continuing to rely on the finger counting method in an addition problem, or they might display poor calculation fluency. Often there are associated reading and language difficulties.

The numerical competence in kindergarten was compared to mathematic achievement in third grade. The findings indicated that children from low-income families lagged behind their peers upon entering kindergarten and they were overrepresented in the group, in which the growth trajectory displayed subsequent low levels of mathematical ability. Moreover, the study showed that numerical competence predicted mathematical ability, emphasizing the need to teach young children number sense. On average, children from low-income families start using their fingers about a year later than children from medium-income families and use them for a longer period of time. Other studies have shown that instructional programs can bridge the gap in mathematic performance caused by socioeconomic status, although it is not known which aspects of the programs are effective and which ones are not.

Levine et al. (2010) conduct a longitudinal study of 44 pre-school children. The study shows that the relationship between children's knowledge of numerical cardinality was predicted by their exposure to number talk in their home environment. The study started when the children were 14 months old and mainly consisted of a 90-minute visit every four months until the children were 36 months of age. The visits focused on the natural interactions of one parent with the child. The number words of parents and children were counted and the children's understanding of cardinality was measured using a Point-to-X task. The parents' number talk significantly predicted the children's knowledge of cardinality at 46 months of age. Thus, parents should be encouraged to engage in number talk with their young children, since knowledge of cardinality is part of a set of numerical abilities, which also predicts later mathematical achievement.

Clements & Sarama (2011) review research-based mathematic interventions for preschoolers. These interventions are especially important for children from a low socio-economic background, since they might have insufficient opportunities to further develop the necessary cognitive functions necessary for learning more complex mathematics. They include "Rightstart", "Pre-K Mathematics" or "Building Blocks", all aimed at 3- to 5-year-olds. For example, "Rightstart" aims at developing certain mathematical ideas such as quantity comparison, counting competencies and initial notions of changing set size before integrating them. The program included some games that highlighted different types of quantities. There is some empirical evidence that this program significantly improved the mathematical performance of particularly children from low-resource backgrounds compared to their peers.

"Building Blocks", on the other hand, is based on the mathematics of children's everyday activities and uses building blocks, art, stories, puzzles and games. It aims at developing the number sense as well as spatial and geometric concepts. Again, there is some empirical support that this program fosters mathematical skills in children from low-income families. These programs contain learning trajectories and rely on the entire curriculum, including text, software and professional development. Learning trajectories are defined as "directions for successful learning and teaching". In order to attain a goal, the children are taught at successive levels of thinking and integrating the different concepts, skills and problem solving.

The authors of the article studied these intervention frameworks on a large scale using "Building Blocks". They not only looked at the intervention curriculum, but also included collaborations between the administration, the teachers and the children's families, as well as professional development. Teachers implemented the intervention over the course of two years, aided by professional development and web applications. This experiment had a strong positive effect on children's mathematical abilities, including children from low-income backgrounds, generally supporting the notion that such educational interventions are beneficial to children's mathematical development.

Most importantly, Clement and Sarama (2011) propose the concept of the learning trajectory, which helps to develop appropriate instructions with respect to the conceptual goal, and children's developmental progression. Consequently, it is crucial to know the cognitive processes, which are necessary for a particular mathematical competence, and design instructions, tasks and opportunities so that they foster the development of those processes a higher order of thinking can be achieved. It was shown that the knowledge early childhood teachers had, positively correlated with the children's achievement.



The Parmenides Foundation's Proposal

Learning to Think (L2T) – The Art of Mathematical Thinking

Our proposal aims at integrating the reviewed information into our existing L2T framework. Firstly, we will give a brief overview of the motivation of our approach. Secondly, we will detail the foundations of our model. Thirdly, we will deduce and model some practical implications for teaching mathematics.

A long lasting problem: In 1959 Max Wertheimer, the famous Gestaltist, visited mathematic lessons in classrooms. Wertheimer was interested in the nature of productive thinking, which he strictly demarcated from “blind” and reproductive thinking. In one classroom the geometrical concept of a parallelogram was introduced. The students learned to determine the height and base of a parallelogram and to calculate the area accordingly. The teacher encouraged the children to repeatedly do this for a variety of parallelograms. Wertheimer observed the scene and after a while he drew a parallelogram that was rotated by 90 degrees and asked the children to calculate the area. Most of the children were puzzled and claimed that they do not know how to do this, since they never were told. Wertheimer concluded that children had no insight in the nature of a parallelogram and did not see the shape and the interrelationship between shape, sizes and equation. The easiest way

would have been if children had rotated the parallelogram by -90 degrees. Wertheimer realized that it might be necessary to represent the problem in a more general way. The problem could be understood very easily, if the problem solver had the insight that on the left side of the parallelogram something is “too much” that “is missing” on the other side (Wertheimer, 1959, Figure 7), and by conveying the “disturbed” figure into a “good Gestalt”, children immediately realize what is meant by the area of a parallelogram and the close connection to the “easier” and more familiar shape of a rectangle.

Wertheimer's example aims at supporting mathematical thinking by attaining insight into the underlying mathematical principles (e.g. what is meant by an area?). That is, the full comprehension of the underlying law or idea, and as a result not a blind procedural, but an informed conceptual and declarative knowledge, that can also very easily be used when faced with rotated parallelograms.

The Parmenides Learning to Think project pursues exactly the same goal. That is, to identify, describe and explain the basic cognitive operators of mathematical thinking that underlie a mathematical thought and to develop methods of fostering them in a child-oriented

Figure 7: Wertheimer's parallelogram problem (1959). To calculate the area of the shape, one can realize that a triangle can be drawn on the left hand side (gray) that, when moved to the right hand side (black triangle), will form a perfect rectangle, thus, rendering calculation of the figure straightforward.

a) Realization



b) Solution



and sustainable way. As mentioned above, one core problem when analyzing the misconception of children's, but also of teacher's mathematical thinking was the rigid application of algorithms (e.g. Nunes, 2001; Ryan, J. & McCrae, B. (2005/2006); etc.), and the lack of structural understanding of the mathematical concepts. As other authors demonstrate, instructions that focus exactly on structural aspects (Mulligan and Mitchelmore, 2009) reveal a positive effect on children's achievements.

We basically distinguish between two processes that are important in thinking: Analytical and constellatory processes (Evans, 2008, Figure 9). Our model incorporates several aspects of the reviewed state of the art. On the one hand, it was shown that the representation of abstract and non-abstract representations of mathematical objects is an important ability, and can even be fostered, as the dyscalculia literature has shown (e.g. mental number line – constellatory thinking). On the other hand, the insight into the underlying laws and rules, and the development and understanding of the structure (as mentioned before) is sufficient (analytical thinking).

1. Analytical processes are deliberate operations, which are applied to mental representations. They are slow, limited, but conscious and explicit.
2. The constellatory operator is implicit, automatic, fast and virtually unlimited. Constellations can emerge from the given perceptual stimulus.
3. For example, three given dots are grouped together and form the shape of a triangle. Conceptually, from the mutual interrelationship of the given three dots emerges the concept of a triangle (Figure 8 a)) and activates prior knowledge, meanings and laws (the relationship and relation of angles, sides, etc.)

Conceptually, constellations can also re-structure given information, i.e. given the task to sum up the numbers from 1 to 100 can be done by a stepwise, error-prone and slow strategy: $1 + 2 = 3 + 3 = 6 \dots$, or by realizing the underlying symmetry of the given series and deducing a more elegant, error-free, fast and general principle (Figure 8 b)).

In more detail, we state that there are at least four different mental operators that play a role in mathematical thinking.

Figure 8: Examples, for perceptual a) and conceptual constellations b).

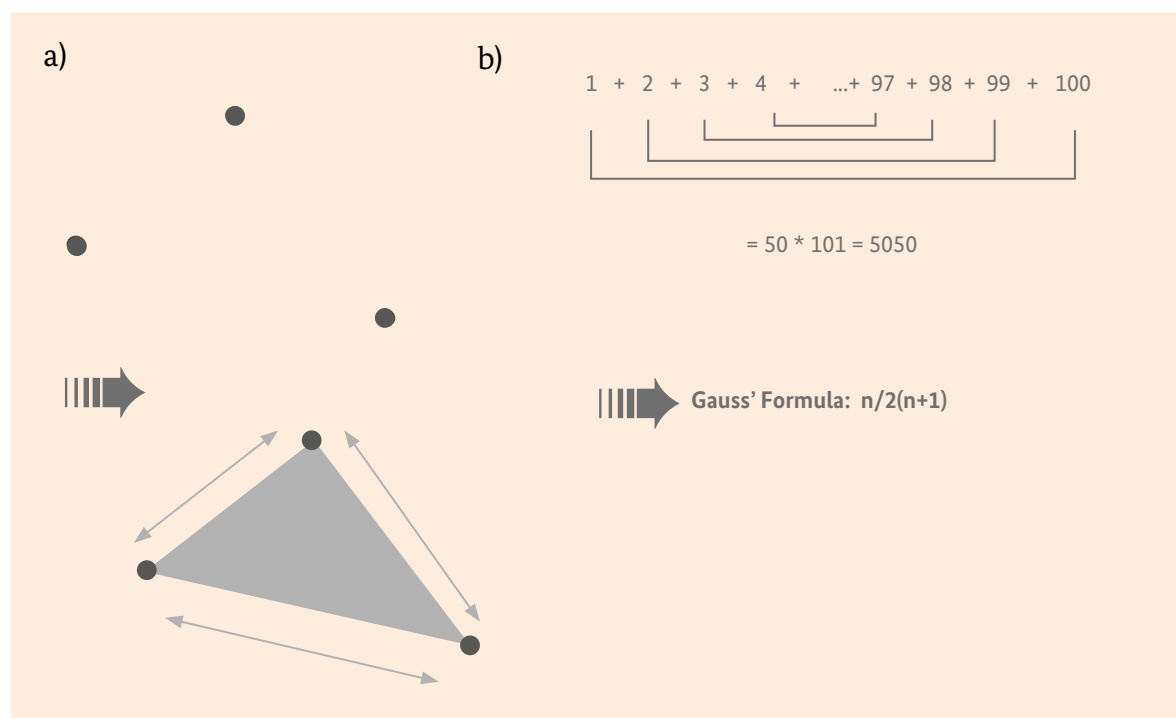
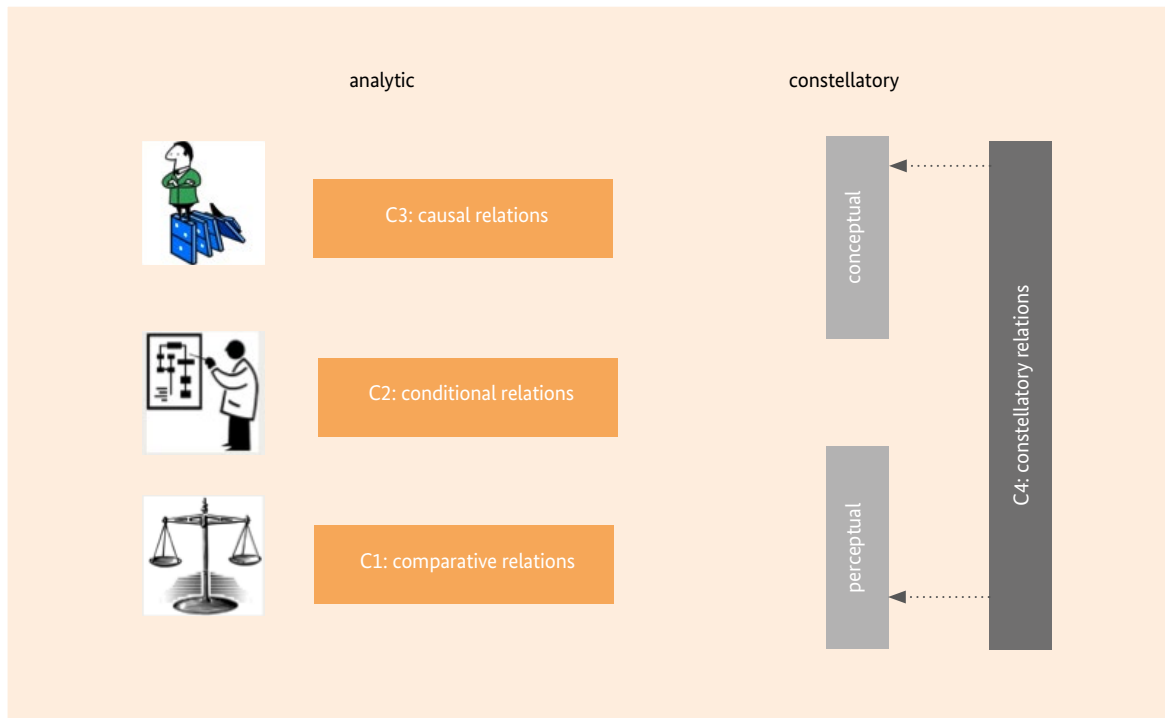


Figure 9: Basic operators of thinking. On the left: three analytic operators (C1-C3). On the right: constellatory operator.



We assume that complex thinking always requires:

- to compare and evaluate given information (comparing: objects, alternatives, thoughts, options, situations, etc.) and see differences or similarities – comparative mode (C1). For the understanding of mathematics this is a very basic operator. As we have seen, e.g. rough estimation plays an important role in the development of mathematical concepts (greater than, less than). Moreover, it is important to see numeric equivalence,
- to deduce rules from given information. That is, if ... then ... statements are extracted in order to understand the relations between the constituents of the situation – conditional mode (C2). This is the basic operator to deduce general laws from given information,
- to understand causal relations. That is, it is necessary to apprehend the relationship between cause, effect and the “hidden force” that intervenes between cause and effect. This operation allows gaining deeper insights into the processes occurring and controlling the world, and enables us to make predictions for upcoming events. This holds true for both the social and the physical domain – causal mode (C3). This is necessary to understand and produce effects, when parameters of a system are changed,
- to appropriately reduce, configure or re-configure complex information. That is, building new configurations, chunks or clusters from

the given information, in order to see new patterns or underlying laws in the apparently unstructured or confusing information – constellatory mode (C4). The basic principle to find new structures in the given information, emerging new meanings and attaining higher order semantic knowledge.

We further assume that the C1, C2, and C3 operators build a hierarchically ordered structure. From a cognitive perspective, C1 is the most basic and simplest operator while C3 is the most demanding. The C4 operator builds an orthogonal dimension. It is conceivable that the C4 operator spans a continuum from simple perceptual grouping processes (Gestalt laws) to complex conceptual re-grouping processes (insight problem solving). Thinking, according to this model, consists of the collaboration of these four basic operations. We assume that the C4 operators are established in early childhood. We propose that early education of those four operators build the ground for mathematical thinking. In the next paragraphs we briefly sketch how to foster numeracy by regarding developmental aspects of different age groups:

Young infants (1-3 years): We follow Lakoff and Nunez (2001) and assume that there is a very close interconnection between the development of mathematical concepts and the body experiences children make, when they are interacting with their environment. For example, infants very early show great pleasure in emptying containers filled with objects. What they

implicitly learn is that a set consist of elements, a set can be changed and, finally, a set can be empty (empty set). They also learn that a container is a distinct space that includes elements and the number of elements can vary. Lakoff and Nunez convincingly demonstrate that embodiment and interaction with objects in the environment drives the development of metaphors and the development of mathematical concepts.

We suggest connecting our dual-process account with the embodiment ideas of Lakoff and Nunez. That is, we will provide opportunities for the psychomotor domain (Cognitope-Principle™, Figure 10) of young children, which are constructed in a way that basic mathematical concepts (e.g. coarse estimation, subitizing, counting, categorization) at different perceptual levels are fostered. The idea is to provide perceptual constellations that elicit a holistic activation of the children's perception and establish rich and general conceptual principles, which establish a number sense in the children.

Figure 10: A simple Cognitope™, in which children can experience being part (element) of an empty container, being inside or outside. When inside, they experience the boundaries, the whole, the small windows, the view from the left or right position, etc.

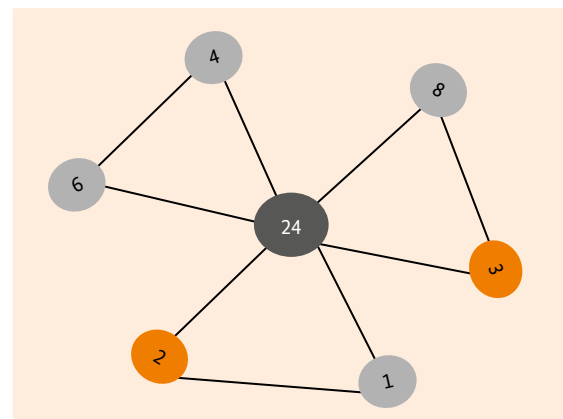


Kindergarten (3-6 years): At the next level, we link motor activities more strongly with increasingly abstract concepts to particularly foster basic thinking operators (C4). The concept of a number line can be introduced as a visual spatial object within a room. Programs like "Building Blocks" (see above) exactly use children's daily activities and provide opportunities and games, in which children have to identify the shape of different objects in a "feely box" for example. That is, they have to compare (C1) and apply and deduce rules (C2) (Clements & Sarama, 2011). At this age, computerized training programs can be an interesting supplementary domain to foster thinking (Building Block software). Moreover, as the reviewed evidence

shows, it is crucial to help children to establish an appropriate number line representation. This can be done by introduction of visual-spatial and imagery tasks, and by perspective taking (bird's eye view).

Primary School: We propose a stronger focus on the basic operators of thinking, which in this age group can be fostered by more complex and numeracy domain specific tasks. We assume that it is worthwhile to support children's representation of numbers and the number line by visual-spatial, mental exercises. Figure 11 is an example of a new, visual-spatial representation of two pairs of numbers, which, when multiplied, equal 24. The idea is that children always find "two friends" of the 24, which, when multiplied, equal 24. That is, firstly, a perceptual constellation of results and operands is provided, which is very easy to memorize. Secondly, the idea of "the problem as a friend" is implemented. Children are motivated to play with friends and not with abstract and lifeless concepts. Thirdly, further discussion about the relationship between the given numbers or the concept of prime numbers can be introduced.

Figure 11: Find "two friends" of the 24, which, when multiplied, equal 24.



Furthermore, a detailed analysis of children's typical errors and misconceptions to determine individual thinking errors should be done. Consequently, interventions could be exactly aligned with the child's particular requirements. Computer programs can help to track children's reasoning processes, and reveal and indicate typical errors or biases.

Individual Motivation and Achievement: An important aspect is to increase children's motivation regarding mathematical thinking. We propose to provide an individualized and visual performance specification, which focuses more strongly on individual developments and resources than on a deficit-oriented, class

wide competition and “frustration” model. Children should learn to increase their self-efficacy and see “problems as a friend”, not as a constant threat to their own competencies.

As suggested by Clements & Sarama 2011, we propose the development of instructional activities that are aligned with the developmental progression and the intended mathematical concept (goal). This requires:

- a) **Culture:** Considering and exploiting cultural typicalities and particularities (e.g. language advantage), and prior knowledge and experience of the children (e.g. contextual calculations – sidewalk sale is used to introduce more abstract mathematical principles).
- b) **Teachers** are required to have detailed knowledge about developmental trajectories, in general, and for individual children, in particular. Teachers have to be trained to teach basic mathematical concepts, patterns and structures.
- c) **Parents** have to be informed about the development of mathematical thinking and the impact of using mathematical terms – parental talk – in daily activities for the development of mathematical concepts. The parents should be encouraged to play games with children that foster mathematical concepts, e.g. Parchessi fostering counting, estimation, etc.
- d) **Diagnostics** that pinpoint potential individual deficits and resources at the level of cognitive processes,

and the detection of misconceptions and systematical thinking errors.

e) **Instructions** that bridge the gap between the taught mathematical concept and the current developmental state.

f) **Training:** Computerized training programs, which foster mathematical thinking in a self-paced way, are required, and with respect to the individual abilities and deficits – adaptive training.

g) **Techniques:** Computer programs that give detailed visual feedback, track changes, motivate to continue, to learn and to improve. Programs which identify, and evaluate key features of mathematical achievements, and determine which elements have to be repeated.

h) **Changed Attitudes:** Mathematics as a field which allows learning problem solving, success, new insights and joy, and not a boring subject that is frustrating or threatening.

In a nutshell, we recommend starting to foster basic numerical concepts, which elicit a reliable number sense, as early as possible. Children should be provided with tasks and instructions which allow them to discover mathematical structures and to attain insights into the underlying mathematical concepts by themselves. The work on mathematical problems should be experienced as joyful and motivating (see Figure 13).

Figure 12: L2T summary.

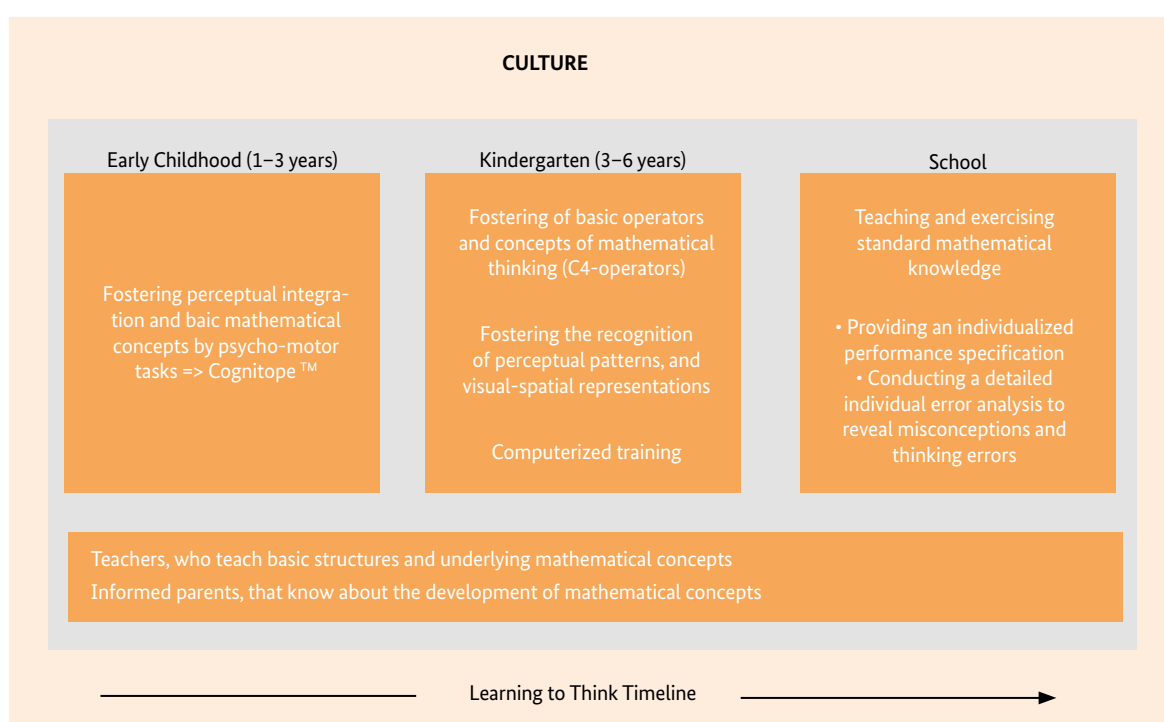
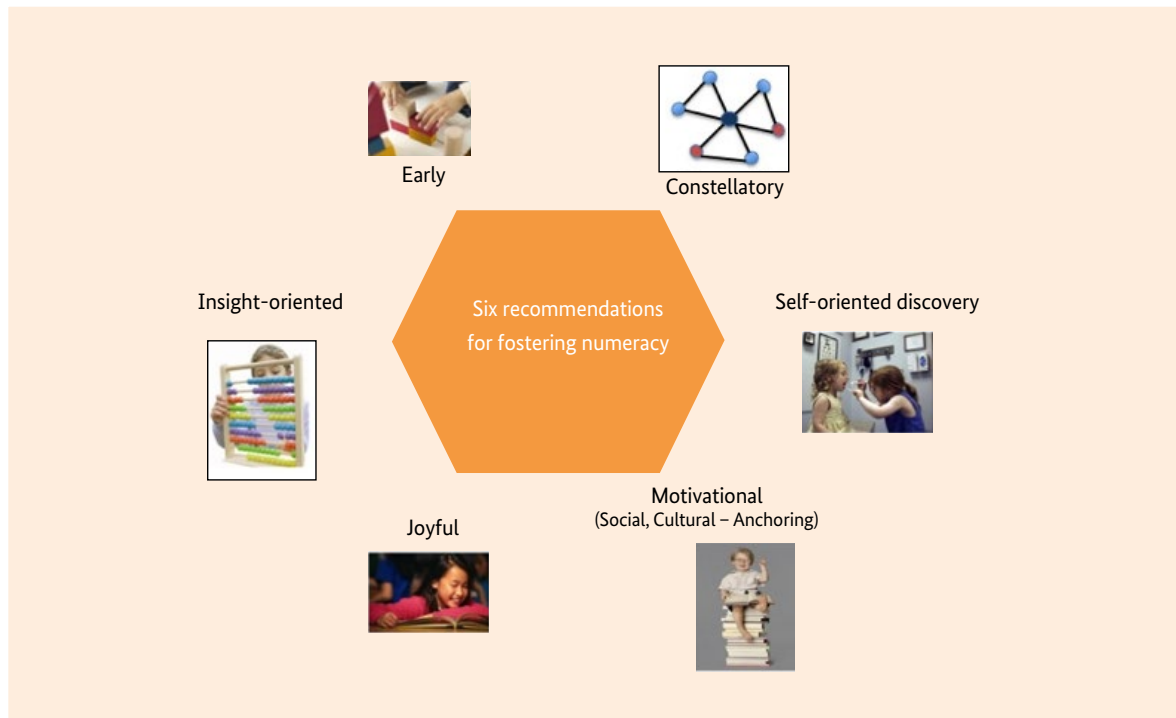


Figure 13: Six recommendations for fostering numeracy.





Motivational Aspects with Special Focus on Low Income Countries

After introducing the Learning to Think approach to teaching numeracy, we now focus in greater detail on the question how to increase the motivation to learn numeracy considering the particular problems in low income countries.

As pointed out in Figure 1, the successful implementation of a new program requires to consider at least motivational aspects, social attitudes, existing stereotypes and cultural aspects. Basically, we have to address the needs of different groups of persons, as there are “teachers”, “children/students” and “parents”. But before we take a look at pure arguments for motivating people to learn or foster mathematic education, we want to mention the “Cash for Delivery” (CoD AID) approach of the Center for Global Development. This approach is still in a testing phase (e.g. Twaweza, an Uwezo supporting organization, tests CoD AID for fostering numeracy and literacy in East Africa), so that nothing concluding can be said, but the first results seem to indicate that CoD AID could be a powerful tool.

To convince the targeted groups of people of the importance of numeracy education, we have to separately address the perspective of each group (multi-perspective-account):

1. Convincing teachers of the importance of numeracy and mathematical skills in general should not be necessary. But to convince and motivate them to apply new and unfamiliar methods might be a great challenge. A way of motivating teachers is pointing out their importance for the latter success of individuals and society, and to pay and educate them accordingly. According to this logic, society will develop, if it has excellent engineers, IT-specialists, technicians, scientists, etc. – and mathematics is crucial for these professions.
2. The title of our desk-study “mathematics – the cheapest science for great minds” is far from being an empty phrase, since a mathematically gifted child

does not need much more than a pencil and paper in order to become an excellent mathematician. What the child additionally mainly needs is the motivation to study mathematics. So it is necessary to inform the children about the unlimited possibilities of success that studying mathematics offers to them, although they stem from a low income country. Furthermore, it has to become clear for the children why mathematics is important for their life, for their career, for their understanding of the world, for their self-efficacy, etc. It has to be pointed out that there are alternative life designs, which are contrasts to the life of their parents, e.g. as farmers. The motivation can also be increased to use more cultural relevant examples of daily activities and contextual information to introduce mathematical concepts. Moreover, this can also be used to increase gender-specific interests.

3. Students’ parents play a crucial role in increasing the motivation for their children to learn numeracy. It can be emphasized that education can increase the future income of their children. When parents are convinced of the importance of mathematical skills, instructions are needed to inform and support parents. The usage of mobile learning applications of parents with their children is an adequate mean, if a suitable income is available. A crucial problem concerning mobile learning applications is to find appropriate and child-oriented programs. Easier and cheaper, as recommended by the mathematical-didactic experts of Ludwig Maximilian University of Munich, is the play of classical board games like the German “Mensch ärgere dich nicht”, the English “Ludo”, the French “Jeu de petits chevaux” and others fostering numeracy in early years with simple efforts. Playing these games is an excellent first numeracy training for young children, and it is very easy and fun for children to build adequate versions of these games on their own. Even cheaper, but just as effective, is the use of parental talk pronouncing mathematical concepts like counting, quantifying or categorization (see our review).



Measurement Approaches and Implications for Teaching Strategies Concerning Numeracy Education in Low Income Countries

The usual way of evaluating different teaching strategies is testing students, who were taught using different strategies, at a certain point in time. That procedure is well established and without any doubts a powerful instrument for that reason. Nevertheless, there are more possibilities of screening and evaluating teaching strategies that can be used additionally, as we show in the following. Our approach consists of two pillars, recording or live streaming of lessons and analyzing the videos, and the use of mobile learning applications. After the introduction of our new approach we will discuss the technological requirements.

Screening and Evaluating Teaching Strategies by Using Cameras

The evaluation of the efficiency of a newly implemented educational instruction is fairly complicated. By recording the mathematical lessons, not only achievement at certain time points can be evaluated and compared, but also the process can be analyzed. This refers to the teacher, as well as to the children. It is interesting to reveal misconceptions, problems and resources. The usage of cameras is not only an excellent method of avoiding that problem. Recording teaching lessons is useful in at least three ways:

1. The range of intervention studies could be extended.
2. The implementation of recommended teaching strategies and applied instructions could be evaluated in a much greater detail.
3. Conceptual problems and individual developmental trajectories could be exactly addressed.

But there are also disadvantages:

1. Technical problems.
2. Resentments against monitoring and “controlling” teacher and children.
3. The data analysis can be extremely sophisticated and costly.

Using Cameras for Widening the Range of Studies in Low Income Countries

In order to evaluate the efficiency of a certain teaching strategy fostering numeracy, conducting a case or intervention study is an important and necessary measure to assure the appropriate application and the correct use of the intended strategies. In low income countries a lack of available trained professionals conducting a study could arise. The use of cameras can face and solve this problem, because the local teachers can be involved in a reasonable and natural way. Instead of having many teams of professional scientists for a long period of time in several low income countries, it is sufficient sending them to the countries, which participate in the study, only for a short period of time to train the local teachers. After the training, the local teachers have to record their lessons with cameras and transmit the videos to a central database, so that the scientists can evaluate the teaching process. If there should be any problems, the scientists can intervene and support the teachers in a detailed and problem-oriented way. With this procedure large studies in low income countries can be executed in a fast and efficient way. However, depending on the country there could be a language problem. If the scientists do not understand the spoken language in the classes, translators would be needed and costs for the study would rise.

Screening the Implementation of Recommended Teaching Strategies

The same procedure as described above can be used for screening, if a recommended teaching strategy is correctly implemented in everyday school life. Of course, it is not necessary to review and assess all videos. It might be sufficient to take samples. Implementing a recommended teaching strategy does not have the same constraints as a study, which has to fulfill scientific standards.

An alternative to recording teaching lessons is online live streaming. For example, if systematical problems occur, it might be worthwhile to live stream a lesson, so that the instructor can intervene instantly or talk to the teacher directly after the lesson. Whether this technique is feasible, depends on the technological infrastructure of the countries in question.

Screening and Evaluating Teaching Strategies by Using Mobile Learning Applications

The second pillar of our approach is the usage of mobile learning applications. These applications have to be easily understandable and adapted to the curriculum, so that the students can use them at home. It is also very important that the applications address children's "play instinct". It can be taken as a fact that people learn better and faster, if they enjoy the learning exercises. In consideration of the fact that the use of mobile learning applications at home has to be voluntary, it is even more important that the applications are entertaining, such as the application used in the M4Girls project in 2008 in South Africa.

The applications must have an offline, as well as an online mode. The offline mode is necessary so that the students can use the applications in areas without internet access, too, and the online mode, obviously, is crucial for sending data, which is produced by the students during the use of the applications, to a central database as often as possible. This data can be used in at least three ways:

1. An implemented teaching strategy can be screened and evaluated concomitantly to the implication, rather than at the end of this process.
2. The level of performance of every individual student can be regarded additional to tests in school.

3. Developmental trajectories, level of achievements, deficits and resources can be visualized and used for increasing children's individual motivation

There is an increasing amount of learning applications, but there are, at least to our knowledge, no appropriate applications available for our purposes. The main reason for that is obviously that we need applications, which are complementary to a certain curriculum and are connected to a central database, so that the data can be analyzed and evaluated by scientists and teachers. Nevertheless we should think of the benefits of such a kind of application for screening and evaluating the implementation process of a new teaching strategy in low income countries. (The second advantage of mobile learning applications mentioned above will be discussed below).

Using mobile learning applications has the big advantage that all data is already digitized. The digitized data can be analyzed efficiently and can provide great details on the individual learning process, for example the already well-known concepts, existing misconceptions, committed errors, attention spans etc. The analyzing software can be adjusted in a manner that the scientists, who execute the implementation process, are informed, if the performance of the participating students does not improve or even begins to decrease. Beside this very useful feature the collected data can be used for the concluding evaluation of the implementation process.

As excellent as the advantages of the usage of cameras and applications are, as serious are some of the obstacles, which have to be overcome before they can be used, as we show in the next chapter.

Technical Requirements for the Usage of Cameras and Applications

Due to developmental differences, low income countries do not possess the same technological infrastructure as industrialized countries. Thus, it has to be examined which technical devices and technologies are available and usable in the targeted countries. The needed technical devices, respectively technologies, are: Cameras (and eventually microphones, depends on the quality of the available cameras), smartphones, mobile learning applications and internet access.

The area-wide usage of tablet PCs or smartphones is by far harder to put into practice than that of cameras.

Ideally, all students have a tablet or smartphone, in order to use the learning applications. According to the UNESCO report “Turning on mobile learning in Africa and the Middle East” (p.12) the proliferation of mobile devices in Africa increased by approximately 30% every year in the last ten years, and there are forecasts that by the end of 2012 more than 70% of the African population will own a mobile device. In the low income countries of Asia the situation seems to be equal or even better considering that the data is from 2010 and 2011 (UNESCO report “Turning on mobile learning in Asia”, p. 9-11).

The negative sides of this actually positive trend are social and gender inequalities concerning the allocation of mobile devices, so that the probability that a rich and/or male person possesses a mobile device is much higher than a poor and/or female person. Another problem is that the majority of the African mobile device owners do not possess smartphones at the moment, but this fact is not as problematic as it maybe *prima facie* seems, because the technical progress of smartphones develops and prices decrease rather quickly. The possibility of equipping all individual students of a school with smartphones is far too expensive, and equipping only some students is impossible, because of social reasons. Nevertheless, students who already possess smartphones can be integrated in studies and sensible mobile learning applications can be recommended to them.

The most serious problem concerning mobile learning applications is the non-existence of appropriate ones for our criteria. This means that they have to be developed and implemented, which in turn means that firstly a curriculum has to be developed to which the applications can be adjusted; both take a lot of time and money. But those efforts, which are necessary in order to get appropriate applications, are well invested. Once a development aid company has a set of qualitatively good applications, the cost for keeping them up to date are relatively low and the benefit for the students is presumably very high.

The availability of permanent internet access is not by all means necessary. Of course, the claim of a constant feedback is not fulfilled, if the internet access is not permanently given. However, it is still fast enough, if the data is sent once a day or every second day at a certain time.

Interestingly, the availability of mobile internet, compared to stationary access is more wide-spread in most low income countries (UNESCO report “Turning on mobile learning - global themes”, p.15). This fact, combined with the fast technical development of mobile devices, gives hope to an area-wide usage of smartphones in the near future of these countries. So let us now have a look at mobile learning.



Mobile Learning, E-profiling and Adaptive Training Configurations

Even if the first mobile learning initiatives were started in Europe in the 1980s (UNESCO report “Turning on mobile learning in Europe”, p.12), mobile learning is still a relatively new phenomenon in most countries. It seems to be clear already that its importance can hardly be overstated because “... there are over 5,9 billion mobile subscriptions worldwide ...” (UNESCO report “Turning on mobile learning - global themes”, p.15) and the market is still growing. Especially for low income countries mobile learning is a great opportunity to generate knowledge, because – as already stated above – getting mobile internet access in these countries is easier than getting stationary internet access.

In the remainder we focus on two initiatives, which have implications for numeracy. Furthermore, we introduce an application in development at Parmenides Foundation at the moment, of which we think it will be very efficient.

A very elaborated and successful project is MoMath, which was founded by the South African Department of Education and Nokia and won the SAFIPA (South African Finland Knowledge Partnership Program) reward for Best Social Impact in 2011. It started in South Africa in 2009 through a cooperation of the South African Department of Education, Nokia, Mxit (biggest African social network), Cell C (mobile provider in South Africa), MTN (communications company in Africa), Maskew Miller Longman (textbook publisher in South Africa), and SAFIPA. MoMath addresses students in the 10th grade to improve their mathematical skills. In the beginning MoMath involved 260 students, at the end of 2011 the number of participants had grown to about 25,000 students, 500 teachers, and 172 schools; to keep the costs low all students have to use their own mobile device. Since more than 10,000 math exercises, which cover the whole curriculum, are completely text based, no smartphones are needed for the transmission. In 2010, first results of the project were published: Competency in mathematics rose 14% and 82% of the participating students used MoMath outside of school. The great success of MoMath led to a new partner for the project, the Commonwealth of Learning, and now the cooperating organizations plan

to bring MoMath to another three African countries. The second mobile learning project mentioned in this chapter is something very different to MoMath, because, on the one hand, it was a small-scale project (one second grade class in Scotland), and, on the other hand, it was a portable video game (“Nintendogs”) used instead of a real mobile learning application. The reason for mentioning this project is to show that the fruitful usage of mobile devices does not necessarily depend on large-scale and deeply elaborate projects, but that it can also be done on one’s own initiative. In this game the player has to look after a puppy including cleaning, playing, taking it to the veterinary and dog contests, and so on. Money is an integrate part of this game; for example, it can be earned with good results in dog contests and spent at the veterinary or the dog toyshop. Understanding numeracy is crucial for successfully playing the game. The children had to write stories about what they experienced with the puppies (they were supported by older children of the seventh grade) and calculate how much money they could spend at the dog toyshop, so that they still had enough money for the veterinarian and things like that. The guided playing of this video game trained the handling of technical devices, writing, mathematics, and social skills; furthermore they were excited about school and loved going in this class.

The Parmenides Foundation is currently developing an application that implements a new, sophisticated and individual learning supervisory system, which is called “Kairos” (ancient Greek notion for the right moment to act). It is based on the fact that for the effortless and sustainable recall of learned knowledge there are huge differences of effectiveness. This means repeating facts too early is often too easy, ineffective and time-consuming. Repeating facts too late makes it hard or impossible to recall the facts; the crucial task is to find the ideal point for repetition. This will increase the individual motivation, and improve and foster learning and the availability of knowledge. The Kairos-application uses algorithms that determine the most appropriate moments for repetition by measuring and analyzing the reaction time when knowledge is repeated. The algorithms and research are developed and investigated by Hedderik van Rijn, University of Groningen.



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